MA612L-Partial Differential Equations

Lecture 13 : Reynolds Transport Theorem

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Recap

Conservation Law

Fundamental conservation law in differential form.

$$u_t(x,t) + \phi_x(x,t) = f(x,t) \tag{1}$$

$$\phi(x,t) = -Ku_x \implies u_t = c^2 u_{xx} \text{ (Heat Equation)}$$

$$u(x,t) = w_t(x,t) \implies u_{tt} = c^2 u_{xx} \text{ (Wave Equation)}$$

Leibniz-Reynolds Transport Theorem

Eulerian Coordinates

Lagrangian vs Eulerian

Eulerian Coordinates: This is also called spatial description. In this, the continuum is described in terms of fields defined at fixed spatial locations. A field such as velocity is expressed as

$$v = v(x, t), x \in \Omega_t$$

where x is the spatial coordinate at time t.

Properties are described as functions of the current position and time, without explicit reference to individual material particles.

Eulerian Coordinates (key Idea)

- Fix attention on specific locations in space, and observe how fluid/material flows through those points.
- Place cameras or sensors at fixed points in a river and measure how water flows past.

$$v = v(x, t)$$

where x is a fixed spatial coordinate and velocity (or other quantity) is what a probe at that point measures as time passes.

Eulerian Coordinates

Lagrangian Coordinates: This is also called the material or particle description. In this, each material particle of the continuum is identified by its position in a reference configuration (initial coordinates) X. The motion is described by a mapping

$$x = \psi(X, t), X \in \Omega_0$$

where ψ is the motion function, X is the material coordinate and x is the current position at time t.
Properties such as velocity, acceleration, and stress are expressed as functions of (X, t)

Lagrangian Coordinates (key Idea)

- Track individual fluid (or material) particles as they move through space and time.
- Think of attaching a GPS to each water droplet and recording where it goes.
- Each particle is “tagged” by its initial position (say at $t = 0$)
- Particle motion is given by its trajectory $x = \psi(X, t)$

X: material coordinates (reference/initial position), x : current position at time t

$$v(x, t) = \frac{\partial \chi(X, t)}{\partial t}$$

Eulerian vs Lagrangian

The Eulerian and Lagrangian views are related through the material derivative:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + (\mathbf{v} \cdot \nabla)\phi$$

which gives the rate of change of a field ϕ (Eulerian) as experienced by a moving particle (Lagrangian).

Eulerian vs Lagrangian

Aspect	Lagrangian Description	Eulerian Description
Viewpoint	Follows individual material particles (particle-based view).	Observes fields at fixed spatial points (field-based view).
Coordinates	Material coordinates $\mathbf{X}$ (initial position of particle).	Spatial coordinates $\mathbf{x}$ (current position in space).
Motion Mapping	$\mathbf{x} = \chi(\mathbf{X}, t)$ where χ is the motion function.	$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$, velocity field at location $\mathbf{x}$.
Velocity	$\mathbf{v}(\mathbf{X}, t) = \dfrac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}$	$\mathbf{v}(\mathbf{x}, t)$ measured at fixed point in space.
Acceleration	$\mathbf{a}(\mathbf{X}, t) = \dfrac{\partial^2 \mathbf{x}(\mathbf{X}, t)}{\partial t^2}$	$\dfrac{D\mathbf{v}}{Dt} = \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}.$
Analogy	GPS tracker on each car to follow its path.	Traffic camera at an intersection measuring passing cars.

