MA612L-Partial Differential Equations

Lecture 14 : Reynolds Transport Theorem

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Recap

Reynolds Transport Theorem

Consider the integration of $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ over a time-dependent region $\Omega(t)$ with boundary $\partial\Omega(t)$, then the Reynolds Transport theorem relates taking the derivative with respect to time as follows.

Theorem 1 (Reynolds Transport Theorem)
Let $\Omega(t) \subset \mathbb{R}^3$ and $\mathbf{f} : \Omega(t) \times [0, \infty) \to U$, then

$$\frac{D}{Dt}\left(\int_{\Omega(t)} \mathbf{f} dV\right) = \int_{\Omega(t)} \frac{\partial \mathbf{f}}{\partial t} dV + \int_{\partial\Omega(t)} (\mathbf{v}.\mathbf{n})\mathbf{f} dS \qquad (1)$$

Here $\mathbf{x}(t)$ is the position of the points in $\Omega(t)$. $\mathbf{n}(\mathbf{x}, \mathbf{t})$ is the outward unit normal vector to $\partial\Omega(t)$. dV and dS are volume and surface elements at $\mathbf{x}$. $\mathbf{v}(\mathbf{x}, \mathbf{t})$ denotes the velocity of the area element. The function $\mathbf{f}$ may be scalar-valued or vector-valued, or tensor-valued. $\frac{D}{Dt}$ is usually called as total derivative or material derivative.

Reynolds Transport Theorem

In fluid dynamics or continuum mechanics, this can be written as follows:
Let B be any property of the fluid and $\beta = \frac{d\mathbf{B}}{dm}$ be the intensive value of B (amount of B per unit mass) in any small element of the fluid, then

$$\left(\frac{d\mathbf{B}}{dt}\right)_{\Omega(t)} = \int_{\Omega(t)} \left[\frac{\partial}{\partial t}(\rho\beta) + \mathbf{v}.\boldsymbol{\nabla}(\rho\beta) + \rho\beta\boldsymbol{\nabla}.\mathbf{v}\right] dV \qquad (2)$$

For the Pictorial Proof using fluid dynamics continuum mechanics approach, refer to the lecture notes of Hyunse Yoon, University of Iowa.

Conservation Laws

Conservation of Mass

Use $B = m$ in (2) then

$$\left(\frac{dm}{dt}\right)_{\Omega(t)} = 0, \quad \beta = \left(\frac{dm}{dm}\right) = 1$$

$$\left(\frac{dm}{dt}\right)_{\Omega(t)} = \int_{\Omega(t)} \left[\frac{\partial}{\partial t}(\rho\beta) + \mathbf{v}.\boldsymbol{\nabla}(\rho\beta) + \rho\beta\boldsymbol{\nabla}.\mathbf{v}\right] dV$$

$$\implies 0 = \int_{\Omega(t)} \left[\frac{\partial \rho}{\partial t} + \mathbf{v}.\boldsymbol{\nabla}(\rho) + \rho\boldsymbol{\nabla}.\mathbf{v}\right] dV$$

Hence

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho\mathbf{v}) = 0} \qquad (3)$$

Conservation of Momentum

Use $B = m\mathbf{v}$ in (2) then

$$\left(\frac{dm\mathbf{v}}{dt}\right)_{\Omega(t)} = \int\limits_{\Omega(t)} \mathbf{f} dV, \;\; \beta = \left(\frac{dm\mathbf{v}}{dm}\right) = \mathbf{v}$$

Here f is an external force.

$$\int\limits_{\Omega(t)} \mathbf{f} dV = \int\limits_{\Omega(t)} \left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \mathbf{v}.\boldsymbol{\nabla}(\rho\mathbf{v}) + \rho\mathbf{v}\boldsymbol{\nabla}.\mathbf{v}\right] dV$$

$$\boxed{\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}.(\rho\mathbf{v}\mathbf{v}) = \boldsymbol{\nabla}.T_s + \mathbf{f}_b} \qquad (4)$$

Here $T_s = [(-p + \lambda\boldsymbol{\nabla}.\mathbf{v})I + 2\mu D]$, $D = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^T)$, p is the pressure, f_b is the body force and T_s is the stress tensor.

Navier Stokes Equation

$$\boxed{\frac{\partial}{\partial t}(\rho \mathbf{v}) + \boldsymbol{\nabla}.(\rho \mathbf{v} \otimes \mathbf{v}) = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{v} + \frac{1}{3}\mu \boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{v}) + \rho \mathbf{g}} \tag{5}$$

The incompressible Navier-Stokes equation is given by

$$\overbrace{\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\text{Variation}} + \underbrace{(\mathbf{v}\cdot\nabla)\mathbf{v}}_{\text{Convection}}}^{\text{Inertia (per volume)}} \overbrace{- \underbrace{\nu \nabla^2 \mathbf{v}}_{\text{Diffusion}} = \underbrace{-\nabla w}_{\text{Internal source}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{g}}_{\text{External source}} \tag{6}$$

