MA612L-Partial Differential Equations

Lecture 15 : d'Alembert's Formula

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

September 12, 2025

भारतीय प्रौद्योगिकी संस्थान तिरुपति
TIRUPATI

d’Alembert’s Formula for Wave Equation

Recall: Transport Equation in $\mathbb{R}^n \times (0, \infty)$

Consider the following inhomogeneous problem

$$\begin{cases} u_t + b.Du = f, \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}), x \in \mathbb{R}^n \end{cases} \quad (1)$$

Then

$$u(\mathbf{x}, t) = g(\mathbf{x} - \mathbf{b}t) + \int_0^t f(\mathbf{x} + (s - t)\mathbf{b}, s)ds, x \in \mathbb{R}^n, t \geq 0$$

solves the IVP (1).

d'Alembert's Formula $n = 1$

Consider the following one-dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u = f \text{ on } \mathbb{R} \times (t = 0) \\ u_t = g \text{ on } \mathbb{R} \times (t = 0) \end{cases} \tag{2}$$

Let us try to find the solution of the wave equation in terms of f and g.
Let us rewrite the wave equation as follows:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = u_{tt} - u_{xx} = 0$$

d'Alembert's Formula

Let

$$v(x,t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x,t)$$

Then the wave equation can be written as

$$v_t(x,t) + cv_x(x,t) = 0 \text{ in } \mathbb{R} \times (0,\infty)$$

This is a simple transport equation. We know the solution to this problem, which is given by

$$v(x,t) = a(x-ct)$$

where

$$v(x,0) = a(x)$$

d'Alembert's Formula

Therefore, we have

$$u_t(x,t) - cu_x(x,t) = a(x-ct) \text{ in } \mathbb{R} \times (0,\infty)$$

This is once again the transport equation. Therefore,

$$u(x,t) = f(x+ct) + \int_0^t a(x-cs-c(s-t))ds, x \in \mathbb{R}, t \geq 0$$

$$u(x,t) = f(x+ct) + \int_0^t a(x-2cs+ct))ds, x \in \mathbb{R}, t \geq 0$$

d'Alembert's Formula

Use $y = x - 2cs + ct$, then $dy = -2cds$, $s = 0 \implies y = x + ct$,
$s = t \implies y = x - ct$

$$u(x,t) = f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} a(y)dy, x \in \mathbb{R}, t \geq 0$$

Now,

$$a(x) = v(x,0) = u_t(x,0) - cu_x(x,0) = g(x) - cf'(x)$$

$$\implies \frac{1}{2c}\int_{x-ct}^{x+ct} a(y)dy = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy - \frac{1}{2}\int_{x-ct}^{x+ct} f'(y)dy$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy - \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct)$$

d'Alembert's Formula

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$

is called d'Alembert's formula. This can be written as

$$u(x,t) = F(x+ct) + G(x-ct)$$

for appropriate F and G. Conversely, any function of this form solves $u_{tt} - c^2 u_{xx} = 0$. Observe the beauty of this equation, the general solution of the one-dimensional wave equation is a sum of the general solution of the transport equations

$$u_t + cu_x = 0 \quad \text{and} \quad u_t - cu_x = 0$$

Wave Equation: Separation Variables

Recollect the solution obtained from the Separation of Variables. (Of course, this was on domain $[0, L] \times (0, \infty)$)

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right) \right] \sin\left(\frac{n\pi}{L} x\right) \tag{3}$$

Using $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ and $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$, we can rewrite this as

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{n\pi}{L}(x+ct)\right) + A_n \sin\left(\frac{n\pi}{L}(x-ct)\right) \right]$$

$$+ \frac{1}{2} \sum_{n=1}^{\infty} \left[B_n \cos\left(\frac{n\pi}{L}(x-ct)\right) - B_n \cos\left(\frac{n\pi}{L}(x+ct)\right) \right]$$

d'Alembert's Formula

Again, you can see that

$$u(x,t) = F(x+ct) + G(x-ct)$$

When $c = 1, g$ is an odd function and f is an even function, then

$$u(0,t) = f(t)$$

When $c = 1, g$ is an an even function and f is an odd function, then

$$u(0,t) = \int_0^t g(y)dy$$

When $g = 0$, you can observe that the initial displacement splits into two parts, one moving to the right with speed c and the other to the left with speed c.

