MA612L-Partial Differential Equations

Lecture 17 : Spherical Means

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Recap

Preliminaries

- $B(\mathbf{x}, r)$ = the closed ball with center $\mathbf{x}$ and radius $r > 0$
- $\alpha(n)$ = Volume of unit ball $B(\mathbf{0}, 1)$ in $\mathbb{R}^n = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$
- $n\alpha(n)$ = Surface area of unit Sphere $B(\mathbf{0}, 1)$ in $\mathbb{R}^n$
- $\displaystyle ⨍_{B(\mathbf{x},r)} f d\mathbf{y} = \frac{1}{\alpha(n) r^n} \int_{B(\mathbf{x},r)} f d\mathbf{y}$ = average of f over the ball $B(\mathbf{x}, r)$
- $\displaystyle ⨍_{\partial B(\mathbf{x},r)} f dS = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(\mathbf{x},r)} f dS$ = avg of f over the sphere $\partial B(\mathbf{x}, r)$

Preliminaries

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Definition 1

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say $\partial\Omega$ is C^k if for each $\mathbf{x}^0 \in \partial\Omega$ there exist $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that upon relabeling and reorienting the coordinate axes if necessary we have

$$\partial\Omega \cap B(\mathbf{x}^0, r) = \{x \in B(\mathbf{x}^0, r) : x_n > \gamma(x_1, x_2, \cdots, x_{n-1})\}$$

Theorem 1 (Gauss-Green Theorem)

Suppose $u \in C^1(\overline{\Omega})$. Then

$$\int_{\Omega} u_{x_i} dx = \int_{\partial\Omega} u\nu^i dS \tag{1}$$

where $i = 1, 2, \cdots, n$.

Preliminaries

Preliminaries

Theorem 2 (Integration by Parts)
Suppose $u, v \in C^1(\overline{\Omega})$. Then

$$\int\limits_{\Omega} u_{x_i} v d\mathbf{x} = -\int\limits_{\Omega} u v_{x_i} d\mathbf{x} + \int\limits_{\partial\Omega} u v \nu^i dS \tag{2}$$

where $i = 1, 2, \cdots, n$.

By the divergence theorem,

$$\int_{\Omega} \operatorname{div} F \, dx = \int_{\partial\Omega} F \cdot \nu \, dS.$$

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Define the vector field $F : \overline{\Omega} \to \mathbb{R}^n$ by

$$F(x) = (0, \ldots, 0, \underbrace{u(x)v(x)}_{i\text{-th component}}, 0, \ldots, 0),$$

Since $u, v \in C^1(\overline{\Omega})$, we have $F \in C^1(\overline{\Omega}; \mathbb{R}^n)$ and

$$\operatorname{div} F = \frac{\partial}{\partial x_i}(uv) = u_{x_i}\, v + u\, v_{x_i}.$$

Applying the divergence theorem (Gauss–Ostrogradsky), the left-hand side becomes

$$\int_\Omega (u_{x_i}\, v + u\, v_{x_i})\, d\mathbf{x},$$

and the right-hand side reduces to

$$\int_{\partial\Omega} F \cdot \nu \, dS = \int_{\partial\Omega} F^i \nu^i \, dS = \int_{\partial\Omega} uv\, \nu^i \, dS.$$

Preliminaries

Hence

$$\int_\Omega u_{x_i}\, v \, d\mathbf{x} + \int_\Omega u \, v_{x_i} \, d\mathbf{x} = \int_{\partial\Omega} uv \, \nu^i \, dS,$$

and rearranging gives the desired identity

$$\int_\Omega u_{x_i}\, v \, d\mathbf{x} = - \int_\Omega u \, v_{x_i} \, d\mathbf{x} + \int_{\partial\Omega} uv \, \nu^i \, dS.$$

In vector form,

$$\int_\Omega (\nabla u) \, v \, d\mathbf{x} = - \int_\Omega u \, \nabla v \, d\mathbf{x} + \int_{\partial\Omega} uv \, \nu \, dS.$$

Preliminaries

Theorem 3 (Green's Formulas)

Suppose $u, v \in C^2(\overline{\Omega})$. Then

1. $$\int_{\Omega} \Delta u d\mathbf{x} = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS$$
2. $$\int_{\Omega} Du.Dv d\mathbf{x} = -\int_{\Omega} u\Delta v d\mathbf{x} + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} u dS$$
3. $$\int_{\Omega} (u\Delta v - v\Delta u) d\mathbf{x} = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS$$

