MA612L-Partial Differential Equations

Lecture 12 : Conservation Laws

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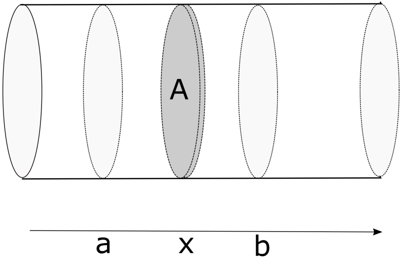
Conservation Law

1. Many PDEs involve in physical phenomena studies about how **certain quantity** changes with time and space
2. These changes usually obey the conservation laws
3. Conservation of Mass, Momentum, and Energy
4. We will prove the Transport theorem for $n-$ dimensional space later.

Let us quantify a certain quantity as $u(x,t)$, that is, $u(x,t)$ denotes the density of a certain quantity (mass, energy, momentum). Note that density is measured in the amount of the quantity per unit volume or unit length.

Conservation Law



Let us consider a thin tube as our domain. Its cross-sectional area is denoted by A.

Assumptions

1. Lateral sides are insulated so that the quantity being studied only varies in the $x-$direction and in time.
2. x, $u(x,t)$ does not vary within the cross section at x.
3. The domain has a constant cross-sectional area A.
4. $\phi = \phi(x,t)$ denotes the flux of the quantity at x at time t. It measures the amount of the quantity crossing the section at x at time t.
5. ϕ is **positive** if the flow is to the right and **negative** if the flow is to the left.
6. $f(x,t)$ denotes the rate at which the quantity is created or destroyed per unit volume within the section at x at time t.
7. f is called **source** if it is positive, **sink** if it is negative.

Conservation Law

- The amount of the quantity at time t in a small section of width dx is $u(x,t)Adx$ for each x.
- The amount of the quantity in arbitrary section $[a,b]$ will be

$$\int_a^b u(x,t)Adx$$

- ϕ is the amount of quantity per unit area per unit time. The actual amount of the quantity crossing the section at x, at time t, is given by $A\phi(x,t)$.
- f is measured in the amount of the quantity per unit volume per unit time.
- The amount of the quantity being created in a small section of width dx for each x is $f(x,t)Adx$ per unit time.

Conservation Law

- The amount of the quantity being created within the arbitrary section $[a, b]$ will be

$$\int_a^b f(x,t)A dx$$

- As per the conservation law, the rate of change of the amount of the quantity in the section $[a, b]$ must be equal to the rate at which the quantity flows in at $x = a$ minus at which it flows out at b plus the rate at which it is created within $[a, b]$.

Therefore,

$$\frac{d}{dt}\int_a^b u(x,t)A dx = A\phi(a,t) - A\phi(b,t) + \int_a^b f(x,t)A dx \qquad (1)$$

Conservation Law

Since A is constant, we can obtain the following fundamental conservation law.

$$\frac{d}{dt}\int_a^b u(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b f(x,t)dx \tag{2}$$

It shows a balance between how much goes in, how much goes out, and how much is changed.

Theorem 1 (Leibniz Rule)

If $a(t), b(t)$ and $F(x,t)$ are continuously differentiable then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(y,t)dy = F(b(t),t)b'(t) - F(a(t),t)a'(t) + \int_{a(t)}^{b(t)} F_t(y,t)dy \tag{3}$$

Taking the derivative inside the integral.

Conservation Laws

From (2) and if we assume u has continuous partial derivatives, then by the Leibniz Rule, we obtain that

$$\frac{d}{dt}\int_a^b u(x,t)dx = \int_a^b u_t(x,t)dx$$

If we further assume that ϕ has continuous partial derivatives, then using the fundamental theorem of calculus, we can get that

$$\phi(a,t) - \phi(b,t) = -\int_a^b \phi_x(x,t)dx \tag{4}$$

Conservation Laws

Therefore (2) becomes

$$\int_a^b [u_t(x,t) + \phi_x(x,t) - f(x,t)]dx = 0$$

Since a and b are arbitrary, the integrand must be 0. Since the integrand is continuous, we have

$$u_t(x,t) + \phi_x(x,t) = f(x,t) \tag{5}$$

This is called the fundamental conservation law in differential form.

