MA612L-Partial Differential Equations

Lecture 11 : Variable Separable

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Separation of Variables

In general, the method described for Heat equations can be applied to linear homogeneous PDEs. Let us derive the Heat equation later. For the time being, let us solve the 1D Heat equation without a source term. Consider the following:

$$u_t = c^2 u_{xx} \quad \text{on } x \in (0, L), t > 0 \tag{1}$$

$$u(x, 0) = f(x), x \in [0, L] \tag{2}$$

$$u(0, t) = 0, t \geq 0 \tag{3}$$

$$u(L, t) = 0, t \geq 0 \tag{4}$$

In general, the method described for Heat equations can be applied to linear homogeneous PDEs. Again, our aim here is to convert the PDE to an ODE or system of ODEs

Product Solution

Definition 1

Given a PDE in $u = u(x, y)$, we say that u is a product solution if

$$u(x, y) = G(x)H(y) \tag{5}$$

for functions G and H. More generally, $u = u(x_1, x_2, \cdots, x_n)$ is a product solution of a PDE

$$F(u, Du, D^2u, \cdots, D^m u) = 0$$

if

$$u(x_1, x_2, \cdots, x_n) = G_1(x_1)G_2(x_2)\cdots G_n(x_n) \tag{6}$$

Simple Example

Example 1

Find all product solutions of the PDE $u_x + u_y = 0$

Solution: Let $u(x, y) = X(x)Y(y)$. Then

$$u_x = X'Y, \quad u_y = XY' \implies X'Y + XY' = 0$$

Let us assume that $X \neq 0, Y \neq 0$ (valid??) and divide by XY to obtain

$$\frac{X'}{X} = -\frac{Y'}{Y} = \nu(Why??) \implies X' - \nu X = 0 \text{ and } Y' + \nu Y = 0$$

$$\implies X(x) = e^{\nu x}, Y(y) = e^{-\nu y} \implies u(x, y) = e^{\nu(x-y)}$$

Since ν is any real constant, any linear combination of these solutions is again a solution.

Simple Example

Example 2

Using separation of variables, obtain the ODEs for the following PDE $3u_{yy} - 5u_{xxxy} + 7u_{xxy} = 0$

Solution: Let $u(x,y) = X(x)Y(y)$. Then

$$u_{yy} = XY'', \quad u_{xxxy} = X'''Y', \quad u_{xxy} = X''Y' \implies 3XY'' - 5X'''Y' + 7X''Y' = 0$$

$$\frac{3Y''}{Y'} = \frac{5X''' - 7X''}{X} = \nu \implies 3Y'' - \nu Y' = 0 \text{ and } 5X''' - 7X'' - \nu X = 0$$

One can solve this using ODE theory.

Heat Equation

Let us obtain a similar one for the Heat equation with initial and boundary conditions (1)-(4). Let $u(x,t) = X(x)T(t)$, then we obtain

$$u_t = X(x)T'(t) \quad u_{xx} = X''(x)T(t)$$

We obtain that

$$X(x)T'(t) = c^2 X''(x)T(t) \implies \frac{T'}{c^2 T} = \frac{X''}{X} = \mu(constant)$$

Therefore, we have

$$\frac{dT}{dt} - c^2 \mu T = 0, \quad \frac{d^2 X}{dx^2} - \mu X = 0$$

where μ is any real constant. (Can it be complex?!)

Heat Equation

Let us look at the boundary conditions.

$$u(0,t) = 0 \implies X(0)T(t) = 0$$

If $T \equiv 0$, then $u(x,t) \equiv 0$, which is a trivial solution, and we are not interested in this solution. Therefore, $T(t) \not\equiv 0$, consequently, $X(0) = 0$. Similarly,

$$u(L,t) = 0 \implies X(L)T(t) = 0$$

If $T \equiv 0$, then $u(x,t) \equiv 0$, which is a trivial solution, and we are not interested in this solution. Therefore, $T(t) \not\equiv 0$, consequently, $X(L) = 0$.