Proof: Exercise

Hints

(1) Take $F = \nabla u$. Then $\operatorname{div} F = \operatorname{div}(\nabla u) = \Delta u$, and

$$\int_\Omega \Delta u \, d\mathbf{x} = \int_\Omega \operatorname{div}(\nabla u) \, d\mathbf{x} = \int_{\partial\Omega} \nabla u \cdot \nu \, dS = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \, dS.$$

(2) Use the product rule

$$\operatorname{div}(u\nabla v) = \nabla u \cdot \nabla v + u \, \Delta v.$$

Integrate over Ω and apply the divergence theorem:

$$\int_\Omega \operatorname{div}(u\nabla v) \, d\mathbf{x} = \int_{\partial\Omega} u\nabla v \cdot \nu \, dS = \int_{\partial\Omega} u \, \frac{\partial v}{\partial \nu} \, dS.$$

Hints

(3) Observe the product-rule identity

$$\operatorname{div}\left(u\nabla v - v\nabla u\right) = u\,\Delta v - v\,\Delta u,$$

since the mixed gradient terms cancel. Applying the divergence theorem gives

$$\int_\Omega \left(u\,\Delta v - v\,\Delta u\right) d\mathbf{x} = \int_{\partial\Omega} \left(u\nabla v - v\nabla u\right) \cdot \nu\, dS = \int_{\partial\Omega} \left(u\,\frac{\partial v}{\partial \nu} - v\,\frac{\partial u}{\partial \nu}\right) dS,$$

Preliminaries

From the above theorem, if we take $\Omega = B(\mathbf{x}, r)$ you can observe that

$$\int_{B(\mathbf{x},r)} \Delta u d\mathbf{x} = \int_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu} dS$$

$$⨍_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu} dS = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu} dS = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(\mathbf{x},r)} \Delta u d\mathbf{x}$$

$$= \frac{r}{n} \left(\frac{1}{\alpha(n) r^n} \int_{B(\mathbf{x},r)} \Delta u d\mathbf{x} \right) = \frac{r}{n} ⨍_{B(\mathbf{x},r)} \Delta u d\mathbf{x}$$

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Let $r = |\mathbf{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}, \mathbf{x} \neq 0$ and $u(x) = v(r)$. Then

$$\frac{\partial r}{\partial x_i} = \frac{1}{2\sqrt{\sum_{i=1}^{n} x_i^2}} 2x_i = \frac{x_i}{r} \implies u_{x_i} = \frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r}$$

$$\frac{\partial^2 r}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{r - x_i \frac{\partial r}{x_i}}{r^2} = \frac{1}{r} - \frac{x_i^2}{r^3}$$

$$\implies u_{x_i x_i} = \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial v}{\partial r}\frac{\partial r}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial v}{\partial r}\right)\frac{\partial r}{\partial x_i} + \frac{\partial v}{\partial r}\frac{\partial^2 r}{\partial x_i^2}$$

$$u_{x_i x_i} = \frac{\partial v'(r)}{\partial r}\left(\frac{x_i}{r}\right)^2 + v'(r)\frac{\partial^2 r}{\partial x_i^2} = v''(r)\left(\frac{x_i}{r}\right)^2 + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

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$$Du = (u_{x_1}, u_{x_2}, \cdots u_{x_n}) = \frac{v'(r)}{r}(x_1, x_2, \cdots, x_n) = \frac{v'(r)}{r}\mathbf{x}$$

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = v''(r) \sum_{i=1}^{n} \left(\frac{x_i}{r}\right)^2 + v'(r) \sum_{i=1}^{n} \frac{1}{r} - v'(r) \sum_{i=1}^{n} \frac{x_i^2}{r^3}$$

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r)$$

Spherical Means

Now suppose $n \geq 2, m \geq 2$ and $u \in C^m(\mathbb{R}) \times [0, \infty)$ solves the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u = f \text{ on } \mathbb{R}^n \times \{t = 0\} \\ u_t = g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases} \quad (3)$$

The general idea of spherical means is as follows:

1. First understand the average of u over a sphere radius r
2. Obtain the Euler-Poisson-Darboux equation and solve it
3. For odd n, convert the wave equation to a one-dimensional wave equation
4. Apply d'Alembert's formula or its variants to obtain the solution.

Spherical Means

Definition 2

Let $x \in \mathbb{R}^n, t > 0, r > 0$. Define

$$U(x; r, t) := \fint_{\partial B(\mathbf{x}, r)} u(\mathbf{y}, t) dS(\mathbf{y}) \tag{4}$$

the average of $u(\mathbf{x}, t)$ over the sphere $\partial B(\mathbf{x}, r)$.

