MA612L-Partial Differential Equations

Lecture 18 : Kirchhoff's Formula

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MA612L
PARTIAL DIFFERENTIAL EQUATIONS

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TIRUPATI

Recap

Preliminaries

- $B(\mathbf{x}, r)$ = the closed ball with center $\mathbf{x}$ and radius $r > 0$
- $\alpha(n)$ = Volume of unit ball $B(\mathbf{0}, 1)$ in $\mathbb{R}^n = \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$
- $n\alpha(n)$ = Surface area of unit Sphere $B(\mathbf{0}, 1)$ in $\mathbb{R}^n$
- $\displaystyle ⨍_{B(\mathbf{x},r)} f\,d\mathbf{y} = \frac{1}{\alpha(n)r^n}\int_{B(\mathbf{x},r)} f\,d\mathbf{y}$ = average of f over the ball $B(\mathbf{x}, r)$
- $\displaystyle ⨍_{\partial B(\mathbf{x},r)} f\,dS = \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(\mathbf{x},r)} f\,dS$ = avg of f over the sphere $\partial B(\mathbf{x}, r)$

Recap

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r)$$

Now suppose $n \geq 2, m \geq 2$ and $u \in C^m(\mathbb{R}) \times [0, \infty)$ solves the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u = f \text{ on } \mathbb{R}^n \times \{t = 0\} \\ u_t = g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases} \quad (1)$$

Let $x \in \mathbb{R}^n, t > 0, r > 0$. Define

$$U(x; r, t) := \fint_{\partial B(\mathbf{x}, r)} u(\mathbf{y}, t) dS(\mathbf{y}) \quad (2)$$

the average of $u(\mathbf{x}, t)$ over the sphere $\partial B(\mathbf{x}, r)$.

Recap

$$U_r(x;r,t) = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y} \implies \lim_{r\to 0^+} U_r(x;r,t) = 0$$

$$U_{rr}(x;r,t) = \fint_{\partial B(\mathbf{x},r)} \Delta u dS + \left(\frac{1}{n} - 1\right) \fint_{B(\mathbf{x},r)} \Delta u d\mathbf{y}$$

$$\lim_{r\to 0^+} U_{rr}(x;r,t) = \frac{1}{n}\Delta u(x,t)$$

Fix $x \in \mathbb{R}^n$ and let u satisfy (1). Then $U \in C^m(\overline{\mathbb{R}}_+ \times [0,\infty])$ and

$$\begin{cases} U_{tt} - U_{rr} - \frac{n-1}{r} U_r = 0 \text{ in } \mathbb{R}_+ \times (0,\infty) \\ U = F \text{ on } \mathbb{R}_+ \times \{t=0\} \\ U_t = G \text{ on } \mathbb{R}_+ \times \{t=0\} \end{cases} \qquad (3)$$

Recap

Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\phi \in C^{k+1}(\mathbb{R})$. Then prove that for $k = 1, 2, \cdots$,

$$\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\frac{d\phi(r)}{dr}\right)$$

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi(r)}{dr^j}$$

where β_j^k are independent of ϕ and $\beta_0^k = \prod_{j=1}^{k}(2j-1)$

Kirchhoff’s Formula

Wave Equation Solution for odd n

Now, let us transform the Euler-Poisson-Darboux equation into the usual one-dimensional wave equation. Let us find a solution for $n = 3$. Let $u \in C^2(\mathbb{R}^3 \times [0, \infty))$. Define

$$\overline{U}(x; r, t) := rU, \overline{F}(r) := rF, \overline{G}(r) := rG,$$

Let us prove that $\overline{U}$ solves

$$\begin{cases} \overline{U}_{tt} - \overline{U}_{rr} = 0 \text{ in } \mathbb{R}_+ \times (0, \infty) \\ \overline{U} = \overline{F} \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ \overline{U}_t = \overline{G} \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ \overline{U} = 0 \text{ on } \{r = 0\} \times (0, \infty) \end{cases} \tag{4}$$

Kirchhoff's Formula

Claim: $\overline{U}_{tt} - \overline{U}_{rr} = 0$

$$\begin{aligned}
\overline{U}_{tt} &= rU_{tt} \\
&= r[U_{rr} + \frac{2}{r}U_r] \\
&= rU_{rr} + 2U_r \\
&= (U + rU_r)_r \\
&= \overline{U}_{rr}
\end{aligned}$$