Heat Equation

So, let us consider the following problem,

$$X'' - \mu X = 0, x \in [0, L]$$
$$X(0) = 0$$
$$X(L) = 0$$

Now, let us consider three cases, $\mu > 0, = 0, < 0.$
Case 1: $\mu = 0$
Then

$$X'' = 0 \implies X' = a \implies X = ax + b$$

$$X(0) = 0 \implies b = 0$$

$$X(L) = 0 \implies aL = 0, L \neq 0 \implies a = 0$$

$$X(x) \equiv 0 \implies u(x,t) = X(x)T(t) = 0 \implies u \equiv 0$$

Heat Equation

We are not interested in the solution $u \equiv 0$

Case 2: $\mu > 0$**, let** $\mu = \nu^2$

Then

$$X'' - \nu^2 X \implies m^2 - \nu^2 = 0 \quad (\text{ auxiliary equation })$$

$$\implies m = \pm\nu \implies X(x) = Ae^{\nu x} + Be^{-\nu x}$$

$$X(0) = 0 \implies A + B = 0 \implies A = -B \implies X(x) = A(e^{\nu x} - e^{-\nu x})$$

$$X(L) = 0 \implies A(e^{\nu L} - e^{-\nu L}) = 0$$

$$\implies e^{\nu L} = e^{-\nu L} \quad \text{or} \quad A = 0$$

$$\implies L = 0 \quad \text{or} \quad \nu = 0 \quad \text{or} \quad A = 0$$

$$L \neq 0, \nu = 0 \implies \mu = 0 \quad (\text{Case 1, discussed}) \implies A = 0$$

$$X(x) \equiv 0 \implies u(x,t) = X(x)T(t) = 0 \implies u \equiv 0$$

Heat Equation

Again, we are not interested in the solution $u \equiv 0$

Case 3: $\mu < 0$**, let** $\mu = -\nu^2$

Then

$$X'' + \nu^2 X \implies m^2 + \nu^2 = 0 \quad (\text{ auxiliary equation })$$

$$\implies m = \pm \nu i \implies X(x) = A\cos(\nu x) + B\sin(\nu x)$$

$$X(0) = 0 \implies A + 0 = 0 \implies A = 0 \implies X(x) = B\sin(\nu x)$$

$$X(L) = 0 \implies B\sin(\nu L) = 0 \implies \nu_n = \frac{n\pi}{L}, n \in \mathbb{Z}$$

Let

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right), n \in \mathbb{Z}$$

Heat Equation

Observe that

$$X_n(x) = -B_n \sin\left(\frac{|n|\pi}{L}x\right), n < 0$$

$$\implies X_n(x) = C_n \sin\left(\frac{n\pi}{L}x\right), n > 0$$

and we are not interested in

$$X_n(x) = 0, n = 0$$

Therefore,

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right), n \in \mathbb{N}$$

Heat Equation

Therefore, we have only case 3 with a non-trivial solution. That is, when $\mu = -\nu_n^2 = -\left(\frac{n\pi}{L}\right)^2$. Since for each n, we obtain a different ν. We have,

$$\frac{dT_n}{dt} + \left(\frac{cn\pi}{L}\right)^2 T_n = 0, t > 0, n \in \mathbb{N}$$

$$\implies T_n(t) = K_n e^{-\lambda_n^2 t}, n \in \mathbb{N}, \lambda_n = \frac{cn\pi}{L}$$

Therefore, we have

$$u_n(x,t) = X_n(x)T_n(t) = C_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi}{L}x\right), n \in \mathbb{N}$$

These are called eigenfunctions of the heat equation corresponding to the eigenvalue $\lambda_n = \frac{cn\pi}{L}$

Heat Equation

Now, the linear combination of these is again a solution (Proof?? Convergence??). Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi}{L}x\right) \tag{7}$$

Now, let us apply the initial condition to obtain the solution of the problem (1)-(4)

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

Fourier Series History

This is how the Fourier Series history started. Fourier developed the trigonometric series during a war in Egypt and later.

- Joseph Fourier introduced his series idea while studying heat conduction.
- His work came right after he served as an engineer in Napoleon's Egyptian campaign (1798–1801).
- During that expedition, Fourier was tasked with military logistics: planning fortifications, managing supplies, and analyzing terrain.
- The Egyptian heat inspired him to model how heat spreads in metal and stone, leading to his mathematical breakthrough.

"Every function hides a symphony — Fourier teaches us how to listen."

