MA612L-Partial Differential Equations

Lecture 10 : Shock Waves and General Solution

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Rarefaction - Recap

Rarefaction

$$u_t + uu_x = 0, x \in \mathbb{R}, t > 0$$

Suppose

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases} \tag{1}$$

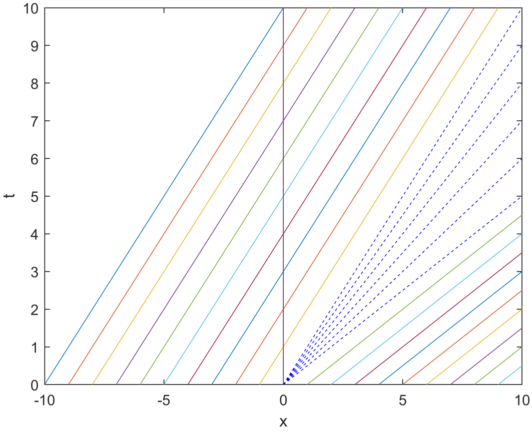
$$x(t) = \begin{cases} t + x(0) & \text{if } x(0) < 0 \\ 2t + x(0) & \text{if } x(0) > 0 \end{cases}$$

Solving for t, we have

$$t = \begin{cases} x - x(0) & \text{if } x(0) < 0 \\ \frac{1}{2}(x - x(0)) & \text{if } x(0) > 0 \end{cases} \tag{2}$$

Rarefaction

The characteristic lines corresponding to the initial condition (1). These lines are two families of characteristic lines with different slopes.



Rarefaction

Imagine that there are infinitely many characteristics originating from the origin with slopes ranging between $\frac{1}{2}$ and 1. The proper way to see this is to notice that in the case of $x(0) = 0$ implies that

$$u = \frac{x}{t} \quad \text{if } t < x < 2t$$

This type of waves, which arise from decompression or **rarefaction** of the medium due to the increasing gap formed between the wave fronts traveling at different speeds, are called **rarefaction waves**. Putting all the pieces together, we can write the solution of Burger's equation satisfying the initial condition as follows

$$u(x,t) = \begin{cases} 1 & \text{if } x < t \\ \frac{x}{t} & \text{if } t < x < 2t \\ 2 & \text{if } x > 2t \end{cases} \tag{3}$$

Rarefaction

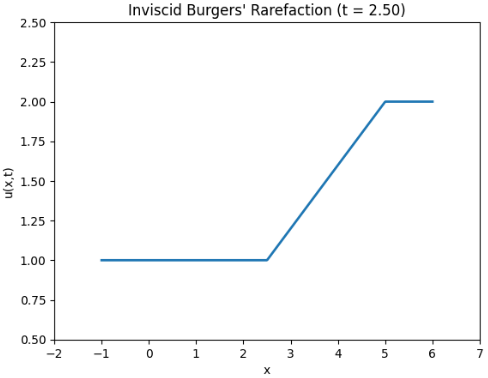


Figure 1: Rarefaction

Rarefaction

Imagine a highway with cars moving at different speeds:

- To the left of $x = 0$ (say, before a toll plaza), cars are moving slowly at a speed of 1 unit.
- To the right of $x = 0$ (after they leave the toll and accelerate), cars are moving faster at speed 2 units.

At time $t = 0$, there's a sharp boundary: all cars to the left are slower, all cars to the right are faster.
This is like when a traffic jam clears: cars at the front speed up first, then cars behind gradually accelerate, causing the jam to "dissolve" into a smooth transition.

Shock Waves

Shock waves

- It is the complete opposite phenomenon of rarefaction.

$$u_t + uu_x = 0, x \in \mathbb{R}, t > 0$$

- Here, it has faster moving from left to right, catching up to a slower wave.