Therefore, the solution is given by for $0 \leq r \leq t$

$$\overline{U}(x; r, t) = \frac{1}{2}[\overline{F}(r+t) - \overline{F}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \overline{G}(y)dy$$

Kirchhoff's Formula

Since

$$U(x;r,t) := \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y},t)dS(\mathbf{y})$$

we have $\lim_{r\to 0^+} U(x;r,t) = u(x,t)$

$$\implies u(x,t) = \frac{\overline{U}(x;r,t)}{r}$$

$$= \lim_{r\to 0^+} \left[\frac{1}{2r}[\overline{F}(r+t) - \overline{F}(t-r)] + \frac{1}{2r} \int_{-r+t}^{r+t} \overline{G}(y)dy \right]$$

$$= \overline{F}'(t) + \overline{G}(t)$$

Kirchhoff's Formula

$$u(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x},t)} f dS \right) + t \fint_{\partial B(\mathbf{x},t)} g dS$$

Now,

$$\fint_{\partial B(\mathbf{x},t)} f(y) dS(y) = \fint_{\partial B(\mathbf{0},1)} f(\mathbf{x} + t\mathbf{z}) dS(\mathbf{z})$$

$$\implies \frac{\partial}{\partial t}\left(\fint_{\partial B(\mathbf{x},t)} f dS \right) = \fint_{\partial B(\mathbf{0},1)} Df(\mathbf{x}+t\mathbf{z}).\mathbf{z} dS(\mathbf{z}) = \fint_{\partial B(\mathbf{x},t)} Df(\mathbf{y}).\left(\frac{\mathbf{y}-\mathbf{x}}{t}\right) dS(\mathbf{y})$$

Kirchhoff's Formula

$$\frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x},t)} f dS\right) = \fint_{\partial B(\mathbf{x},t)} f dS + t\frac{\partial}{\partial t}\left(\fint_{\partial B(\mathbf{x},t)} f dS\right)$$

$$\implies \frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x},t)} f dS\right) = \fint_{\partial B(\mathbf{x},t)} f dS + t \fint_{\partial B(\mathbf{x},t)} Df(\mathbf{y}).\left(\frac{\mathbf{y}-\mathbf{x}}{t}\right) dS(\mathbf{y})$$

$$\implies \frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x},t)} f(y) dS(y)\right) = \fint_{\partial B(\mathbf{x},t)} f dS + \fint_{\partial B(\mathbf{x},t)} Df(\mathbf{y}).\left(\mathbf{y}-\mathbf{x}\right) dS(\mathbf{y})$$

Kirchhoff's Formula

$$u(x,t) = \fint_{\partial B(\mathbf{x},t)} tg(\mathbf{y}) + f(\mathbf{y}) + Df(\mathbf{y}).(\mathbf{y}-\mathbf{x})dS(\mathbf{y}), (\mathbf{x} \in \mathbb{R}^3, t > 0)$$

This is Kirchhoff's formula for the solution of the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = f \text{ on } \mathbb{R}^n \times \{t=0\} \\ u_t = g \text{ on } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{5}$$

Note for $n = 2$, this transformation will not work to convert the Euler-Poisson-Darboux equation into a one-dimensional equation. However, we will try to use $\mathbf{x} = (x_1, x_2, 0) \in \mathbb{R}^3$.

Wave Equation Solution for odd n

Now, let us transform the Euler-Poisson-Darboux equation into the usual one-dimensional wave equation. Let us find a solution for $n = 2k+1$. Let $u \in C^{k+1}(\mathbb{R}^n \times [0,\infty))$ solves the initial-value problem. For $r > 0, t \geq 0$ write

$$\overline{U}(r,t) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right)$$

$$\overline{F}(r) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}F(x;r)\right)$$

$$\overline{G}(r) := \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}G(x;r)\right)$$