Conservation of Energy

Use $B = E = m\left(u + \frac{1}{2}\mathbf{v}.\mathbf{v}\right)$ in (2) then

$$\left(\frac{dE}{dt}\right)_{\Omega(t)} = \dot{Q} - \dot{W}, \quad \beta = \left(\frac{dE}{dm}\right) = e$$

$$\boxed{\frac{\partial}{\partial t}(\rho e) + \boldsymbol{\nabla}.(\rho\mathbf{v}e) = -\boldsymbol{\nabla}.\dot{q}_s - p\nabla.\mathbf{v} + \boldsymbol{\nabla}.(\boldsymbol{\tau}.\mathbf{v}) + \mathbf{f}_b.\mathbf{v} + \dot{q}_v} \qquad (7)$$

Here p: pressure, $\dot{q}_s$: rate of heat transfer per unit area across the surface area, $\dot{q}_v$: rate of heat source or sink within material volume per unit volume, $\dot{Q}$: net rate of heat transferred to the material element, $\dot{W}$: net rate of work done by the material volume, τ: viscous stress tensor

Heat Equation

The conservation of total internal energy can be written as

$$\underbrace{\frac{\partial}{\partial t}\left(\rho\left[e+\frac{1}{2}v^2\right]\right)}_{\text{Rate of increase of Energy per unit volume}}+\underbrace{\boldsymbol{\nabla}.\left[\rho\mathbf{v}\left(e+\frac{1}{2}v^2\right)\right]}_{\text{Convection energy into point by flow}}=-\underbrace{\boldsymbol{\nabla}.\mathbf{q}}_{\text{net heat flux}}+\underbrace{\nabla.(\sigma.\mathbf{v})}_{\text{Work of Surface forces}}+\underbrace{\rho\boldsymbol{v}.\boldsymbol{F}}_{\text{Work of body forces}} \tag{8}$$

$$\boxed{\frac{\partial}{\partial t}(\rho c_p T)+\mathbf{v}.\boldsymbol{\nabla}T(T\mathbf{v})=\boldsymbol{\nabla}.(k\boldsymbol{\nabla}T)+\beta T\left(\frac{\partial p}{\partial t}+\mathbf{v}.\nabla p\right)+\boldsymbol{\tau}:\boldsymbol{\nabla}\mathbf{v}+\boldsymbol{Q}} \tag{9}$$

Here c_p: specific heat capacity, k: thermal conductivity

d’Alembert’s Formula for Wave Equation

Recall: Transport Equation in $\mathbb{R}^n \times (0, \infty)$

Consider the following inhomogeneous problem

$$\begin{cases} u_t + b.Du = f, \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}), x \in \mathbb{R}^n \end{cases} \tag{10}$$

Then

$$u(\mathbf{x}, t) = g(\mathbf{x} - \mathbf{b}t) + \int_0^t f(\mathbf{x} + (s - t)\mathbf{b}, s) ds, x \in \mathbb{R}^n, t \geq 0$$

solves the IVP (10).

d'Alembert's Formula $n = 1$

Consider the following one-dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u = f \text{ on } \mathbb{R} \times (t = 0) \\ u_t = g \text{ on } \mathbb{R} \times (t = 0) \end{cases} \tag{11}$$

Let us try to find the solution of the wave equation in terms of f and g.
Let us rewrite the wave equation as follows:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0$$

d'Alembert's Formula

Let

$$v(x,t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x,t)$$

Then the wave equation can be written as

$$v_t(x,t) + cv_x(x,t) = 0 \text{ in } \mathbb{R} \times (0,\infty)$$

This is a simple transport equation. We know the solution to this problem, which is given by

$$v(x,t) = a(x - ct)$$

where

$$v(x,0) = a(x)$$

d’Alembert’s Formula

Therefore, we have

$$u_t(x,t) - cu_x(x,t) = a(x-ct) \text{ in } \mathbb{R}\times(0,\infty)$$

This is once again the transport equation. Therefore,

$$u(x,t) = f(x+ct) + \int_0^t a(x - cs - c(s-t))ds, x \in \mathbb{R}, t \geq 0$$

$$u(x,t) = f(x+ct) + \int_0^t a(x - 2cs + ct))ds, x \in \mathbb{R}, t \geq 0$$

d'Alembert's Formula

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Use $y = x - 2cs + ct$, then $dy = -2cds$, $s = 0 \implies y = x + ct$, $s = t \implies y = x - ct$

$$u(x,t) = f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} a(y)dy, x \in \mathbb{R}, t \geq 0$$