Consider the following initial condition for Burger's equation

$$u(x,0) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{4}$$

Shock waves

The characteristic lines are

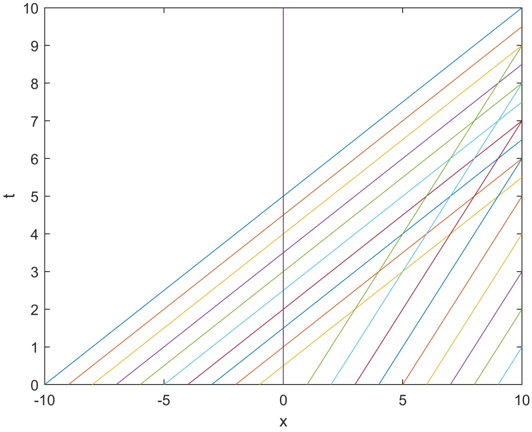
$$x(t) = \begin{cases} 2t + x(0) & \text{if } x(0) < 0 \\ t + x(0) & \text{if } x(0) > 0 \end{cases}$$

Solving for t, we have

$$t = \begin{cases} \frac{1}{2}(x - x(0)) & \text{if } x(0) < 0 \\ x - x(0) & \text{if } x(0) > 0 \end{cases} \tag{5}$$

Shock wave

The characteristic lines corresponding to the initial condition (4). These lines are two families of characteristic lines with different slopes.



Shock wave

Remarks

1. The characteristic lines originating at $x(0) < 0$ have smaller slope compared the characteristic lines originating from $x(0) > 0$
2. Characteristics from two families intersect
3. It leads to a problem as we can't trace back the correct characteristics to an initial value
4. At the intersection points, u becomes multivalued
5. This phenomenon is called **shock waves**
6. The faster-moving wave catches up to the slower-moving wave to form a multivalued wave.

Shock wave

Imagine a highway with cars moving at different speeds:

- To the left of $x = 0$ (after they leave the toll and accelerate), cars are moving faster at speed 2 units.
- To the right of $x = 0$ (say, before a toll plaza), cars are moving slowly at a speed of 1 unit.

At time $t = 0$, there's a sharp boundary: there's a sharp boundary between fast and slow cars.

Shock wave

- The fast cars from the left catch up with the slow cars on the right.
- This creates a traffic shock wave: cars pile up, compressing traffic density.
- Instead of spreading out (like in the rarefaction case), the transition region sharpens into a shock front that moves over time.

This models a traffic jam forming and moving backward along the road:

Shock wave

There are several examples of shock waves.

Examples 1

Examples

1. Moving shock - Balloon bursting, Shock tube
2. Detonation wave - TNT explosive or high explosive
3. Bow shock - Space Shuttle return, bullets
4. Attached shock - Supersonic wedges
5. Normal shock (at 90°) - Oblique Shock - Bow Shock, R-H
6. Supernova, an asteroid hitting Earth's atmosphere

Shock wave

$$u(x,t) = \begin{cases} 2 & \text{if } x < \frac{3}{2}t \\ 1 & \text{if } x > \frac{3}{2}t \end{cases} \tag{6}$$

To derive this, we must discuss scalar conservation laws, weak derivatives, test functions, entropy conditions, R-H conditions, and Riemann problems in the latter part of our course. Let us see one more theorem and see nonlinear PDEs again with Charpit's methods.

Shock Wave

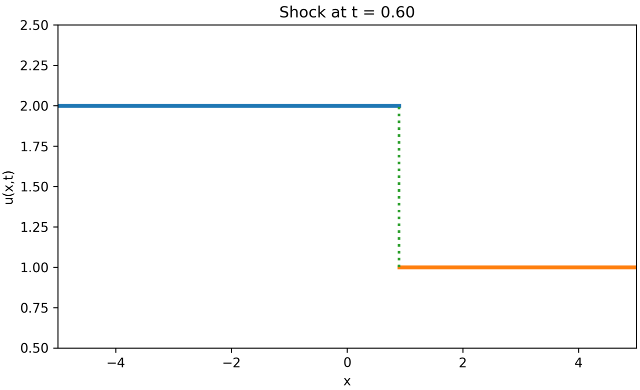


Figure 2: Shock Wave

Exercises

Exercise 1: Shock Waves and Rarefaction (Hard)

Solve Burger's equation for the following initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - x & \text{if } x \in (0,1) \\ 0 & \text{if } x > 1 \end{cases}$$

and then for

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

General Solutions for Quasilinear PDEs

Inverse Function Theorem

Theorem 2 (Inverse Function Theorem (Rudin))
Suppose $\mathbf{f}$ is a C^1 mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $\mathbf{f}'(\mathbf{a})$ is invertible for some $\mathbf{a} \in E$ and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then

1. there exist open sets U and V in $\mathbb{R}^n$ such that $\mathbf{a} \in U, \mathbf{b} \in V, \mathbf{f}$ is one-to-one on U and $\mathbf{f}(U) = V$
2. if $\mathbf{g}$ is the inverse of $\mathbf{f}$, defined in V by $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}, \mathbf{x} \in U$, then $\mathbf{g} \in C^1(V)$.