Eulerian vs Lagrangian

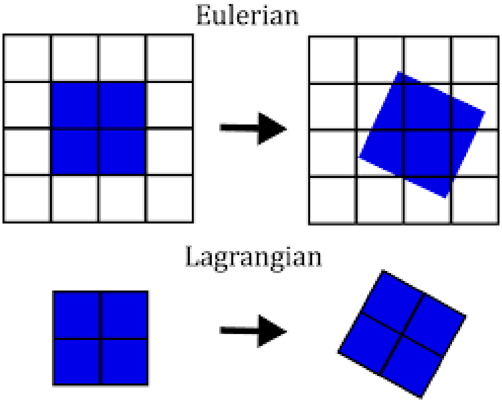

Figure 1: Source:Lagrangian approach in Computational Fluid Dynamics, Alex Tall

Eulerian vs Lagrangian

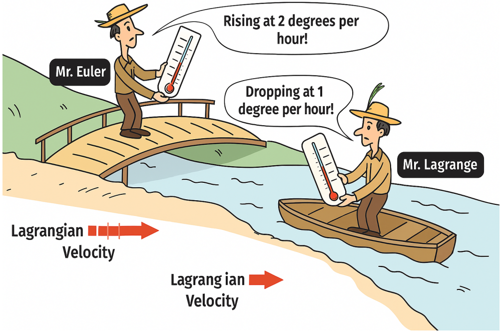

Figure 2: Source: http://www.flowillustrator.com/fluid-dynamics/basics/lagrangian-eulerian-viewpoints.php

Reynolds Transport Theorem

Theorem 1 (Leibniz Rule)

If $a(t), b(t)$ and $F(x,t)$ are continuously differentiable then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)dx = F(b(t),t)\frac{db(t)}{dt} - F(a(t),t)\frac{da(t)}{dt} + \int_{a(t)}^{b(t)} \frac{\partial F(x,t)}{\partial t}dx \quad (2)$$

1. Taking the derivative inside the integral or differentiation under the integral sign
2. Reynolds Transport theorem or Reynolds Theorem is a three-dimensional generalization of the Leibniz Integral rule.

Reynolds Transport Theorem

Consider the integration of $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ over a time-dependent region $\Omega(t)$ with boundary $\partial\Omega(t)$, then the Reynolds Transport theorem relates taking the derivative with respect to time as follows.

Theorem 2 (Reynolds Transport Theorem)
Let $\Omega(t) \subset \mathbb{R}^3$ and $\mathbf{f} : \Omega(t) \times [0, \infty) \to U$, then

$$\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f} dV\right) = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t} dV + \int_{\partial\Omega(t)} (\mathbf{v}.\mathbf{n})\mathbf{f} dS \quad (3)$$

Here $\mathbf{x}(t)$ is the position of the points in $\Omega(t)$. $\mathbf{n}(\mathbf{x}, \mathbf{t})$ is the outward unit normal vector to $\partial\Omega(t)$. dV and dS are volume and surface elements at $\mathbf{x}$. $\mathbf{v}(\mathbf{x}, \mathbf{t})$ denotes the velocity of the area element. The function $\mathbf{f}$ may be scalar-valued or vector-valued, or tensor-valued. $\frac{D}{Dt}$ is usually called as total derivative or material derivative.

Reynolds Transport Theorem: Interpretation

The rate of change of any quantity of interest for a system equals the rate of change within the control volume plus the net flow across the boundaries.

1. What is happening to the property inside the control volume?
2. What is flowing in/out of the control surface flux?
3. The LHS is the Lagrangian view
4. The RHS is an Eulerian view

Reynolds Transport Theorem (Proof)

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With the help of Gauss's divergence theorem, we can write (3) as follows:

$$\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f} dV\right) = \int_{\Omega(t)} \left(\frac{\partial \mathbf{f}}{\partial t} + \nabla \mathbf{f}.\mathbf{v} + \mathbf{f}\nabla.\mathbf{v}\right) dV \tag{4}$$

Proof: Let Ω_0 be reference configuration of the region $\Omega(t)$. Let The motion and the deformation gradient are given by:

$$\mathbf{x} = \varphi(\mathbf{X}, t)\ ; \qquad \Longrightarrow \qquad \boldsymbol{F}(\mathbf{X}, t) = \boldsymbol{\nabla}_{\circ}\varphi$$