Similarly

$$\begin{cases} F(x; r, t) := \fint_{\partial B(\mathbf{x}, r)} f(\mathbf{y}, t) dS(\mathbf{y}) \\ G(x; r, t) := \fint_{\partial B(\mathbf{x}, r)} g(\mathbf{y}, t) dS(\mathbf{y}) \end{cases} \tag{5}$$

Spherical Means

(4) can also be written as

$$U(x;r,t) = \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y},t)dS(\mathbf{y}) = \fint_{\partial B(\mathbf{0},1)} u(\mathbf{x}+r\mathbf{z},t)dS(\mathbf{z})$$

Differentiating it w.r.t r, we obtain the following:

$$\begin{aligned} U_r(x;r,t) &= \fint_{\partial B(\mathbf{0},1)} Du(\mathbf{x}+r\mathbf{z},t).\mathbf{z}dS(\mathbf{z}) \\ &= \fint_{\partial B(\mathbf{x},r)} Du(\mathbf{y},t).\frac{\mathbf{y}-\mathbf{x}}{r}dS(\mathbf{y}) \\ &= \fint_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu}dS(\mathbf{y}) = \frac{r}{n}\fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y} \end{aligned}$$

Spherical Means

Since

$$U_r(x;r,t) = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y} \implies \lim_{r\to 0^+} U_r(x;r,t) = 0$$

Exercise 1: Simple

Prove the following:

$$U_{rr}(x;r,t) = \fint_{\partial B(\mathbf{x},r)} \Delta u dS + \left(\frac{1}{n} - 1\right) \fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y}$$

$$\lim_{r\to 0^+} U_{rr}(x;r,t) = \frac{1}{n}\Delta u(x,t)$$

Euler-Poisson-Darboux Equation

Euler-Poisson-Darboux Equation

Theorem 4 (Euler-Poisson-Darboux Equation)
Fix $x \in \mathbb{R}^n$ and let u satisfy (3). Then $U \in C^m(\overline{\mathbb{R}}_+ \times [0, \infty])$ and

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ U = F \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ U_t = G \text{ on } \mathbb{R}_+ \times \{t = 0\} \end{cases} \tag{6}$$

Here $U_{rr} + \frac{n-1}{r} U_r$ represents the radial part of the Laplacian Δ in polar coordinates.

Euler-Poisson-Darboux Equation

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Proof: We have already proved that

$$\lim_{r \to 0^+} U_r(x; r, t) = 0$$

Also, from the exercise, you can get that

$$\lim_{r \to 0^+} U_{rr}(x; r, t) = \frac{1}{n} \Delta u(x, t)$$

By computing through U_{rrr}, U_{rrrr}, etc., we can obtain that $U \in C^m(\overline{\mathbb{R}}_+ \times [0, \infty])$ Now,

$$U_r(x; r, t) = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y} = \frac{r}{n} \fint_{B(\mathbf{x},r)} u_{tt} d\mathbf{y}$$

Euler-Poisson-Darboux Equation

Proof (continued):

$$U_r(x;r,t) = \frac{r}{n}\frac{1}{\alpha(n)r^n}\int_{B(\mathbf{x},r)} u_{tt}d\mathbf{y} = \frac{1}{n\alpha(n)r^{n-1}}\int_{B(\mathbf{x},r)} u_{tt}d\mathbf{y}$$

$$\implies r^{n-1}U_r = \frac{1}{n\alpha(n)}\int_{B(\mathbf{x},r)} u_{tt}d\mathbf{y}$$

Now differentiating w.r.t r, we obtain

$$(r^{n-1}U_r)_r = \frac{1}{n\alpha(n)}\int_{\partial B(\mathbf{x},r)} u_{tt}dS$$

Euler-Poisson-Darboux Equation

Proof (continued):

$$(n-1)r^{n-2}U_r + r^{n-1}U_{rr} = r^{n-1}\left(\frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(\mathbf{x},r)} u_{tt}dS\right)$$

$$(n-1)r^{n-2}U_r + r^{n-1}U_{rr} = r^{n-1}\fint_{\partial B(\mathbf{x},r)} u_{tt} = r^{n-1}U_{tt}$$