Heat Equation

The coefficients C_n are obtained by integrating on both sides by exploiting the orthogonality properties of the set $\{\sin mx : m \in \mathbb{N}\}$. It is known that

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0, \quad \text{if } \; n \neq m$$

From the analogy of the Fourier Series, we can obtain that

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \in \mathbb{N}$$

The solution can be obtained by assuming that $f(x)$ is piecewise continuous on the interval $[0, L]$ and has one-sided derivatives $\forall x \in (0, L)$. Since (7) has exponential factor with time, as $t \to \infty$, all terms in $u(x, t)$, that is, $u_n(x, t) \to 0$. The rate of decay increases with n.

Exercise

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Exercise 1: Insulated Bar

Find the temperature $u(x,t)$ in a laterally insulated copper bar $80cm$ long if the initial temperature is $100\sin\left(\frac{\pi x}{80}\right)^{\circ}C$ and the ends are kept at $0°C$. How long will it take for the maximum temperature in the bar to drop to $50°C$? Assume that $c^2 = 1.158cm^2/sec$.

Exercise 2: Initial Conditions

Find the temperature $u(x,t)$ in a laterally insulated bar with $L = \pi, c = 1$ and $f(x) = 1 - \frac{x}{\pi}$

Exercise

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Exercise 3: With Heat Generation

Find the temperature $u(x,t)$ in a laterally insulated bar with heat generation H which is modelled by

$$u_t = c^2 u_{xx} + H$$

where $H > 0, L = \pi, c = 1$ and $f(x) = 1 - \frac{x}{\pi}$

Exercise 4: With Heat Generation

Derive the solution of the heat equation similarly for the following initial and boundary conditions

$$u(x,0) = f(x), u_x(0,t) = 0, u_x(L,t) = 0$$

Separation of Variables
Wave Equation

Wave Equation

Let us consider and solve the 1D Wave equation in a similar fashion

$$u_{tt} = c^2 u_{xx} \quad \text{on } x \in (0, L), t > 0 \tag{8}$$

$$u(x, 0) = f(x), x \in [0, L] \tag{9}$$

$$u_t(x, 0) = g(x), x \in [0, L] \tag{10}$$

$$u(0, t) = 0, t \geq 0 \tag{11}$$

$$u(L, t) = 0, t \geq 0 \tag{12}$$

Wave Equation

Let us obtain a similar one for the wave equation with initial and boundary conditions (8)-(12). Let $u(x,t) = X(x)T(t)$, then we obtain

$$u_t = X(x)T''(t) \quad u_{xx} = X''(x)T(t)$$

We obtain that

$$X(x)T''(t) = c^2 X''(x)T(t) \implies \frac{T'}{c^2 T} = \frac{X''}{X} = \mu(constant)$$

Therefore, we have

$$\frac{d^2T}{dt^2} - c^2\mu T = 0, \quad \frac{d^2X}{dx^2} - \mu X = 0$$

where μ is any real constant. (Can it be complex?!)

Wave Equation

Let us look at the boundary conditions.

$$u(0,t) = 0 \implies X(0)T(t) = 0$$

If $T \equiv 0$, then $u(x,t) \equiv 0$, which is a trivial solution, and we are not interested in this solution. Therefore, $T(t) \not\equiv 0$, consequently, $X(0) = 0$. Similarly,

$$u(L,t) = 0 \implies X(L)T(t) = 0$$

If $T \equiv 0$, then $u(x,t) \equiv 0$, which is a trivial solution, and we are not interested in this solution. Therefore, $T(t) \not\equiv 0$, consequently, $X(L) = 0$.

Wave Equation

So, let us consider the following problem,

$$\begin{aligned} X'' - \mu X = 0, x \in [0, L] \\ X(0) = 0 \\ X(L) = 0 \end{aligned}$$

Now, let us consider three cases, $\mu > 0, = 0, < 0$.
Case 1: $\mu = 0$
Then

$$X'' = 0 \implies X' = a \implies X = ax + b$$

$$X(0) = 0 \implies b = 0$$

$$X(L) = 0 \implies aL = 0, L \neq 0 \implies a = 0$$

$$X(x) \equiv 0 \implies u(x,t) = X(x)T(t) = 0 \implies u \equiv 0$$

Wave Equation

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We are not interested in the solution $u \equiv 0$