Conservation Laws

Generally, the conservation law can be written as

$$u_t + \phi_x = f \tag{6}$$

where ϕ and f are function of x, t, u. When ϕ and f are functions of u, it will lead to a nonlinear model. As an example, we can obtain the advection or transport equation and Burger's equation.
Suppose

$$\phi = cu$$

then we get the transport equation where c is the speed of the fluid, when $f(x, t) = 0$, it is called advection equation with no source

$$u_t + cu_x = 0 \tag{7}$$

General Advection Equation

Advection usually refers to the transport of a certain substance. A model where the flux is proportional to the density is called an advection model. The most general form of the advection equation is given by

$$u_t + cu_x + \alpha u = f(x,t) \tag{8}$$

where α and c are constants. Here, cu_x is the term related to the flux. αu and $f(x,t)$ are correpsonding to the source term.

Diffusion

Diffusion is the transport of a material by molecular motion. This phenomenon can be explained as follows:
Suppose you are standing in a row to buy a ticket to watch a film or cricket. When the person at the back pushes, this will impact everyone. When a person in the middle of the row is strong enough to produce an opposite force, then the rest of the row ahead of him may not get any disturbance. Similarly, if there is a discontinuity at one place, there is a chance that it may not get disturbed.
In the same way, when a fluid moves at a high speed, it will cause a disturbance to the adjacent fluid that moves at a slow speed. If the high-speed fluid is "stronger enough" (like a Tsunami or a Flood) than the adjacent fluid or materials, it can even swallow or sweep away materials on its way. However, if the object adjacent is strong enough (like a dam wall), it can't be swept away, and equilibrium can be formed. Both advection and diffusion processes can occur.

Diffusion Equation

Now, let us consider the heat equation in the same way. When one end of the rod is heated, the molecules get agitated, which causes their adjacent molecules to be agitated. The flow is always from more agitated molecules to less agitated molecules to agitate. Therefore, the heat flows from the hot end to the cold end. Agitation of molecules depends on the temperature, density, and many other physical parameters. The higher the temperature or density, the higher the agitation. That is, a higher density gradient leads to more flow. Mathematically, the flux (a measure of the flow) ϕ is proportional to the gradient of the concentration of the substance. In the 1D case, the gradient is the derivative with respect to x. Hence

$$\phi(x,t) = -Ku_x$$

Here $K > 0$ is a proportionality or diffusion constant, so usually denoted by k^2.

Diffusion Equation

1. If $u_x > 0$, then the density increases from left to right. Therefore, the flow should be from right to left, that is, negative.
2. If $u_x < 0$, then the density decreases from right to left. Therefore, the flux is from left to right, that is, positive.
   The heat equation is given by

$$u_t - k^2 u_{xx} = f(x,t)$$

If $f = 0$, we obtain the general diffusion equation or Fick's Law

$$u_t - k^2 u_{xx} = 0$$

Advection-Diffusion Equation

If there is advection and diffusion in a process, it can be modelled as

$$u_t + cu_x - k^2 u_{xx} = 0$$

For example, consider the ant story or the pollutant in a river. Advection occurs due to the river flow with speed c. Since pollutants are flowing, they can also diffuse depending on the flow speed.