Then

$$\overline{U}(r,0) = \overline{F}(r), \overline{U}_t(r,0) = \overline{G}(r)$$

Wave Equation Solution for odd n

Let us prove that $\overline{U}$ solves

$$\begin{cases} \overline{U}_{tt} - \overline{U}_{rr} = 0 \text{ in } \mathbb{R}_+ \times (0,\infty) \\ \overline{U} = \overline{F} \text{ on } \mathbb{R}_+ \times \{t=0\} \\ \overline{U}_t = \overline{G} \text{ on } \mathbb{R}_+ \times \{t=0\} \\ \overline{U} = 0 \text{ on } \{r=0\} \times (0,\infty) \end{cases} \tag{6}$$

$$\begin{aligned} \overline{U}_{rr} &= \left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U\right) \\ &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}U_r\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U_{rr} + 2kr^{2k-2}U_r\right) \end{aligned}$$

Wave Equation Solution for odd n

$$\overline{U}_{rr} = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[r^{2k-1}\left(U_{rr} + \frac{n-1}{r}U_r\right)\right] = \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U_{tt}\right) = \overline{U}_{tt}$$

From the Euler-Poisson-Darboux equation, it follows that

$$\overline{U} = \overline{F}, \overline{U}_t = \overline{G}$$

Also,

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U(r)\right) = \sum_{k=0}^{j-1}\beta_j^k r^{j+1}\frac{\partial^j U(r)}{\partial r^j} = 0$$

when $r = 0$

$$\implies \overline{U}(r,0) = \overline{F}(r), \overline{U}_t(r,0) = \overline{G}(r)$$

Wave Equation Solution for odd n

Therefore, the solution is given by

$$\overline{U}(r,t) = \frac{1}{2}[\overline{F}(r+t) - \overline{F}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \overline{G}(y)dy$$

Recall, $u(x,t) = \lim_{r\to 0^+} U(x;r,t)$. Also,

$$\begin{aligned}\overline{U}(r,t) &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}U(x;r,t)\right)\\ &= \sum_{j=0}^{k-1} \beta_j^k r^{j+1}\frac{\partial^j U(r)}{\partial r^j}\end{aligned}$$

Wave Equation Solution for odd n

$$u(x,t) = \lim_{r\to 0^+} U(x;r,t) = \lim_{r\to 0^+} \frac{\overline{U}(r,t)}{\beta_0^k r}$$

$$= \lim_{r\to 0^+} \frac{1}{\beta_0^k}\left[\frac{\overline{F}(r+t) - \overline{F}(t-r)}{2r} + \frac{1}{2r}\int_{-r+t}^{r+t} \overline{G}(y)dy\right]$$

$$= \frac{1}{\beta_0^k}[\overline{F}'(t) + \overline{G}(t)]$$

Wave Equation Solution for odd n

$$u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(\mathbf{x},t)} f\,dS\right) + \left(\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(\mathbf{x},t)} g\,dS\right)\right] \tag{7}$$

here n is odd, $n = 2k+1$ and $\gamma_n = \prod_{j=1}^{k}(2k-1)$

Wave Equation Solution for odd n

Theorem 1

Assume n is an odd integer, $n \geq 3, m = \frac{n+1}{2}, f \in C^{m+1}(\mathbb{R}^n), g \in C^m(\mathbb{R}^n)$ and define u by above. Then

1. $u \in C^2(\mathbb{R}^n \times [0, \infty))$
2. $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times [0, \infty)$
3. $\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0)\\ \mathbf{x}\in\mathbb{R}^n, t>0}} u(\mathbf{x}, t) = f(\mathbf{x}^0)$
4. $\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0)\\ \mathbf{x}\in\mathbb{R}^n, t>0}} u_t(\mathbf{x}, t) = g(\mathbf{x}^0)$

In (3) and (4) for each point $\mathbf{x}^0 \in \mathbb{R}^n$

Proof: Exercise.