Case 2: $\mu > 0$**, let** $\mu = \nu^2$

Then

$$X'' - \nu^2 X \implies m^2 - \nu^2 = 0 \quad (\text{ auxiliary equation })$$

$$\implies m = \pm\nu \implies X(x) = Ae^{\nu x} + Be^{-\nu x}$$

$$X(0) = 0 \implies A + B = 0 \implies A = -B \implies X(x) = A(e^{\nu x} - e^{-\nu x})$$

$$X(L) = 0 \implies A(e^{\nu L} - e^{-\nu L}) = 0$$

$$\implies e^{\nu L} = e^{-\nu L} \quad \text{or} \quad A = 0$$

$$\implies L = 0 \quad \text{or} \quad \nu = 0 \quad \text{or} \quad A = 0$$

$$L \neq 0, \nu = 0 \implies \mu = 0 \quad (\text{Case 1, discussed}) \implies A = 0$$

$$X(x) \equiv 0 \implies u(x,t) = X(x)T(t) = 0 \implies u \equiv 0$$

Wave Equation

Again, we are not interested in the solution $u \equiv 0$
Case 3: $\mu < 0$**, let** $\mu = -\nu^2$
Then

$$X'' + \nu^2 X \implies m^2 + \nu^2 = 0 \quad (\text{ auxiliary equation })$$

$$\implies m = \pm \nu i \implies X(x) = A\cos(\nu x) + B\sin(\nu x)$$

$$X(0) = 0 \implies A + 0 = 0 \implies A = 0 \implies X(x) = B\sin(\nu x)$$

$$X(L) = 0 \implies B\sin(\nu L) = 0 \implies \nu_n = \frac{n\pi}{L}, n \in \mathbb{Z}$$

Let

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right), n \in \mathbb{Z}$$

Wave Equation

Observe that

$$X_n(x) = -B_n \sin\left(\frac{|n|\pi}{L}x\right), n < 0$$

$$\implies X_n(x) = C_n \sin\left(\frac{n\pi}{L}x\right), n > 0$$

and we are not interested in

$$X_n(x) = 0, n = 0$$

Therefore,

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right), n \in \mathbb{N}$$

Wave Equation

Therefore, we have only case 3 with a non-trivial solution. That is, when $\mu = -\nu_n^2 = -\left(\frac{n\pi}{L}\right)^2$. Since for each n, we obtain a different ν. We have,

$$\frac{d^2T_n}{dt^2} + \left(\frac{cn\pi}{L}\right)^2 T_n = 0, t > 0, n \in \mathbb{N}$$

$$\implies T_n(t) = A_n \cos \lambda_n t + B_n \sin \lambda_n t, n \in \mathbb{N}, \lambda_n = \frac{n\pi}{L}$$

Note: There are notation abuses for A_n and B_n. Therefore, we have

$$u_n(x,t) = X_n(x)T_n(t) = (A_n \cos \lambda_n t + B_n \sin \lambda_n t) \sin\left(\frac{n\pi}{L}x\right), n \in \mathbb{N}$$

These are called eigenfunctions of the heat equation corresponding to the eigenvalue $\lambda_n = \frac{cn\pi}{L}$. The set $\{\lambda_i, i \in \mathbb{N}\}$ is called the spectrum.

Wave Equation

Now, the linear combination of these is again a solution (Proof?? Convergence??). Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (A_n \cos \lambda_n t + B_n \sin \lambda_n t) \sin\left(\frac{n\pi}{L} x\right) \qquad (13)$$

Now, let us apply the initial condition to obtain the solution of the problem (8)-(12)

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right) = f(x)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \lambda_n \sin\left(\frac{n\pi}{L} x\right) = g(x)$$

Wave Equation

The coefficients A_n and B_n are obtained by integrating on both sides by exploiting the orthogonality properties of the set $\{\sin mx : m \in \mathbb{N}\}$. The solution of the wave equation is given by

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \lambda_n t + B_n \sin \lambda_n t) \sin\left(\frac{n\pi}{L} x\right) \tag{14}$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \in \mathbb{N}$$

$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \in \mathbb{N}$$

Separation of Variables
Laplace Equation

Laplace Equation

Let us consider and solve the 1D Wave equation in a similar fashion

$$u_{xx} + u_{yy} = 0 \quad \text{on } x \in (0, a), y \in (0, b) \tag{15}$$

$$u(x, 0) = 0, x \in [0, a] \tag{16}$$

$$u(x, b) = f(x), x \in [0, a] \tag{17}$$

$$u(0, y) = 0, y \in [0, b] \tag{18}$$

$$u(a, y) = 0, y \in [0, b] \tag{19}$$

This is a boundary value problem. Here, we have given only Dirichlet boundary conditions.