d'Alembert's Formula

Theorem 1

Assume $f \in C^2(\mathbb{R}), g \in C^1(\mathbb{R}), c = 1$ and define u by d'Alembert's. Then,

1. $u \in C^2(\mathbb{R} \times [0, \infty))$
2. $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times [0, \infty)$
3. $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ t>0}} u(x,t) = f(x^0)$
4. $\lim\limits_{\substack{(x,t)\to(x^0,0)\\ t>0}} u_t(x,t) = g(x^0)$

In (3) and (4) for each point $x^0 \in \mathbb{R}$

The proof follows immediately from the above discussion and is left as an exercise.

Exercise

Exercise 1: CSIR-June-2019

Let u be the solution of (2) where $c = 1, f, g$ are in $C^2(\mathbb{R})$ and satisfy the following conditions:

(1) $f(x) = g(x) = 0$ for $x \leq 0$ (2) $0 < f(x) \leq 1$ for $x > 0$

(3) $g(x) > 0$ for $x > 0$ (4) $\int\limits_0^\infty g(x)dx < \infty$

Then, which of the following statements are true? Justify your answer

1. $u(x,t) = 0$ for all $x \leq 0$ and $t > 0$
2. u is bounded on $\mathbb{R} \times (0, \infty)$
3. $u(x,t) = 0$ whenever $x + t < 0$
4. $u(x,t) = 0$ for some (x,t) satisfying $x + t > 0$

d'Alembert's Formula in the half-line

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Let us consider the following wave equation on the half-line $\mathbb{R}_+$. Note $c = 1$ here.

$$\begin{cases} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ u = f \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ u_t = g \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ u = 0 \text{ on } \{x = 0\} \times (0, \infty) \end{cases} \tag{4}$$

where f and g are given, with $f(0) = g(0) = 0$.

d'Alembert's Formula in the half-line

Let us convert (4) by extending u, g, h to all $\mathbb{R}$ by **odd reflection**. That is, we set

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & (x \geq 0, t \geq 0) \\ -u(-x,t) & (x \leq 0, t \geq 0) \end{cases}$$

$$\tilde{f}(x) := \begin{cases} f(x) & (x \geq 0) \\ -f(-x) & (x \leq 0) \end{cases}$$

$$\tilde{g}(x) := \begin{cases} g(x) & (x \geq 0) \\ -g(-x) & (x \leq 0) \end{cases}$$

d'Alembert's Formula in the half-line

Then (4) becomes

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ \tilde{u} = \tilde{f} \text{ on } \mathbb{R} \times \{t = 0\} \\ \tilde{u}_t = \tilde{g} \text{ on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{5}$$

Hence, d'Alembert's formula becomes

$$\tilde{u}(x,t) = \frac{1}{2}[\tilde{f}(x+t) + \tilde{f}(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{g}(y)dy \tag{6}$$

d'Alembert's Formula in the half-line

The solution obtained from the **reflection method** for (4) becomes

$$u(x,t) = \begin{cases} \dfrac{1}{2}[f(x+t)+f(x-t)] + \dfrac{1}{2}\displaystyle\int_{x-t}^{x+t} g(y)dy, & \text{if } x \geq t \geq 0 \\ \dfrac{1}{2}[f(x+t)-f(t-x)] + \dfrac{1}{2}\displaystyle\int_{-x+t}^{x+t} g(y)dy, & \text{if } 0 \leq x \leq t \end{cases} \tag{7}$$

1. If $g \equiv 0$, we can understand the formula (7) as saying that an initial displacement f splits into two parts. One moving to the right with speed one and another moving to the left with speed one
2. The latter reflects off the point $x = 0$, where the vibrating string is held fixed.

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in