Let $J(\mathbf{X}, t) = \det[\boldsymbol{F}(\mathbf{X}, t)]$. Then, integrals in the current and the reference configurations are related by

$$\int_{\Omega(t)} \mathbf{f}(\mathbf{x}, t)\, \mathrm{dV} = \int_{\Omega_0} \mathbf{f}[\varphi(\mathbf{X}, t), t]\, J(\mathbf{X}, t)\, \mathrm{dV}_0 = \int_{\Omega_0} \hat{\mathbf{f}}(\mathbf{X}, t)\, J(\mathbf{X}, t)\, \mathrm{dV}_0$$

Reynolds Transport Theorem (Proof)

The time derivative of an integral over a volume is defined as :

$$\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f}(\mathbf{x},t)\,\mathsf{dV}\right) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{\Omega(t+\Delta t)} \mathbf{f}(\mathbf{x},t+\Delta t)\,\mathsf{dV} - \int_{\Omega(t)} \mathbf{f}(\mathbf{x},t)\,\mathsf{dV}\right)$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{\Omega_0} \hat{\mathbf{f}}(\mathbf{X},t+\Delta t)\,J(\mathbf{X},t+\Delta t)\,\mathsf{dV}_0\right)$$

$$-\left(\int_{\Omega_0} \hat{\mathbf{f}}(\mathbf{X},t)\,J(\mathbf{X},t)\,\mathsf{dV}_0\right)$$

$$= \int_{\Omega_0}\left[\lim_{\Delta t\to 0}\frac{\hat{\mathbf{f}}(\mathbf{X},t+\Delta t)\,J(\mathbf{X},t+\Delta t) - \hat{\mathbf{f}}(\mathbf{X},t)\,J(\mathbf{X},t)}{\Delta t}\right]\mathsf{dV}_0$$

$$= \int_{\Omega_0}\frac{\partial}{\partial t}[\hat{\mathbf{f}}(\mathbf{X},t)\,J(\mathbf{X},t)]\,\mathsf{dV}_0$$

Reynolds Transport Theorem (Proof)

Since Ω_0 is independent of time, we have

$$\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f}(\mathbf{x},t)\ \mathsf{dV}\right) = \int_{\Omega_0} \left(\frac{\partial}{\partial t}[\hat{\mathbf{f}}(\mathbf{X},t)]\ J(\mathbf{X},t) + \hat{\mathbf{f}}(\mathbf{X},t)\ \frac{\partial}{\partial t}[J(\mathbf{X},t)]\right)\ \mathsf{dV}_0$$

Now, the time derivative of $\det \boldsymbol{F}$ is given by

$$\begin{aligned}\frac{\partial J(\mathbf{X},t)}{\partial t} = \frac{\partial}{\partial t}(\det \boldsymbol{F}) &= (\det \boldsymbol{F})(\boldsymbol{\nabla} \cdot \mathbf{v}) \\ &= J(\mathbf{X},t)\ \boldsymbol{\nabla} \cdot \mathbf{v}(\varphi(\mathbf{X},t),t) \\ &= J(\mathbf{X},t)\ \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x},t)\end{aligned}$$

Reynolds Transport Theorem (Proof)

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Therefore,

$$\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f}(\mathbf{x},t)\,\mathrm{dV}\right) = \int_{\Omega_0}\left(\frac{\partial}{\partial t}[\hat{\mathbf{f}}(\mathbf{X},t)]\,J(\mathbf{X},t) + \hat{\mathbf{f}}(\mathbf{X},t)\,J(\mathbf{X},t)\,\boldsymbol{\nabla}\cdot\mathbf{v}\right)\mathrm{dV}_0$$

$$= \int_{\Omega_0}\left(\frac{\partial}{\partial t}[\hat{\mathbf{f}}(\mathbf{X},t)] + \hat{\mathbf{f}}(\mathbf{X},t)\,\boldsymbol{\nabla}\cdot\mathbf{v}\right) J(\mathbf{X},t)\,\mathrm{dV}_0$$

$$= \int_{\Omega(t)}\left(\frac{D\mathbf{f}}{Dt} + \mathbf{f}\,\boldsymbol{\nabla}\cdot\mathbf{v}\right)\mathrm{dV}$$