Laplace Equation

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Let us obtain a similar one for the Laplace equation with boundary conditions (15)-(19). Let $u(x,y) = X(x)Y(y)$, then we obtain

$$u_{xx} = X''(x)Y(y) \quad u_{yy} = X(x)Y''(y)$$

We obtain that

$$X''(x)Y(y) = -X(x)Y''(y) \implies \frac{X''}{X} = -\frac{Y''}{Y} = \mu(constant)$$

Therefore, we have

$$\frac{d^2X}{dx^2} - \mu X = 0, \quad \frac{d^2Y}{dy^2} + \mu Y = 0$$

where μ is any real constant.

Laplace Equation

Let us look at the left and right boundary conditions.

$$u(0, y) = 0 \implies X(0)Y(y) = 0$$

If $Y \equiv 0$, then $u(x, y) \equiv 0$, which is a trivial solution, and we are not interested in this solution. Therefore, $Y(y) \not\equiv 0$, consequently, $X(0) = 0$. Similarly,

$$u(a, y) = 0 \implies X(a)Y(y) = 0$$

If $Y \equiv 0$, then $u(x, t) \equiv 0$, which is a trivial solution, and we are not interested in this solution. Therefore, $Y(y) \not\equiv 0$, consequently, $X(a) = 0$.

Laplace Equation

MA612L PARTIAL DIFFERENTIAL EQUATIONS

So, let us consider the following problem,

$$X'' - \mu X = 0, x \in [0, a]$$
$$X(0) = 0$$
$$X(a) = 0$$

Now, let us consider three cases, $\mu > 0, = 0, < 0$.

Case 1: $\mu = 0$

Then

$$X'' = 0 \implies X' = c \implies X = cx + b$$

$$X(0) = 0 \implies b = 0$$

$$X(a) = 0 \implies ca = 0, a \neq 0 \implies c = 0$$

$$X(x) \equiv 0 \implies u(x,t) = X(x)T(t) = 0 \implies u \equiv 0$$

Laplace Equation

We are not interested in the solution $u \equiv 0$

Case 2: $\mu > 0$**, let** $\mu = \nu^2$

Then

$$X'' - \nu^2 X \implies m^2 - \nu^2 = 0 \quad \text{(auxiliary equation)}$$

$$\implies m = \pm\nu \implies X(x) = Ae^{\nu x} + Be^{-\nu x}$$

$$X(0) = 0 \implies A + B = 0 \implies A = -B \implies X(x) = A(e^{\nu x} - e^{-\nu x})$$

$$X(a) = 0 \implies A(e^{\nu a} - e^{-\nu a}) = 0$$

$$\implies e^{\nu a} = e^{-\nu a} \quad \text{or} \quad A = 0$$

$$\implies a = 0 \quad \text{or} \quad \nu = 0 \quad \text{or} \quad A = 0$$

$$a \neq 0, \nu = 0 \implies \mu = 0 \quad \text{(Case 1, discussed)} \implies A = 0$$

$$X(x) \equiv 0 \implies u(x,t) = X(x)T(t) = 0 \implies u \equiv 0$$

Laplace Equation

Again, we are not interested in the solution $u \equiv 0$

Case 3: $\mu < 0$**, let** $\mu = -\nu^2$

Then

$$X'' + \nu^2 X \implies m^2 + \nu^2 = 0 \quad (\text{ auxiliary equation })$$

$$\implies m = \pm \nu i \implies X(x) = A\cos(\nu x) + B\sin(\nu x)$$

$$X(0) = 0 \implies A + 0 = 0 \implies A = 0 \implies X(x) = B\sin(\nu x)$$

$$X(a) = 0 \implies B\sin(\nu a) = 0 \implies \nu_n = \frac{n\pi}{a}, n \in \mathbb{Z}$$