Poisson's Formula

Homogeneous Problem

Homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u = f \text{ on } \mathbb{R}^n \times \{t = 0\} \\ u_t = g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{8}$$

1. $n = 1$: d'Alembert's Formula from Transport Equation basics
2. $n = 3$: Spherical Means $\implies$ Euler-Poisson-Darboux Equation $\implies$ Kirchhoff's formula
3. $n = 2k + 1$: Extend this idea using the identity provided in the exercise
4. $n = 2$ and $n = 2k$: Poisson's formula

Poisson's Formula

Assume that $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solves (8) for $n = 2$. Let

$$\mathbf{x} = (x_1, x_2) \quad \text{and} \quad \overline{\mathbf{x}} = (x_1, x_2, 0)$$

$$\overline{u}(\overline{\mathbf{x}}, t) := u(\mathbf{x}, t), \quad \overline{f}(\overline{\mathbf{x}}, t) := f(\mathbf{x}, t), \quad \overline{g}(\overline{\mathbf{x}}, t) := g(\mathbf{x}, t)$$

Then

$$\begin{cases} \overline{u}_{tt} - \Delta \overline{u} = 0 \ \text{ in } \ \mathbb{R}^3 \times (0,\infty) \\ \overline{u} = \overline{f} \ \text{ on } \ \mathbb{R}^3 \times \{t = 0\} \\ \overline{u}_t = \overline{g} \ \text{ on } \ \mathbb{R}^3 \times \{t = 0\} \end{cases} \tag{9}$$

$$u(x,t) = \overline{u}(\overline{\mathbf{x}}, t) = \frac{\partial}{\partial t}\left(t \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{f} d\overline{S} \right) + t \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{g} dS$$

Poisson's Formula

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Here $\partial\overline{B}(\overline{\mathbf{x}},t)$ and $\overline{B}(\overline{\mathbf{x}},t)$ denote respectively the ball and sphere in $\mathbb{R}^3$ with center $\mathbf{x}$, radius $t>0$. $d\overline{\mathbf{S}}$ denotes two-dimensional surface measure on $\overline{B}(\overline{\mathbf{x}},t)$. Now,

$$t \fint_{\partial\overline{B}(\overline{\mathbf{x}},t)} \overline{f}d\overline{S} = \frac{1}{4\pi t}\int_{\partial\overline{B}(\overline{\mathbf{x}},t)} \overline{f}d\overline{S} = \frac{2}{4\pi t}\int_{B(\mathbf{x},t)} f(\mathbf{y})\sqrt{1+|D\gamma(\mathbf{y})|^2}d\mathbf{y}$$

Here, $\gamma(\mathbf{y}) = \sqrt{t^2 - |\mathbf{y}-\mathbf{x}|^2}$ for $\mathbf{y}\in B(\mathbf{x},t)$. The factor 2 appears since $\partial\overline{B}(\overline{\mathbf{x}},t)$ has two hemispheres. Also,

$$D\gamma(y) = -\frac{|\mathbf{y}-\mathbf{x}|}{\gamma(\mathbf{y})} \implies \sqrt{1+D\gamma(\mathbf{y})^2} = \frac{t}{\gamma(\mathbf{y})}$$

Poisson's Formula

Therefore,

$$t \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{f} d\overline{S} = \frac{1}{2\pi} \int_{B(\mathbf{x},t)} \frac{f(\mathbf{y})}{\gamma(\mathbf{y})} d\mathbf{y} = \frac{t^2}{2} \fint_{B(\mathbf{x},t)} \frac{f(\mathbf{y})}{\gamma(\mathbf{y})} d\mathbf{y}$$

Similarly,

$$t \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{g} d\overline{S} = \frac{t^2}{2} \fint_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{\gamma(\mathbf{y})} d\mathbf{y}$$

If we take $\mathbf{y} = \mathbf{x} + t\mathbf{z}$, then $\gamma(\mathbf{y}) = t\sqrt{1 - |\mathbf{z}|^2}$. Hence,

$$t \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{f} d\overline{S} = \frac{t^2}{2} \fint_{B(\mathbf{x},t)} \frac{f(\mathbf{y})}{\gamma(\mathbf{y})} d\mathbf{y} = \frac{t}{2} \fint_{B(\mathbf{0},1)} \frac{f(\mathbf{x} + t\mathbf{z})}{\sqrt{1 - |\mathbf{z}|^2}} d\mathbf{z}$$