Let us rewrite this theorem for $\mathbb{R}^2$. This will be used for the existence and uniqueness theorem for quasilinear PDE.

Inverse Function Theorem

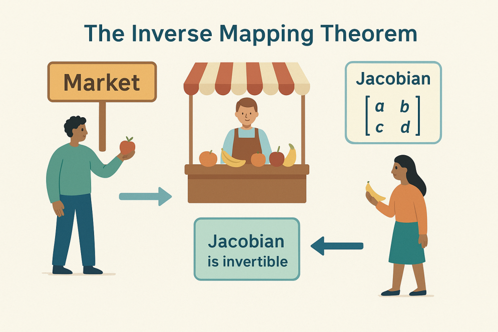


Figure 3: Inverse Function Theorem

Implicit Function Theorem

Theorem 3 (Implicit Function Theorem (Rudin))
(Refer to Class Lecture)

Theorem 4 (Rank Theorem (Rudin))
(Refer to Class Lecture)

General Solution

Consider the quasilinear PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{7}$$

Suppose that $P(x, y, u) \in \Omega, v \neq 0$ The characteristic curve

$$\Gamma : \begin{cases} x = x(s) \\ y = y(s) \\ u = u(s) \end{cases}$$

can be represented as the intersection of two surfaces

$$\begin{aligned} \Gamma &= S_1 \cap S_2 \\ S_1 &: \phi(x, y, u) = C_1 \\ S_2 &: \psi(x, y, u) = C_2 \end{aligned} \tag{8}$$

for which n_ϕ and n_ψ are linearly independent at each P.

General Solution

Here $n_\phi = \nabla\phi = (\phi_x, \phi_y, \phi_u)$ and $n_\psi = \nabla\psi = (\psi_x, \psi_y, \psi_u)$. n_ϕ and n_ψ are simply the normal vectors to the surfaces S_1 and S_2 at P.

Definition 1 (First Integral)
A continuously differentiable function $\phi(x, y, u)$ is said to be a first integral of (7) if it is constant on characteristic curves.

Definition 2 (Functionally Independent)
The first two integrals $\phi(x, y, u)$ and $\psi(x, y, u)$ of (7) are functionally independent if

$$rank \begin{bmatrix} \phi_x & \phi_y & \phi_u \\ \psi_x & \psi_y & \psi_u \end{bmatrix} = 2$$

that is, if n_ϕ and n_ψ are linearly independent.

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Why Linearly Independent

Since the curve Γ is the intersection of two surfaces, its tangent vector T is given by

$$T = n_\phi \times n_\psi$$

- T exactly defines the direction of the vector of the characteristic curve.
- If n_ϕ and n_ψ are linearly dependent, the two surfaces are tangent to each other and do not define a unique curve.
- Therefore, linear independence ensures that their intersection is indeed a well-defined 1D curve through P

General Solution

Suppose $\phi(x, y, u)$ and $\psi(x, y, u)$ are functionally independent integrals and

$$\begin{aligned} \phi(x(s), y(s), u(s)) &= C_1 \\ \psi(x(s), y(s), u(s)) &= C_2 \end{aligned} \implies \begin{aligned} \phi_x \frac{dx}{ds} + \phi_y \frac{dy}{ds} + \phi_u \frac{du}{ds} &= 0 \\ \psi_x \frac{dx}{ds} + \psi_y \frac{dy}{ds} + \psi_u \frac{du}{ds} &= 0 \end{aligned}$$

$$\begin{aligned} \implies \quad \phi_x a(x, y, u) + \phi_y b(x, y, u) + \phi_u c(x, y, u) &= 0 \\ \psi_x a(x, y, u) + \psi_y b(x, y, u) + \psi_u c(x, y, u) &= 0 \end{aligned}$$

Therefore, ϕ and ψ are functionally independent first integrals iff

$$\frac{a(x, y, u)}{\begin{vmatrix} \phi_y & \phi_u \\ \psi_y & \psi_u \end{vmatrix}} = \frac{b(x, y, u)}{\begin{vmatrix} \phi_u & \phi_x \\ \psi_u & \psi_x \end{vmatrix}} = \frac{c(x, y, u)}{\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}} \qquad (9)$$