Hence

$$U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0$$

Spherical Means

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Exercise 2: Some Identities

Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\phi \in C^{k+1}(\mathbb{R})$. Then prove that for $k = 1, 2, \cdots$,

$$\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi(r)}{dr}\right)$$

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi(r)}{dr^j}$$

where β_j^k are independent of ϕ and $\beta_0^k = \prod_{j=1}^{k}(2j-1)$

Kirchhoff’s Formula

Wave Equation Solution for odd n

Now, let us transform the Euler-Poisson-Darboux equation into the usual one-dimensional wave equation. Let us find a solution for $n = 3$. Let $u \in C^2(\mathbb{R}^3 \times [0, \infty))$

$$\overline{U} := rU, \overline{F} := rF, \overline{G} := rG,$$

Let us prove that $\overline{U}$ solves

$$\begin{cases} \overline{U}_{tt} - \overline{U}_{rr} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ \overline{U} = \overline{F} \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ \overline{U}_t = \overline{G} \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ \overline{U} = 0 \text{ on } \{r = 0\} \times (0, \infty) \end{cases} \tag{7}$$

Kirchhoff's Formula

Claim: $\overline{U}_{tt} - \overline{U}_{rr} = 0$

$$\begin{aligned}
\overline{U}_{tt} &= rU_{tt} \\
&= r[U_{rr} + \frac{2}{r}U_r] \\
&= rU_{rr} + 2U_r \\
&= (U + rU_r)_r \\
&= \overline{U}_{rr}
\end{aligned}$$

Therefore, the solution is given by for $0 \leq r \leq t$

$$\overline{U}(x; r, t) = \frac{1}{2}[\overline{F}(r+t) - \overline{F}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \overline{G}(y)dy$$

Kirchhoff's Formula

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Since

$$U(x;r,t) := \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y},t)dS(\mathbf{y})$$

we have $\lim_{r\to 0^+} U(x;r,t) = u(x,t)$

$$\begin{aligned}\implies u(x,t) &= \frac{\overline{U}(x;r,t)}{r} \\ &= \lim_{r\to 0^+}\left[\frac{1}{2r}[\overline{F}(r+t)-\overline{F}(t-r)] + \frac{1}{2r}\int_{-r+t}^{r+t}\overline{G}(y)dy\right] \\ &= \overline{F}'(t) + \overline{G}(t)\end{aligned}$$

Kirchhoff's Formula

$$u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x},t)} f dS \right) + t \fint_{\partial B(\mathbf{x},t)} g dS$$

Now,

$$\fint_{\partial B(\mathbf{x},t)} f(y) dS(y) = \fint_{\partial B(\mathbf{0},1)} f(\mathbf{x} + t\mathbf{z}) dS(\mathbf{z})$$

$$\implies \frac{\partial}{\partial t}\left(\fint_{\partial B(\mathbf{x},t)} f dS \right) = \fint_{\partial B(\mathbf{0},1)} Df(\mathbf{x}+t\mathbf{z}).\mathbf{z} dS(\mathbf{z}) = \fint_{\partial B(\mathbf{x},t)} Df(\mathbf{y}).\left(\frac{\mathbf{y}-\mathbf{x}}{t}\right) dS(\mathbf{y})$$

Kirchhoff's Formula

$$\frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x},t)} f\,dS\right) = \fint_{\partial B(\mathbf{x},t)} f\,dS + t\frac{\partial}{\partial t}\left(\fint_{\partial B(\mathbf{x},t)} f\,dS\right)$$

$$\implies \frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x},t)} f\,dS\right) = \fint_{\partial B(\mathbf{x},t)} f\,dS + t\fint_{\partial B(\mathbf{x},t)} Df(\mathbf{y}).\left(\frac{\mathbf{y}-\mathbf{x}}{t}\right) dS(\mathbf{y})$$

$$\implies \frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x},t)} f(y)\,dS(y)\right) = \fint_{\partial B(\mathbf{x},t)} f\,dS + \fint_{\partial B(\mathbf{x},t)} Df(\mathbf{y}).\left(\mathbf{y}-\mathbf{x}\right) dS(\mathbf{y})$$

Kirchhoff's Formula

$$u(x,t) = \fint_{\partial B(\mathbf{x},t)} tg(\mathbf{y}) + f(\mathbf{y}) + Df(\mathbf{y}).(\mathbf{y}-\mathbf{x})dS(\mathbf{y}), (\mathbf{x} \in \mathbb{R}^3, t > 0)$$

This is Kirchhoff's formula for the solution of the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = f \text{ on } \mathbb{R}^n \times \{t=0\} \\ u_t = g \text{ on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad (8)$$

Note for $n=2$, this transformation will not work to convert the Euler-Poisson-Darboux equation into a one-dimensional equation. However, we will try to use $\mathbf{x} = (x_1, x_2, 0) \in \mathbb{R}^3$.