Poisson's Formula

$$\frac{\partial}{\partial t}\left(t \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{f}\, d\overline{S}\right) = \frac{\partial}{\partial t}\left(\frac{t}{2}\fint_{B(\mathbf{0},1)} \frac{f(\mathbf{x}+t\mathbf{z})}{\sqrt{1-|\mathbf{z}|^2}}d\mathbf{z}\right)$$

$$= \frac{1}{2}\fint_{B(\mathbf{0},1)} \frac{f(\mathbf{x}+t\mathbf{z})}{\sqrt{1-|\mathbf{z}|^2}}d\mathbf{z} + \frac{t}{2}\fint_{B(\mathbf{0},1)} \frac{Df(\mathbf{x}+t\mathbf{z}).\mathbf{z}}{\sqrt{1-|\mathbf{z}|^2}}d\mathbf{z}$$

$$= \frac{t}{2}\fint_{B(\mathbf{x},t)} \frac{f(\mathbf{y})}{\gamma(\mathbf{y})}d\mathbf{y} + \frac{t}{2}\fint_{B(\mathbf{x},t)} \frac{Df(\mathbf{y}).(\mathbf{y}-\mathbf{x})}{\gamma(\mathbf{y})}d\mathbf{y}$$

Poisson's Formula

$$u(\mathbf{x},t) = \frac{t}{2} \fint_{B(\mathbf{x},t)} \frac{f(\mathbf{y}) + t2g(\mathbf{y}) + Df(\mathbf{y}).(\mathbf{y}-\mathbf{x})}{\gamma(\mathbf{y})} d\mathbf{y} \qquad (10)$$

This is Poisson formula for the solution of the initial value problems (9) in two dimensions. This technique of obtaining the solution of problem in three dimensions and dropping to two dimension is called **method of descent.** Note that for odd n, the data f and g at a given point $x \in \mathbb{R}^n$ affect the solution u only on the boundary $\{(\mathbf{y},t) | t > 0, |\mathbf{x}-\mathbf{y}| = t\}$ of the cone $C = \{(\mathbf{y},t) | t > 0, |\mathbf{x}-\mathbf{y}| < t\}$. However, for even n, the data f and g within all of C.

Huygen's Principle

For a disturbance originating at x propagates along a sharp wavefront in odd dimensions, but in even dimensions continues to have effects even after the leading edge of the wavefront passes. This is called Huygen's principle. Note that for even n, you need the information of f and g in the entire ball, whereas for odd n, you require the information only on the sphere.

Wave Equation Solution for even n

Suppose $u \in C^m$ is a solution of (8) and $m = \frac{n+2}{2}$. We use the same trick of method of descent. Let

$$\mathbf{x} = (x_1, x_2, \cdots, x_{n+1}) \quad \text{and} \quad \overline{\mathbf{x}} = (x_1, x_2, \cdots, x_n, 0)$$

$$\overline{u}(\overline{\mathbf{x}}, t) := u(\mathbf{x}, t), \quad \overline{f}(\overline{\mathbf{x}}, t) := f(\mathbf{x}, t), \quad \overline{g}(\overline{\mathbf{x}}, t) := g(\mathbf{x}, t)$$

Then we obtain

$$u(x,t) = \frac{1}{\gamma_{n+1}} \left[\left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n-1} \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{f} d\overline{S} \right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n-1} \fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{g} d\overline{S} \right) \right] \tag{11}$$

Wave Equation Solution for even n

Doing the same way as we did in Poisson's formula, we obtain

$$u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(\mathbf{x},t)} \frac{f(\mathbf{y})}{\gamma(\mathbf{y})}\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(\mathbf{x},t)} \frac{g(\mathbf{y})}{\gamma(\mathbf{y})}\right)\right] \tag{12}$$

Here $n = 2k$ is even and $\gamma_n = \prod_{j=1}^{k}(2j)$

Wave Equation Solution for even n

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Theorem 2

Assume n is an even integer, $n \geq 2, m = \frac{n+2}{2}, f \in C^{m+1}(\mathbb{R}^n), g \in C^m(\mathbb{R}^n)$ and define u by above. Then

1. $u \in C^2(\mathbb{R}^n \times [0, \infty))$
2. $u_{tt} - \Delta u = 0$ in $\mathbb{R}^n \times [0, \infty)$
3. $\lim\limits_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0) \\ \mathbf{x}\in\mathbb{R}^n, t>0}} u(\mathbf{x}, t) = f(\mathbf{x}^0)$
4. $\lim\limits_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0) \\ \mathbf{x}\in\mathbb{R}^n, t>0}} u_t(\mathbf{x}, t) = g(\mathbf{x}^0)$

In (3) and (4) for each point $\mathbf{x}^0 \in \mathbb{R}^n$

Proof: Exercise.