If the pollutant also decays at a rate $\lambda > 0$, then the advection-diffusion-decay equation is given by

$$u_t + cu_x - k^2 u_{xx} = -\lambda u$$

Heat Equation

The heat equation is similar to the diffusion model. If we consider ρ as density, C as specific heat, T as temperature and u as energy density, then

$$u(x,t) = \rho C T(x,t)$$

Hence, by the diffusion equation, we have

$$\rho C T(x,t) - k^2 T_{xx} = 0$$

or

$$\frac{dT}{dt} = k^2 \frac{d^2 T}{dx^2}$$

(Warning! Abuse of notations)

Heat Equation in n-dimension

How do we obtain it in $n-$dimension? Let Ω_s represent any smooth subregion of Ω. Then the rate of change of the total quantity (say temperature) within Ω_s is the negative of the flux through $\partial\Omega_s$:

$$\frac{d}{dt}\int_{\Omega_s} \mathbf{u} d\mathbf{x} = -\int_{\partial\Omega_s} \mathbf{F}.\nu dS$$

where F denotes the flux density. Therefore, we have

$$\int_{\Omega_s} u_t d\mathbf{x} = -\int_{\partial\Omega_s} \mathbf{F}.\nu dS = -\int_{\Omega_s} div\mathbf{F} d\mathbf{x}$$

(Gauss divergence theorem)

Heat Equation in n-dimension

$$u_t = -div\mathbf{F}$$

In many cases, $\mathbf{F}$ is proportional to the gradient Du, but points in the opposite direction as flow is from hot to cold. Hence,

$$u_t = -div\mathbf{F}(\mathbf{Du})$$

For small Du, we have $\mathbf{F}(\mathbf{Du}) \approx -c^2 Du$ and we obtain

$$u_t = c^2 \Delta u$$

Let us derive the fundamental solution of the heat equation later.

From Fundamental Conservation Law

- In the case of the heat equation, we have considered $u(x,t)$ as the energy density.
- Now, for the wave equation, we consider u as the momentum density.
- This is natural because waves in a medium are driven by forces that transmit momentum.

Therefore, let

$$u(x,t) = w_t(x,t)$$

where $w(x,t)$ is the displacement of the medium and w_t is the velocity.

From Fundamental Conservation Law

- Flux represents how momentum is transported.
- Momentum flows because of internal forces (like tension in a string or stress in a solid).
- These forces are related to spatial derivatives of displacement.
- From Hooke's law–type reasoning, the restoring force is proportional to $-w_{xx}$, that is, momentum flux occurs as forces are transmitted spatially, which is proportional to $-w_{xx}$.

From Fundamental Conservation Law

Hence, the momentum flux is proportional to the negative gradient of displacement

$$\phi = -c^2 w_x$$

where c is the wave speed.
This means momentum flows from regions of high slope in w toward low slope, consistent with how tension forces act.
Hence, the wave equation is given by

$$w_{tt} = c^2 w_{xx} \quad \text{or abusively} \quad u_{tt} = c^2 u_{xx}$$

Wave Equation: Alternative Derivation

- Consider a thin string of length l. For example, a string on a guitar.
- Disturb the string so that it undergoes relatively small transverse vibrations
- Let ρ denote the density of the string (unit mass per unit length)
- Assume that ρ is constant.
- Due to disturbance, the string displaces, which we denote by $v(x,t)$.
- Let $T(x,t)$ denote the magnitude of the tension force and $\mathbf{T}(x,t)$ denote the tension force.
- Consider the arbitrary interval $[a,b]$, a small portion of the string

Wave Equation: Alternative Derivation

From Newton's second law of motion,

$$F = ma$$

Since v is the displacement and $m = \rho \times \text{Volume} = \rho(b-a)$, we have

$$F = \rho(b-a)v_{tt}$$

Let us ignore the gravity, air resistance, and other external forces, except the tension force. Further, let us assume that the string is perfectly flexible and the tension force has the direction of the tangent vector along the string.

