MA612L-Partial Differential Equations

Lecture 1 : Introduction: What is PDE?

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Course Details

Course Details

Until Mid-Term

- First Order and Second Order PDE
- IVP, BVP Homogeneous, nonhomogeneous
- Fundamental Solution, Green's Function
- Energy Methods
- Transport, Laplace, Heat and Wave Equations
- D'Alembert's Solution, Fourier Method, Poisson Integral
- PDE problems in Cartesian and Polar Coordinates on rectangular, Circular and annular regions

Course Details..

After Mid-Term

- Non-linear PDE
- Complete Integrals, Envelopes
- Characteristic ODE
- Hamilton-Jacobi Equation
- Conservation laws, weak solution, uniqueness, Riemann Problems
- Second order PDEs-Classifications
- Canonical form, Lax Milgram Theorem
- Maximum-minimum principles

Marks

Marks

Components	Marks
Test - 1	20
Test - 2	20
End Semester	60

Preliminiaries

What is a derivative?

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Definition 1 (Derivative (Rudin))
Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x), \tag{1}$$

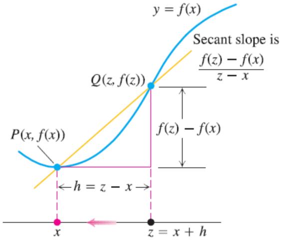
and define

$$f'(x) = \lim_{t \to x} \phi(t) \tag{2}$$

provided this limit exists. We associate with a function f a function f' whose domain is the set of points x at which the limit of (2) exists; f' is called the derivative of f. If f' is defined at a point x, we say that f is differentiable at x. If f' is defined at every $x \in E \subset [a, b]$, we say that f is differentiable on E.

What is a derivative?

What is the pictorial representation? Could you interpret?



Derivative of f at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

What is a derivative?

Definition 2 (Derivative (Thomas Calculus))
The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provide the limit exists.

Examples

Examples

$$f(x) = 4x^2 \implies f'(x) = 8x$$

$$y = e^{5x} \implies \frac{dy}{dx} = 5e^{5x}$$

$$y = \sin x + x^2 \implies \frac{dy}{dx} = \cos x + 2x$$

Examples

Failure

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

f is not differentiable at $x = 0$, because

$$\phi(t) = \sin \frac{1}{t}$$

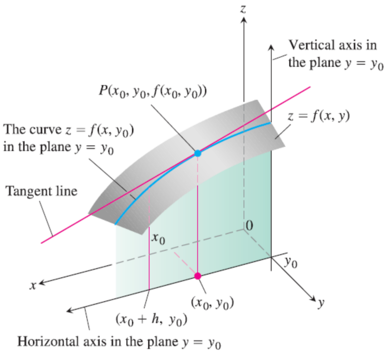
does not tend to any limit as $t \to 0$.

Partial Derivatives

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Definition 3 (Partial Derivative (Thomas Calculus))
Let $f(x, y)$ be a real valued function defined on a domain $D \subseteq \mathbb{R}^2$. Let $(a, b) \in D$. If C is the curve of intersection of the surface $z = f(x, y)$ with the plane $y = b$, then the slope of the tangent line to C at $(a, b, f(a, b))$ is the partial derivative of $f(x, y)$ with respect to x at (a, b).

Partial Derivatives



Partial Derivatives

Definition 4 (Partial Derivative (Thomas Calculus))
The partial derivative of $f(x, y)$ with respect to x at the point (a, b) is

$$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \left.\frac{df(x, b)}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

provided the limit exists.

Partial Derivatives

Definition 5 (Partial Derivative (Thomas Calculus))

The partial derivative of $f(x, y)$ with respect to x at the point (a, b) is

$$f_y(a,b) = \frac{\partial f}{\partial y}\bigg|_{(a,b)} = \frac{df(a,y)}{dy}\bigg|_{y=b} = \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}$$

provided the limit exists.

Partial Derivatives

Example

Find $\partial z/\partial x$ and $\partial z/\partial y$ where $z = f(x, y)$ is defined by

$$x^3 + y^3 + z^3 - 6xyz = 1$$

$$\frac{\partial z}{\partial x} = -\frac{x^2 - 2yz}{z^2 - 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{y^2 - 2xz}{z^2 - 2xy}$$

Partial Derivatives

Example

Find $u_x \pm u_y$ and $u_{xx} \pm u_{yy}$ if $u = e^{x-y}$.