Inhomogeneous Problem

Inhomogeneous Problem

Now let us consider the following inhomogeneous problem

$$\begin{cases} u_{tt} - \Delta u = h \text{ in } \mathbb{R}^n \times (0, \infty) \\ u = 0 \text{ on } \mathbb{R}^n \times \{t = 0\} \\ u_t = 0 \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{13}$$

How do we solve this?

Transport Equation in $\mathbb{R}^n \times (0, \infty)$

Consider the following inhomogeneous problem

$$\begin{cases} u_t + b.Du = f, \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}), x \in \mathbb{R}^n \end{cases} \tag{14}$$

Using the same $z(s)$, we obtain

$$u(\mathbf{x}, t) = g(\mathbf{x} - \mathbf{b}t) + \int_0^t f(\mathbf{x} + (s - t)\mathbf{b}, s)ds, x \in \mathbb{R}^n, t \geq 0$$

solves the IVP (14).

d'Alembert's Formula $n = 1$

Consider the following one dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u = f \text{ on } \mathbb{R} \times (t = 0) \\ u_t = g \text{ on } \mathbb{R} \times (t = 0) \end{cases} \tag{15}$$

$$u(x,t) = f(x+ct) + \int_0^t a(x - 2cs + ct))ds, x \in \mathbb{R}, t \geq 0$$

Duhamel's principle

Let us see more details of Duhamel's principle obtained while solving Heat equation from Fundamental solution. For the moment, let us assume that we define a new function. Let $u = u(\mathbf{x}, t; s)$ be the solution

$$\begin{cases} u_{tt}(, ; s) - \Delta u(.; s) = 0 \text{ in } \mathbb{R}^n \times (s, \infty) \\ u(, ; s) = 0 \text{ on } \mathbb{R}^n \times \{t = s\} \\ u_t(.; s) = g(., s) \text{ on } \mathbb{R}^n \times \{t = s\} \end{cases} \tag{16}$$

Now, set

$$u(\mathbf{x}, t) := \int_0^t u(\mathbf{x}, t; s) ds \quad (\mathbf{x} \in \mathbb{R}^n, t \geq 0) \tag{17}$$

Then Duhamel's principle guarantees that $u(x, t)$ is a solution of (13)

Solution of inhomogeneous wave equation

Theorem 3

Assume $n \geq 2, m = \lfloor \frac{n}{2} \rfloor, h \in C^{m+1}(\mathbb{R}^n \times [0,\infty))$ and define u by (17). Then

1. $u \in C^2(\mathbb{R}^n \times [0,\infty))$
2. $u_{tt} - \Delta u = h$ in $\mathbb{R}^n \times [0,\infty)$
3. $\displaystyle\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0)\\ \mathbf{x}\in\mathbb{R}^n, t>0}} u(\mathbf{x},t) = 0$
4. $\displaystyle\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0)\\ \mathbf{x}\in\mathbb{R}^n, t>0}} u_t(\mathbf{x},t) = 0$

In (3) and (4) for each point $\mathbf{x}^0 \in \mathbb{R}^n$

Solution of inhomogeneous wave equation

Proof: If n is odd, then $m+1=\frac{n+1}{2}$ and as per the theorem from Kirchhoff's formula lecture, we have $u(.,.;s)\in C^2(\mathbb{R}^n\times[0,\infty))$ for each $s\geq 0$. Therefore, $u\in C^2(\mathbb{R}^n\times[0,\infty))$. If n is even, $m+1=\frac{n+2}{2}$.Hence $u\in C^2(\mathbb{R}^n\times[0,\infty))$ by theorem 2. Now,

$$u_t(\mathbf{x},t)=u(\mathbf{x},t;t)+\int_0^t u_t(\mathbf{x},t;s)ds=\int_0^t u_t(\mathbf{x},t;s)ds$$

$$u_{tt}(\mathbf{x},t)=u_t(\mathbf{x},t;t)+\int_0^t u_{tt}(\mathbf{x},t;s)ds=h(\mathbf{x},t)+\int_0^t u_{tt}(\mathbf{x},t;s)ds$$