Now, the material derivative is given by

$$\frac{D\mathbf{f}}{Dt} = \frac{\partial\mathbf{f}}{\partial t} + \boldsymbol{\nabla}\mathbf{f}\cdot\mathbf{v}$$

Reynolds Transport Theorem (Proof)

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Therefore,

$$\boxed{\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f}\ \mathrm{dV}\right) = \int_{\Omega(t)} \left(\frac{\partial \mathbf{f}}{\partial t} + \boldsymbol{\nabla}\mathbf{f}\cdot\mathbf{v} + \mathbf{f}\ \boldsymbol{\nabla}\cdot\mathbf{v}\right)\ \mathrm{dV}}$$

Using the identity

$$\boldsymbol{\nabla}\cdot(\mathbf{v}\otimes\mathbf{w}) = \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{w}) + \boldsymbol{\nabla}\mathbf{v}\cdot\mathbf{w}$$

We then have

$$\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f}\ \mathrm{dV}\right) = \int_{\Omega(t)} \left(\frac{\partial \mathbf{f}}{\partial t} + \boldsymbol{\nabla}\cdot(\mathbf{f}\otimes\mathbf{v})\right)\ \mathrm{dV}$$

Reynolds Transport Theorem (Proof)

Using the divergence theorem and the identity

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{n} = (\mathbf{b} \cdot \mathbf{n})\mathbf{a}$$

we have

$$\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f} \,\mathrm{dV}\right) = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t} \,\mathrm{dV} + \int_{\partial\Omega(t)} (\mathbf{f} \otimes \mathbf{v}) \cdot \mathbf{n} \,\mathrm{dS} \tag{5}$$

$$= \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t} \,\mathrm{dV} + \int_{\partial\Omega(t)} (\mathbf{v} \cdot \mathbf{n})\mathbf{f} \,\mathrm{dS} \tag{6}$$

Hence the proof.

Reynolds Transport Theorem

In fluid dynamics or continuum mechanics, this can be written as follows: Let B be any property of the fluid and $\beta = \frac{d\mathbf{B}}{dm}$ be the intensive value of B (amount of B per unit mass) in any small element of the fluid, then

$$\left(\frac{d\mathbf{B}}{dt}\right)_{\Omega(t)} = \int_{\Omega(t)} \left[\frac{\partial}{\partial t}(\rho\beta) + \mathbf{v}.\boldsymbol{\nabla}(\rho\beta) + \rho\beta\boldsymbol{\nabla}.\mathbf{v}\right] dV \tag{7}$$

For the Pictorial Proof using fluid dynamics continuum mechanics approach, refer to the lecture notes of Hyunse Yoon, University of Iowa.

Conservation Laws

Conservation of Mass

Use $B = m$ in (7) then

$$\left(\frac{dm}{dt}\right)_{\Omega(t)} = 0, \quad \beta = \left(\frac{dm}{dm}\right) = 1$$

$$\left(\frac{dm}{dt}\right)_{\Omega(t)} = \int\limits_{\Omega(t)} \left[\frac{\partial}{\partial t}(\rho\beta) + \mathbf{v}.\boldsymbol{\nabla}(\rho\beta) + \rho\beta\boldsymbol{\nabla}.\mathbf{v}\right] dV$$

$$\implies 0 = \int\limits_{\Omega(t)} \left[\frac{\partial \rho}{\partial t} + \mathbf{v}.\boldsymbol{\nabla}(\rho) + \rho\boldsymbol{\nabla}.\mathbf{v}\right] dV$$

Hence

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho\mathbf{v}) = 0} \tag{8}$$

Conservation of Momentum

Use $B = m\mathbf{v}$ in (7) then

$$\left(\frac{dm\mathbf{v}}{dt}\right)_{\Omega(t)} = \int_{\Omega(t)} \mathbf{f}dV, \;\; \beta = \left(\frac{dm\mathbf{v}}{dm}\right) = \mathbf{v}$$