General Solution

Theorem 5 (General Solution)
If $\phi(x, y, u) = C_1$ and $\psi(x, y, u) = C_2$ be two independent solutions of the ODEs

$$C : \begin{cases} \frac{dx}{dt} = a(x, y, u) \\ \frac{dy}{dt} = b(x, y, u) \\ \frac{du}{dt} = c(x, y, u) \end{cases}$$

and $\phi_u^2 + \psi_u^2 \neq 0$, then the general solution to (7) is given by

$$f(\phi(x, y, u), \psi(x, y, u)) = 0$$

where f is an arbitrary function.

General Solution

Proof: Let $u = u(x, y)$ be a function for which

$$f(\phi(x, y, u(x, y)), \psi(x, y, u(x, y))) = 0$$

Differentiating it with respect to x, y, we have

$$f_\phi(\phi_x + \phi_u u_x) + f_\psi(\psi_x + \psi_u u_x) = 0$$

$$f_\phi(\phi_y + \phi_u u_y) + f_\psi(\psi_y + \psi_u u_y) = 0$$

If $(f_\phi, f_\psi) \neq (0, 0)$, then

$$\begin{vmatrix} \phi_x + \phi_u u_x & \psi_x + \psi_u u_x \\ \phi_y + \phi_u u_y & \psi_y + \psi_u u_y \end{vmatrix} = 0$$

General Solution

Proof (Contd): On simplification,

$$(\phi_u\psi_y - \phi_y\psi_u)u_x + (\phi_x\psi_u - \phi_u\psi_x)u_y = \phi_y\psi_x - \phi_x\psi_y \qquad (10)$$

By comparing (10) and (9) we can obtain that

$$au_x + bu_y = c$$

Conversely, suppose $u = u(x, y)$ is a solution of (7), $\phi(x, y, u)$ and $\psi(x, y, u)$ are functionally independent first integrals of (7). Then, by (9), we obtain (10). Now, we have functions $\Phi = \phi(x, y, u(x, y))$ and $\Psi = \psi(x, y, u(x, y))$.

General Solution

Proof (Contd):

$$\begin{aligned}\begin{vmatrix}\Phi_x & \Psi_x \\ \Phi_y & \Psi_y\end{vmatrix} &= \begin{vmatrix}\phi_x + \phi_u u_x & \psi_x + \psi_u u_x \\ \phi_y + \phi_u u_y & \psi_y + \psi_u u_y\end{vmatrix} \\ &= (\phi_u\psi_y - \phi_y\psi_u)u_x + (\phi_x\psi_u - \phi_u\psi_x)u_y - \phi_y\psi_x - \phi_x\psi_y \\ &= \lambda(au_x + bu_y - c) \\ &= 0\end{aligned}$$

From the rank theorem of Calculus, it follows that one of the functions Φ and Ψ can be expressed as a function of the other. That is, there exists a function g such that

$$\psi(x, y, u(x, y)) = g(\phi(x, y, u(x, y)))$$

$$\implies f(\phi(x, y, u), \psi(x, y, u)) = 0$$

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Examples

Example 6

Show that

$$(y + 2ux)u_x - (x + 2uy)u_y = \frac{1}{2}(x^2 - y^2)$$

with

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = s \\ u = u_0(s) = 0 \end{cases}$$

has exactly one solution.

Solution: The characteristic equations are

$$C : \begin{cases} \frac{dx}{dt} = y + 2ux \\ \frac{dy}{dt} = -(x + 2uy) \\ \frac{du}{dt} = 0.5(x^2 - y^2) \end{cases}$$

Examples

Solution (Contd): One First integral we can obtain from

$$\frac{xdx + ydy}{2u(x^2 - y^2)} = \frac{2du}{x^2 - y^2}$$

$$\implies \phi(x, y, u) = x^2 + y^2 - 4u^2 = C_1$$

We can obtain another independent first integral from

$$\frac{ydx + xdy}{y^2 - x^2} = \frac{2du}{x^2 - y^2}$$

$$\implies \psi(x, y, u) = xy + 2u = C_2$$

The general integral solution is given by

$$x^2 + y^2 - 4u^2 = g(xy + 2u)$$

Examples

Solution (Contd): For the given Cauchy data, we have

$$2s^2 = C_1, s^2 = C_2 \implies C_1 = 2C_2$$

$$\implies f(\phi, \psi) = \phi - 2\psi$$

$$\implies x^2 + y^2 - 4u^2 = 2(xy + 2u)$$

$$\implies x^2 + y^2 - 2xy = 4u^2 + 4u$$

$$\implies u = \frac{1}{2}\left[\sqrt{(x-y)^2+1} - 1\right]$$

It is the only solution that satisfies all conditions.