Solution of inhomogeneous wave equation

Proof (continued): Also,

$$\Delta u(\mathbf{x},t) = \int_0^t \Delta u(\mathbf{x},t;s)ds = \int_0^t u_{tt}(\mathbf{x},t;s)ds = u_{tt} - h(\mathbf{x},t)$$

$$\implies u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) = h(\mathbf{x},t), \quad (x \in \mathbb{R}^n, t > 0)$$

Also, $u(\mathbf{x},0) = u_t(\mathbf{x},0) = 0$. Hence the theorem.
The solution of the inhomogeneous problem is the sum of d'Alembert's formula or Kirchhoff's formula or Poisson's formula and (17).

Solution of inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = h \text{ in } \mathbb{R} \times (0, \infty) \\ u = 0 \text{ on } \mathbb{R} \times \{t = 0\} \\ u_t = 0 \text{ on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{18}$$

Then by d'Alembert's formula

$$u(x,t;s) = \frac{1}{2} \int_{x-t+s}^{x+t+s} h(y,s)dy, \quad u(x,t) = \frac{1}{2} \int_{0}^{t} \int_{x-t+s}^{x+t+s} h(y,s)dyds$$

$$u(x,t) = \frac{1}{2} \int_{0}^{t} \int_{x-s}^{x+s} h(y,t-s)dyds$$

Solution of inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = h \ \text{ in } \ \mathbb{R}^3 \times (0, \infty) \\ u = 0 \ \text{ on } \ \mathbb{R}^3 \times \{t = 0\} \\ u_t = 0 \ \text{ on } \ \mathbb{R}^3 \times \{t = 0\} \end{cases} \tag{19}$$

By Kirchhoff's formula

$$u(\mathbf{x}, t; s) = (t - s) \fint_{\partial B(\mathbf{x}, t-s)} h(\mathbf{y}, s) dS$$

$$u(x, t) = \int_0^t (t - s) \left(\fint_{\partial B(\mathbf{x}, t-s)} h(\mathbf{y}, s) dS \right) ds$$

Solution of inhomogeneous wave equation

$$\implies u(x,t) = \frac{1}{4\pi}\int_0^t \int_{\partial B(\mathbf{x},t-s)} \frac{h(\mathbf{y},s)}{t-s}\,dS\,ds$$

$$\implies u(x,t) = \frac{1}{4\pi}\int_0^t \int_{\partial B(\mathbf{x},r)} \frac{h(\mathbf{y},t-r)}{r}\,dS\,dr$$

$$\implies u(x,t) = \frac{1}{4\pi}\int_{\partial B(\mathbf{x},t)} \frac{h(\mathbf{y},t-|\mathbf{y}-\mathbf{x}|)}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}$$

Exercises

Exercise 1: Compact Support

Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^3 \times (0, \infty) \\ u = f \text{ on } \mathbb{R}^3 \times \{t = 0\} \\ u_t = g \text{ on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

where g, h are smooth and have compact support. Show there exists constants C such that

$$|u(x,t)| \leq \frac{C}{t} \quad (x \in \mathbb{R}^3, t > 0)$$

Energy Methods

Solution of inhomogeneous wave equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with a smooth boundary $\partial\Omega$ and set $\Omega_T = \Omega \times (0, T], \Gamma_T = \overline{\Omega}_T - \Omega_T$ where $T > 0$. Let us solve the following IVP/BVP

$$\begin{cases} u_{tt} - \Delta u = h \ \text{ in } \ \Omega_T \\ u = f \ \text{ on } \ \Gamma_T \\ u_t = g \ \text{ on } \ \Omega \times \{t = 0\} \end{cases} \tag{20}$$

Theorem 4 (Uniqueness for Wave Equation)
There exists at most one function $u \in C^2(\overline{\Omega}_T)$ solving (20).