Wave Equation: Alternate Derivation

If we represent the position of the string at a fixed time by parametric equations, then we have (x as a parameter)

$$x = x(t), \quad v = v(x,t)$$

The tangent vector is given by

$$(1, v_x)$$

and the unit tangent vector is given by

$$\left(\frac{1}{\sqrt{1+v_x^2}}, \frac{v_x}{\sqrt{1+v_x^2}} \right)$$

Wave Equation

Hence, the tension force can be written as

$$\mathbf{T}(x,t) = T(x,t)\left(\frac{1}{\sqrt{1+v_x^2}}, \frac{v_x}{\sqrt{1+v_x^2}}\right) = \frac{1}{\sqrt{1+v_x^2}}(T, Tv_x)$$

Since vibrations are small, we can consider v_x is small and hence by Taylor series, we have $\sqrt{1+v_x^2} \approx 1$. Hence

$$\mathbf{T}(x,t) = (T, Tv_x)$$

Since there is no motion along the $x-$axis, we get that

$$T(b,t) - T(a,t) = 0$$

Wave Equation

Therefore,

$$F = T(b,t)v_x(b,t) - T(a,t)v_x(a,t)$$

Hence,

$$\rho(b-a)v_{tt} = T(b,t)v_x(b,t) - T(a,t)v_x(a,t)$$

$$\implies \rho v_{tt} = \frac{T(b,t)v_x(b,t) - T(a,t)v_x(a,t)}{b-a}$$

Since b and a are arbitrary, allowing $b \to a$ yields

$$\rho v_{tt} = T v_{xx} \quad \text{or abusively} \quad u_{tt} = c^2 u_{xx}$$

where $c^2 = T/\rho$.

Wave Equation

The Wave Equation with air resistance is given by

$$u_{tt} + ru_t = c^2 u_{xx}, \quad r > 0$$

The Wave Equation with transverse elastic force (force proportional to the displacement) is given by

$$u_{tt} + ku = c^2 u_{xx}, \quad k > 0$$

The Wave Equation with an external force is given by

$$u_{tt} = c^2 u_{xx} + f(x,t)$$

Wave Equation

The general wave equation is given by

$$u_{tt} + ku + ru_t = c^2 u_{xx} + f(x,t)$$

The wave equation in three dimensions is given by

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$$

In $n-$dimension

$$u_{tt} = c^2 \Delta u + f, \mathbf{x} \in \mathbb{R}^n, t > 0$$

where $u : \Omega \cup \partial\Omega \times [0,\infty) \to \mathbb{R}, f : \Omega \times [0,\infty) \to \mathbb{R}$

Wave Equation in n-dimension

How do we obtain it in $n-$dimension? Let Ω_s represent any smooth subregion of Ω. Then the acceleration within Ω_s is given by

$$\frac{d^2}{dt^2}\int_{\Omega_s} \mathbf{u} d\mathbf{x} = \int_{\Omega_s} u_{tt} d\mathbf{x}$$

The net contact force is given by

$$-\int_{\partial\Omega_s} \mathbf{F}.\nu dS$$

where F denotes the force acting on Ω_s through $\partial\Omega_s$ and the mass density is taken to be unity.

Wave Equation in n-dimension

Therefore, we have

$$\int_{\Omega_s} u_{tt} d\mathbf{x} = -\int_{\partial\Omega_s} \mathbf{F}.\nu dS$$

By Gauss's divergence theorem, we have

$$u_{tt} = -div\mathbf{F}$$

For elastic bodies, $\mathbf{F}$ is a function of the displacement gradient Du. Hence,

$$u_{tt} = -div\mathbf{F}(\mathbf{Du})$$

For small Du, we have $\mathbf{F}(\mathbf{Du}) \approx -c^2 Du$ and we obtain

$$u_{tt} = c^2 \Delta u$$

Laplace Equation in n-dimension

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Mostly Laplace equation arrives as a particular case of the heat or wave equation when u is stationary or time independent. Researchers study how the wave equation or heat equation reaches a stationary state by setting $u_t = 0$ or $u_{tt} = 0$.
The Laplace equation in n-dimension is given by

$$\Delta u = 0$$

The Poisson equation is given by

$$\Delta u = f$$

Definition 1
A C^2 function u satisfying the Laplace equation is called a harmonic function.

Thanks

Doubts and Suggestions

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