Now,

$$a(x) = v(x,0) = u_t(x,0) - cu_x(x,0) = g(x) - cf'(x)$$

$$\implies \frac{1}{2c}\int_{x-ct}^{x+ct} a(y)dy = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy - \frac{1}{2}\int_{x-ct}^{x+ct} f'(y)dy$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy - \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct)$$

d'Alembert's Formula

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$

is called d'Alembert's formula. This can be written as

$$u(x,t) = F(x+ct) + G(x-ct)$$

for appropriate F and G. Conversely, any function of this form solves $u_{tt} - c^2 u_{xx} = 0$. Observe the beauty of this equation, the general solution of the one-dimensional wave equation is a sum of the general solution of the transport equations

$$u_t + cu_x = 0 \quad \text{and} \quad u_t - cu_x = 0$$

Wave Equation: Separation Variables

Recollect the solution obtained from the Separation of Variables. (Of course, this was on domain $[0, L] \times (0, \infty)$)

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right) \tag{12}$$

Using $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$, we can rewrite this as

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{n\pi}{L}(x+ct)\right) + A_n \sin\left(\frac{n\pi}{L}(x-ct)\right) \right]$$
$$+ \frac{1}{2}\sum_{n=1}^{\infty} \left[B_n \cos\left(\frac{n\pi}{L}(x-ct)\right) - B_n \cos\left(\frac{n\pi}{L}(x+ct)\right) \right]$$

d'Alembert's Formula

Again, you can see that

$$u(x,t) = F(x+ct) + G(x-ct)$$

When $c = 1, g$ is an odd function and f is an even function, then

$$u(0,t) = f(t)$$

When $c = 1, g$ is an an even function and f is an odd function, then

$$u(0,t) = \int_0^t g(y)dy$$

When $g = 0$, you can observe that the initial displacement splits into two parts, one moving to the right with speed c and the other to the left with speed c.

d'Alembert's Formula

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Theorem 2

Assume $f \in C^2(\mathbb{R}), g \in C^1(\mathbb{R}), c = 1$ and define u by d'Alembert's. Then,

1. $u \in C^2(\mathbb{R} \times [0, \infty))$
2. $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times [0, \infty)$
3. $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u(x,t) = f(x^0)$
4. $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ t>0}} u_t(x,t) = g(x^0)$

In (3) and (4) for each point $x^0 \in \mathbb{R}$

The proof follows immediately from the above discussion and is left as an exercise.

Exercise

Exercise 1: CSIR-June-2019

Let u be the solution of (11) where $c = 1$, f, g are in $C^2(\mathbb{R})$ and satisfy the following conditions:

(1) $f(x) = g(x) = 0$ for $x \leq 0$ (2) $0 < f(x) \leq 1$ for $x > 0$

(3) $g(x) > 0$ for $x > 0$ (4) $\int_0^\infty g(x)dx < \infty$

Then, which of the following statements are true? Justify your answer

1. $u(x,t) = 0$ for all $x \leq 0$ and $t > 0$
2. u is bounded on $\mathbb{R} \times (0, \infty)$
3. $u(x,t) = 0$ whenever $x + t < 0$
4. $u(x,t) = 0$ for some (x,t) satisfying $x + t > 0$

d'Alembert's Formula in the half-line

Let us consider the following wave equation on the half-line $\mathbb{R}_+$. Note $c = 1$ here.

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ u = f \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ u_t = g \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 \text{ on } \{x = 0\} \times (0, \infty) \end{cases} \tag{13}$$

where f and g are given, with $f(0) = g(0) = 0$.

d'Alembert's Formula in the half-line

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Let us convert (13) by extending u, g, h to all $\mathbb{R}$ by **odd reflection**. That is, we set

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & (x \geq 0, t \geq 0) \\ -u(-x,t) & (x \leq 0, t \geq 0) \end{cases}$$

$$\tilde{f}(x) := \begin{cases} f(x) & (x \geq 0) \\ -f(-x) & (x \leq 0) \end{cases}$$

$$\tilde{g}(x) := \begin{cases} g(x) & (x \geq 0) \\ -g(-x) & (x \leq 0) \end{cases}$$

d'Alembert's Formula in the half-line

Then (13) becomes

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ \tilde{u} = \tilde{f} \text{ on } \mathbb{R} \times \{t = 0\} \\ \tilde{u}_t = \tilde{g} \text{ on } \mathbb{R} \times \{t = 0\} \end{cases} \quad (14)$$

Hence d'Alembert's formula becomes

$$\tilde{u}(x, t) = \frac{1}{2}[\tilde{f}(x + t) + \tilde{f}(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} \tilde{g}(y) dy \quad (15)$$

d'Alembert's Formula in the half-line

The solution obtained from the **reflection method** for (13) becomes

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int\limits_{x-t}^{x+t} g(y)dy, & \text{if } x \geq t \geq 0 \\ \frac{1}{2}[f(x+t) - f(t-x)] + \frac{1}{2}\int\limits_{-x+t}^{x+t} g(y)dy, & \text{if } 0 \leq x \leq t \end{cases} \tag{16}$$

1. If $g \equiv 0$, we can understand the formula (16) as saying that an initial displacement f splits into two parts. One moving to the right with speed one and another moving to the left with speed one
2. The latter reflects off the point $x = 0$, where the vibrating string is held fixed.

Thanks

Doubts and Suggestions

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