Proof: Suppose u_1 and u_2 are two solution such that $u_1, u_2 \in C^2(\overline{\Omega}_T)$. Define $w := u_1 - u_2$.

Solution of inhomogeneous wave equation

Proof (continued): Then

$$\begin{cases} w_{tt} - \Delta w = h \text{ in } \Omega_T \\ w = 0 \text{ on } \Gamma_T \\ w_t = 0 \text{ on } \Omega \times \{t = 0\} \end{cases} \tag{21}$$

Now, define the energy as

$$E(t) := \frac{1}{2} \int_{\Omega} w_t^2(\mathbf{x}, t) + |Dw(\mathbf{x}, t)|^2 d\mathbf{x}, \quad (0 \leq t \leq T)$$

$$\frac{dE}{dt} = \int_{\Omega} w_t w_{tt} + Dw.Dw_t d\mathbf{x} = \int_{\Omega} w_t(w_{tt} - \Delta w) d\mathbf{x} = 0$$

Solution of inhomogeneous wave equation

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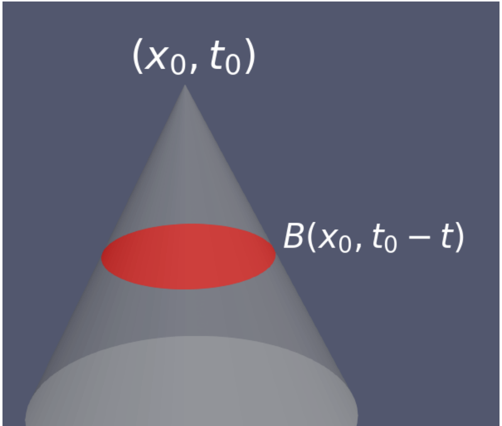
Proof (continued): There is no boundary term since $w = 0$ and hence $w_t = 0$ on $\partial\Omega \times [0, T]$. Thus for all $0 \le t \le T$, $E(t) = E(0) = 0$. Therefore, $w_t \equiv 0, Dw \equiv 0$. Since $w \equiv 0$ on $\Omega \times \{t = 0\}$, $u_1 \equiv u_2$ in Ω_T.



Solution of inhomogeneous wave equation

Suppose $u \in C^2$ solves

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

Let $x_0 \in \mathbb{R}^n, t_0 > 0$, and consider the cone

$$C = \{(x, t) : 0 \le t \le t_0, |\mathbf{x} - \mathbf{x}_0| \le t_0 - t\}$$

Theorem 5 (Finite Propagation Speed)
If $u \equiv u_t \equiv 0$ on $B(\mathbf{x}_0, t_0)$, then $u \equiv 0$ within the cone C.

Proof: Define the energy as

$$E(t) := \frac{1}{2} \int\limits_{\Omega} u_t^2(\mathbf{x}, t) + |Du(\mathbf{x}, t)|^2 d\mathbf{x}, \quad (0 \le t \le t_0)$$

Solution of inhomogeneous wave equation

Proof (continued):

$$\frac{dE}{dt} = \int_{B(\mathbf{x}_0,t_0-t)} u_t u_{tt} + Du.Du_t d\mathbf{x} - \int_{\partial B(\mathbf{x}_0,t_0-t)} u_t^2 + |Du|^2 dS$$

$$= \int_{B(\mathbf{x}_0,t_0-t)} u_t(u_{tt} - \Delta u) d\mathbf{x} + \int_{\partial B(\mathbf{x}_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t dS$$

$$- \int_{\partial B(\mathbf{x}_0,t_0-t)} u_t^2 + |Du|^2 dS$$

$$= \int_{\partial B(\mathbf{x}_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t dS - u_t^2 - |Du|^2 dS$$

Solution of inhomogeneous wave equation

Proof (continued): Now, by Cauchy-Schwarz and Cauchy inequalities.

$$\left|\frac{\partial u}{\partial \nu}u_t\right| \leq |u_t||Du| \leq \frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2$$

Now, $\frac{dE}{dt} \leq 0$ and so, $E(t) \leq E(0) = 0$ for all $0 \leq t \leq t_0$. Thus $u_t \equiv 0, Du \equiv 0$ and hence $u \equiv 0$ within the cone C.

Thanks

Doubts and Suggestions

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