$$u_x = e^{x-y} \implies u_{xx} = e^{x-y}$$

$$u_y = -e^{x-y} \implies u_{yy} = e^{x-y}$$

$$u_x + u_y = 0 \quad \text{and} \quad u_{xx} + u_{yy} = 2u$$

$$u_x - u_y = 2u \quad \text{and} \quad u_{xx} - u_{yy} = 0$$

History of PDE

History

- 1694: Leibniz used partial process, not explicitly, δm for $\partial m/\partial x$ and ϑm for $\partial m/\partial y$, a letter to de l'Hospital.
- 1694: Leibniz used partial differential equations to find envelope of the circles $x^2 + y^2 + b^2 = 2bx + ab$
- 1717: Hermann used PDE in problem of orthogonal trajectories to plane curves
- 1752: d'Alembert introduced one dimensional wave equation as a model of vibrating string
- 1759: Euler extended d'Alembert's work
- 1762: D. Bernoulli extended to 2 and 3 dimensional wave equations
- 1780: Laplace studied gravitational potential fields

History

- 1755: Euler equation of incompressible flows
- 1760: Minimal surface equation by Lagrange
- 1775: Monge-Ampere equation by Monge
- 1813: Laplace and Poisson equations by Poisson
- 1828: by Green
- 1839: by Gauss

History

- Navier-Stokes Equation for fluid flows: 1822-1827 by Navier, 1831 by Poisson and 1845 by Stokes
- Linear Elasticity, Navier 1821 and Cauchy 1822
- Maxwell's equation: 1864
- Helmholtz equation and Eigenvalue problem for the Laplace operator: 1860
- Plateau problem: 1840
- Korteweg-De Vires equation: 1896

History of Solutions

- Method of separation of variables: 1747 by d'Alembert, 1748 by Euler for wave equation, 1748 by Laplace and Legendre for Laplace equation and 1811-1824 by Fourier for heat equation.
- Infinite series solution in 1870's $\Longrightarrow$ Fourier Series and Fourier Integrals
- Real Harmonic and Analytic function of a single complex variable from Riemann (1851) to C. Neumann, Schwarz, Christoffel (1870)
- Green's function and special singular solutions for Laplace equation: 1835
- Dirichlet Principle: 1833 - 1851, by Green, Gauss, Kelvin an Riemann

History of Solutions

- Power Series method by Euler, d'Alembert, Laplace and others
- Power series method for nonlinear PDEs: 1840 by Cauchy
- Convergent power series to general systems: 1875 by Kowalewsky and Simplified by Goursat in 1898
- Existence of Dirichlet principle/integral questioned by Riemann $\Longrightarrow$ 23 Problems by Hilbert known as Hilbert Problems (regularity, existence etc)
- Method of integral equations by Neumann 1877, then by Poincare, Fredholm and Hilbert
- Picard's successive approximation, 1880's

History: 1900-today

- Hilbert, Levi, Lesbegue, Fubini, Zaremba, Tonelli, Courant
- Ascoli's theorem, Sobolev spaces, Weak solutions, distributions, well-posedness, ill-posedness, regularity, parametrix method
- Leray-Schauder theory, Singular Integrals, energy methods, Hilbert transform, Weyl lemma, hypoellipticity, pseudo-differential operator, Hille-Yosida theory, spectral theories, maximum principle

Preliminaries-II

Examples

Let us revisit the ODE example again

Examples

$$f(x) = 4x^2 \implies f'(x) = 8x \overset{?}{\implies} f(x) = 4x^2$$

$$y = e^{5x} \implies \frac{dy}{dx} = 5e^{5x} \overset{?}{\implies} y = e^{5x}$$

$$y = \sin x + x^2 \implies \frac{dy}{dx} = \cos x + 2x \overset{?}{\implies} y = \sin x + x^2$$

Examples

Let us revisit the ODE example again

Examples

$$f(x) = 4x^2 \implies f'(x) = 8x \implies f(x) = 4x^2 + C_1$$

$$y = e^{5x} \implies \frac{dy}{dx} = 5e^{5x} \implies y = e^{5x} + C_2$$

$$y = \sin x + x^2 \implies \frac{dy}{dx} = \cos x + 2x \implies y = \sin x + x^2 + C_3$$

What do these C_1, C_2 and C_3 represent? Note that, the derivative measures the slope of tangent lines of a given curve. However, when you know a tangent line, you end up with a family of curves or no curve.

Examples

Let us revisit the PDE example again

Examples

$$u = e^{x-y} \implies u_x + u_y = 0 \overset{?}{\implies} u = e^{x-y}$$

$$u = e^{x-y} \implies u_{xx} + u_{yy} = 2u \overset{?}{\implies} u = e^{x-y}$$

Will you get a unique answer?

Examples

In fact, what we will see in our next class

Examples

$$u_x + u_y = 0 \implies u = f(x-y)^1$$

$$u_{xx} + u_{yy} = 2u \quad \text{Helmholtz equation* with } k^2 < 0$$

Will you get a unique answer? [1] Using method of characteristics. *Separation of variable can be used to find solution, however, we need more assumptions and BCs to solve this problem.