MA612L-Partial Differential Equations

Lecture 16 : Spherical Means

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Spherical Means

Preliminaries

- $B(\mathbf{x}, r)$ = the closed ball with center $\mathbf{x}$ and radius $r > 0$
- $\alpha(n)$ = Volume of unit ball $B(\mathbf{0}, 1)$ in $\mathbb{R}^n = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ [Proof]
- $n\alpha(n)$ = Surface area of unit Sphere $B(\mathbf{0}, 1)$ in $\mathbb{R}^n$
- $\displaystyle ⨍_{B(\mathbf{x},r)} f d\mathbf{y} = \frac{1}{\alpha(n) r^n} \int_{B(\mathbf{x},r)} f d\mathbf{y}$ = average of f over the ball $B(\mathbf{x}, r)$
- $\displaystyle ⨍_{\partial B(\mathbf{x},r)} f dS = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(\mathbf{x},r)} f dS$ = avg of f over the sphere $\partial B(\mathbf{x}, r)$

Preliminaries

Definition 1
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say $\partial\Omega$ is C^k if for each $\mathbf{x}^0 \in \partial\Omega$ there exist $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that upon relabeling and reorienting the coordinate axes if necessary we have

$$\partial\Omega \cap B(\mathbf{x}^0, r) = \{x \in B(\mathbf{x}^0, r) : x_n > \gamma(x_1, x_2, \cdots, x_{n-1})\}$$

The normal derivative is $\dfrac{\partial u}{\partial \nu} = \nu.Du$, where $u \in C^1(\overline{\Omega})$, $\partial\Omega$ is C^1 and $\nu = (\nu^1, \nu^2, \cdots, \nu^n)$ is a unit normal vector pointing outward.

Preliminaries

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Theorem 1 (Gauss-Green Theorem)
Suppose $u \in C^1(\overline{\Omega})$. Then

$$\int_{\Omega} u_{x_i} dx = \int_{\partial\Omega} u\nu^i dS \tag{1}$$

where $i = 1, 2, \cdots, n$.

By the divergence theorem,

$$\int_{\Omega} \operatorname{div} F \, dx = \int_{\partial\Omega} F \cdot \nu \, dS.$$

Preliminaries

Define a vector field $F : \overline{\Omega} \to \mathbb{R}^n$ by

$$F(x) = (0, \ldots, 0, \underbrace{u(x)}_{i\text{-th component}}, 0, \ldots, 0),$$

that is,

$$F^j(x) = \begin{cases} u(x), & j = i, \\ 0, & j \neq i. \end{cases}$$

Since $u \in C^1(\overline{\Omega})$, the components of F are continuously differentiable. The divergence of F is

$$\operatorname{div} F = \sum_{j=1}^{n} \frac{\partial F^j}{\partial x_j} = \frac{\partial F^i}{\partial x_i} = u_{x_i}.$$

On the left-hand side, we have

$$\int_\Omega \operatorname{div} F \, dx = \int_\Omega u_{x_i} \, dx,$$

and on the right-hand side,

$$\int_{\partial\Omega} F \cdot \nu \, dS = \int_{\partial\Omega} \sum_{j=1}^{n} F^j \nu^j \, dS = \int_{\partial\Omega} F^i \nu^i \, dS = \int_{\partial\Omega} u \, \nu^i \, dS.$$

Hence,

$$\int_\Omega u_{x_i} \, dx = \int_{\partial\Omega} u \, \nu^i \, dS.$$

Thanks

Doubts and Suggestions

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