Let

$$X_n(x) = B_n \sin\left(\frac{n\pi}{a}x\right), n \in \mathbb{Z}$$

Laplace Equation

Observe that

$$X_n(x) = -B_n \sin\left(\frac{|n|\pi}{a}x\right), n < 0$$

$$\implies X_n(x) = C_n \sin\left(\frac{n\pi}{a}x\right), n > 0$$

and we are not interested in

$$X_n(x) = 0, n = 0$$

Therefore,

$$X_n(x) = B_n \sin\left(\frac{n\pi}{a}x\right), n \in \mathbb{N}$$

Laplace Equation

Therefore, we have only case 3 with a non-trivial solution. That is, when $\mu = -\nu_n^2 = -\left(\frac{n\pi}{a}\right)^2$. Since for each n, we obtain a different ν. We have,

$$\frac{d^2Y_n}{dy^2} - \nu_n^2 Y_n = 0, t > 0, n \in \mathbb{N}$$

$$\implies Y_n(y) = A_n e^{\nu_n y} + B_n e^{-\nu_n y}, n \in \mathbb{N}$$

Note: There are notation abuses for A_n and B_n. Therefore, we have

$$u_n(x,y) = X_n(x)Y_n(y) = (A_n e^{\nu_n y} + B_n e^{-\nu_n y}) \sin(\nu_n x), n \in \mathbb{N}$$

These are called eigenfunctions of the heat equation corresponding to the eigenvalue $\nu_n = \frac{n\pi}{a}$. The set $\{\lambda_i, i \in \mathbb{N}\}$ is called the spectrum.

Laplace Equation

Now, the linear combination of these is again a solution (Proof?? Convergence??). Therefore,

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} (A_n e^{\nu_n y} + B_n e^{-\nu_n y}) \sin(\nu_n x) \tag{20}$$

Now, let us apply the bottom boundary condition $u(x,0) = 0$

$$u(x,0) = \sum_{n=1}^{\infty} (A_n + B_n) \sin(\nu_n x) = 0 \implies A_n = -B_n$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n (e^{\nu_n y} - e^{-\nu_n y}) \sin(\nu_n x) = \sum_{n=1}^{\infty} A_n \sin(\nu_n x) \sinh(\nu_n y)$$

Laplace Equation

Now, let us apply the top boundary condition $u(x,b) = f(x)$ to obtain the solution of the problem (15)-(19)

$$u(x,b) = \sum_{n=1}^{\infty} A_n \sin(\nu_n x) \sinh(\nu_n b) = f(x)$$

The coefficients A_n are obtained by integrating on both sides by exploiting the orthogonality properties of the set $\{\sin mx : m \in \mathbb{N}\}$. The solution of the Laplace equation is given by

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(\nu_n x) \sinh(\nu_n y) \tag{21}$$

$$A_n = \frac{2}{a \sinh(\nu_n b)} \int_0^a f(x) \sin(\nu_n x) dx, \quad n \in \mathbb{N}, \nu_n = \frac{n\pi}{a}$$

Laplace Equation

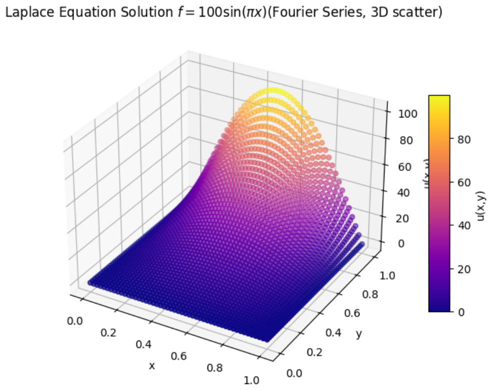

Figure 1: Laplace Equation

Exercise

Exercise 5: Top Boundary

Solve the Laplace equation with $f(x) = 110$, $a = 20, b = 40$

Exercise 6: Top Boundary

Solve the Laplace equation with $f(x) = 1000\sin(\pi x/2)$, $a = b = 2$.

Exercise 7: Top Boundary

Solve the Wave equation for $L = 1, c = 1, f(x) = 0$ and

1. $g(x) = \sin(3\pi x)$
2. $g(x) = x^2(1-x)$

Thanks

Doubts and Suggestions

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