Examples

Solution (Contd):

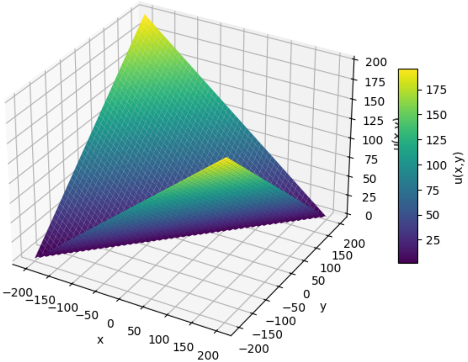

Figure 4: Solution

Examples

Example 7

Find the general solution of the equation

$$(u-y)u_x + yu_y = x + y$$

with

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = 1 \\ u = u_0(s) = 2 + s \end{cases}$$

has exactly one solution.

Solution: The characteristic equations are

$$\frac{dx}{u-y} = \frac{dy}{y} = \frac{du}{x+y}$$

Examples

Solution (Contd): One First integral we can obtain from

$$\frac{dx + du}{u + x} = \frac{dy}{y}$$

$$\implies \phi(x, y, u) = \frac{u + x}{y} = C_1$$

We can obtain another independent first integral from

$$\frac{dx + dy}{u} = \frac{du}{x + y}$$

$$\implies \psi(x, y, u) = (x + y)^2 - u^2 = C_2$$

The general integral solution is given by

$$(x + y)^2 - u^2 = g\left(\frac{u + x}{y}\right)$$

Examples

Solution (Contd): For the given Cauchy data, we have

$$\frac{2s+2}{1} = C_1, (s+1)^2 - (s+2)^2 = C_2$$

$$2s+2 = C_1, -2s-3 = C_2 \implies C_1 + C_2 + 1 = 0$$

$$\implies f(\phi, \psi) = \phi + \psi + 1$$

$$(x+y)^2 - u^2 + 1 + \frac{u+x}{y} = 0, y \neq 0$$

$$\frac{(u+x+y)}{y}(-uy + xy + y^2 + 1) = 0, y \neq 0$$

Examples

Solution (Contd): Either

$$u = -x - y$$

or

$$u = x + y + \frac{1}{y}, y \neq 0$$

$$u = -x - y \implies, u(s) = -s - 1 \neq s + 2 \implies \impliedby$$

$$u = x + y + \frac{1}{y} \implies, u(s) = s + 2$$

Therefore,$u = x + y + \frac{1}{y}, y \neq 0$ is the solution. Further $y = e^t \implies y > 0$

Examples

Solution (Contd):

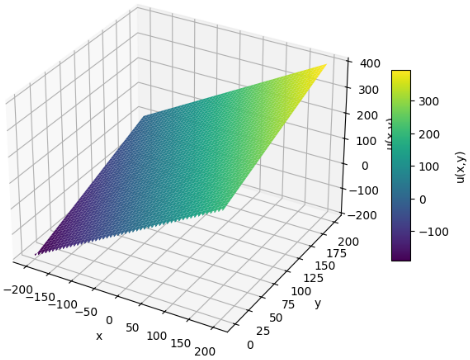

Figure 5: Solution

Exercise

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Exercise 2: General Solution

Find the general solution of the following equations

1. $(x-y)y^2u_x - (x-y)x^2u_y - (x^2+y^2)u = 0$
2. $(y-u)u_x + (u-x)u_y = x-y$
3. $x(y-u)u_x + y(u-x)u_y = (x-y)u$
4. $uu_x + (u^2-x^2)u_y + x = 0$
5. $u_y - \left(\frac{y}{x}u\right)_x = 0$

Let us wrap the first-order linear and quasilinear PDEs for the moment and solve the big three PDEs. Let us begin with the Heat Equation and the separation of variables first.

Thanks

Doubts and Suggestions

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