Here f is an external force.

$$\int_{\Omega(t)} \mathbf{f}dV = \int_{\Omega(t)} \left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \mathbf{v}.\boldsymbol{\nabla}(\rho\mathbf{v}) + \rho\mathbf{v}\boldsymbol{\nabla}.\mathbf{v}\right] dV$$

$$\boxed{\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}.(\rho\mathbf{v}\mathbf{v}) = \boldsymbol{\nabla}.T_s + \mathbf{f}_b} \tag{9}$$

Here $T_s = [(-p + \lambda\boldsymbol{\nabla}.\mathbf{v})I + 2\mu D]$, $D = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^T)$, p is the pressure, f_b is the body force and T_s is the stress tensor.

Navier Stokes Equation

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \boldsymbol{\nabla}.(\rho \mathbf{v} \otimes \mathbf{v}) = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{v} + \frac{1}{3}\mu \boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{v}) + \rho \mathbf{g}} \tag{10}$$

The incompressible Navier-Stokes equation is given by

$$\overbrace{\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\text{Variation}} + \underbrace{(\mathbf{v} \cdot \nabla)\mathbf{v}}_{\text{Convection}}}^{\text{Inertia (per volume)}} \overbrace{- \underbrace{\nu \nabla^2 \mathbf{v}}_{\text{Diffusion}} = \underbrace{-\nabla w}_{\text{Internal source}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{g}}_{\text{External source}} \tag{11}$$

Conservation of Energy

Use $B = E = m\left(u + \frac{1}{2}\mathbf{v}.\mathbf{v}\right)$ in (7) then

$$\left(\frac{dE}{dt}\right)_{\Omega(t)} = \dot{Q} - \dot{W}, \quad \beta = \left(\frac{dE}{dm}\right) = e$$

$$\boxed{\frac{\partial}{\partial t}(\rho e) + \boldsymbol{\nabla}.(\rho\mathbf{v}e) = -\boldsymbol{\nabla}.\dot{q}_s - p\nabla.\mathbf{v} + \boldsymbol{\nabla}.(\boldsymbol{\tau}.\mathbf{v}) + \mathbf{f}_b.\mathbf{v} + \dot{q}_v} \qquad (12)$$

Here p: pressure, $\dot{q}_s$: rate of heat transfer per unit area across the surface area, $\dot{q}_v$: rate of heat source or sink within material volume per unit volume, $\dot{Q}$: net rate of heat transferred to the material element, $\dot{W}$: net rate of work done by the material volume, τ: viscous stress tensor

Heat Equation

The conservation of total internal energy can be written as

$$\underbrace{\frac{\partial}{\partial t}\left(\rho\left[e+\frac{1}{2}v^2\right]\right)}_{\text{Rate of increase of Energy per unit volume}}+\underbrace{\boldsymbol{\nabla}.\left[\rho\mathbf{v}\left(e+\frac{1}{2}v^2\right)\right]}_{\text{Convection energy into point by flow}}=-\underbrace{\boldsymbol{\nabla}.\mathbf{q}}_{\text{net heat flux}}+\underbrace{\nabla.(\sigma.\mathbf{v})}_{\text{Work of Surface forces}}+\underbrace{\rho\boldsymbol{v}.\boldsymbol{F}}_{\text{Work of body forces}} \quad (13)$$

$$\boxed{\frac{\partial}{\partial t}(\rho c_p T)+\mathbf{v}.\boldsymbol{\nabla}T(T\mathbf{v})=\boldsymbol{\nabla}.(k\boldsymbol{\nabla}T)+\beta T\left(\frac{\partial p}{\partial t}+\mathbf{v}.\nabla p\right)+\boldsymbol{\tau}:\boldsymbol{\nabla}\mathbf{v}+\boldsymbol{Q}} \quad (14)$$

Here c_p: specific heat capacity, k: thermal conductivity

Thanks

Doubts and Suggestions

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