Wave Equation Solution for odd n

Now, let us transform the Euler-Poisson-Darboux equation into the usual one-dimensional wave equation. Let us find a solution for $n = 2k+1$. Let $u \in C^{k+1}(\mathbb{R}^n \times [0,\infty))$ solves the initial-value problem. For $r > 0, t \geq 0$ write

$$\overline{U}(r,t) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right)$$

$$\overline{F}(r) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}F(x;r)\right)$$

$$\overline{G}(r) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}G(x;r)\right)$$

Then

$$\overline{U}(r,0) = \overline{F}(r), \overline{U}_t(r,0) = \overline{G}(r)$$

Wave Equation Solution for odd n

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Let us prove that $\overline{U}$ solves

$$\begin{cases} \overline{U}_{tt} - \overline{U}_{rr} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ \overline{U} = \overline{F} \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ \overline{U}_t = \overline{G} \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ \overline{U} = 0 \text{ on } \{r = 0\} \times (0, \infty) \end{cases} \tag{9}$$

$$\begin{aligned} \overline{U}_{rr} &= \left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U\right) \\ &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}U_r\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U_{rr} + 2kr^{2k-2}U_r\right) \end{aligned}$$

Wave Equation Solution for odd n

$$\overline{U}_{rr} = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[r^{2k-1}\left(U_{rr} + \frac{n-1}{r}U_r\right)\right] = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}U_{tt}) = \overline{U}_{tt}$$

From the Euler-Poisson-Darboux equation, it follows that

$$\overline{U} = \overline{F}, \overline{U}_t = \overline{G}$$

Also,

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U(r)\right) = \sum_{k=0}^{j-1}\beta_j^k r^{j+1}\frac{\partial^j U(r)}{\partial r^j} = 0$$

when $r = 0$

$$\implies \overline{U}(r,0) = \overline{F}(r), \overline{U}_t(r,0) = \overline{G}(r)$$

Wave Equation Solution for odd n

Therefore, the solution is given by

$$\overline{U}_{r,t} = \frac{1}{2}[\overline{F}(r+t) - \overline{F}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \overline{G}(y)dy$$

Recall, $u(x,t) = \lim_{r\to 0^+} U(x;r,t)$. Also,

$$\begin{aligned}\overline{U}(r,t) &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right)\\ &= \sum_{j=0}^{k-1} \beta_j^k r^{j+1}\frac{\partial^j U(r)}{\partial r^j}\end{aligned}$$

Wave Equation Solution for odd n

$$u(x,t) = \lim_{r\to 0^+} U(x;r,t) = \lim_{r\to 0^+} \frac{\overline{U}(r,t)}{\beta_0^k r}$$

$$= \lim_{r\to 0^+} \frac{1}{\beta_0^k} \left[\frac{\overline{F}(r+t) - \overline{F}(t-r)}{2r} + \frac{1}{2r} \int_{-r+t}^{r+t} \overline{G}(y)dy \right]$$

$$= \frac{1}{\beta_0^k} [\overline{F}'(t) + \overline{G}(t)]$$

Wave Equation Solution for odd n

$$u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(\mathbf{x},t)} f\,dS\right) + \left(\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(\mathbf{x},t)} g\,dS\right)\right] \tag{10}$$

here n is odd, $n = 2k+1$ and $\gamma_n = \prod_{j=1}^{k}(2k-1)$

Wave Equation Solution for odd n

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Theorem 5

Assume n is an odd integer, $n \geq 3, m = \frac{n+1}{2}, f \in C^{m+1}(\mathbb{R}^n), g \in C^m(\mathbb{R}^n)$ and define u by above. Then

1. $u \in C^2(\mathbb{R}^n \times [0, \infty))$
2. $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times [0, \infty)$
3. $\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0) \\ \mathbf{x}\in\mathbb{R}^n, t>0}} u(\mathbf{x}, t) = f(\mathbf{x}^0)$
4. $\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0) \\ \mathbf{x}\in\mathbb{R}^n, t>0}} u_t(\mathbf{x}, t) = g(\mathbf{x}^0)$

In (3) and (4) for each point $\mathbf{x}^0 \in \mathbb{R}^n$

Proof: Exercise.

Thanks

Doubts and Suggestions

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