MA612L-Partial Differential Equations

Lecture 19 : Wave Equation - Inhomogeneous Problem and Energy Methods

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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Inhomogeneous Problem

Inhomogeneous Problem

Now let us consider the following inhomogeneous problem

$$\begin{cases} u_{tt} - \Delta u = h \text{ in } \mathbb{R}^n \times (0, \infty) \\ u = 0 \text{ on } \mathbb{R}^n \times \{t = 0\} \\ u_t = 0 \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1}$$

How do we solve this?

Transport Equation in $\mathbb{R}^n \times (0, \infty)$

Consider the following inhomogeneous problem

$$\begin{cases} u_t + b.Du = f, \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}), x \in \mathbb{R}^n \end{cases} \tag{2}$$

Using the same $z(s)$, we obtain

$$u(\mathbf{x}, t) = g(\mathbf{x} - \mathbf{b}t) + \int_0^t f(\mathbf{x} + (s-t)\mathbf{b}, s)ds, x \in \mathbb{R}^n, t \geq 0$$

solves the IVP (2).

d'Alembert's Formula $n = 1$

Consider the following one-dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \\ u = f \text{ on } \mathbb{R} \times (t = 0) \\ u_t = h \text{ on } \mathbb{R} \times (t = 0) \end{cases} \tag{3}$$

$$u(x,t) = f(x + ct) + \int_0^t a(x - 2cs + ct))ds, x \in \mathbb{R}, t \geq 0$$

Duhamel's principle

Let us see more details of Duhamel's principle obtained while solving Heat equation from Fundamental solution. For the moment, let us assume that we define a new function. Let $u = u(\mathbf{x}, t; s)$ be the solution

$$\begin{cases} u_{tt}(, ; s) - \Delta u(.; s) = 0 \text{ in } \mathbb{R}^n \times (s, \infty) \\ u(, ; s) = 0 \text{ on } \mathbb{R}^n \times \{t = s\} \\ u_t(.; s) = h(., s) \text{ on } \mathbb{R}^n \times \{t = s\} \end{cases} \tag{4}$$

Now, set

$$u(\mathbf{x}, t) := \int_0^t u(\mathbf{x}, t; s) ds \quad (\mathbf{x} \in \mathbb{R}^n, t \geq 0) \tag{5}$$

Then Duhamel's principle guarantees that $u(x, t)$ is a solution of (1)

Solution of inhomogeneous wave equation

Theorem 1

Assume $n \geq 2, m = \lfloor \frac{n}{2} \rfloor, h \in C^{m+1}(\mathbb{R}^n \times [0,\infty))$ and define u by (5). Then

1. $u \in C^2(\mathbb{R}^n \times [0,\infty))$
2. $u_{tt} - \Delta u = h$ in $\mathbb{R}^n \times [0,\infty)$
3. $\displaystyle\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0) \\ \mathbf{x}\in\mathbb{R}^n, t>0}} u(\mathbf{x},t) = 0$
4. $\displaystyle\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0) \\ \mathbf{x}\in\mathbb{R}^n, t>0}} u_t(\mathbf{x},t) = 0$

In (3) and (4) for each point $\mathbf{x}^0 \in \mathbb{R}^n$

Solution of inhomogeneous wave equation

Proof: If n is odd, then $m+1=\frac{n+1}{2}$ and as per the theorem from Kirchhoff's formula lecture, we have $u(.,.;s) \in C^2(\mathbb{R}^n \times [0,\infty))$ for each $s \geq 0$. Therefore, $u \in C^2(\mathbb{R}^n \times [0,\infty))$. If n is even, $m+1=\frac{n+2}{2}$.Hence $u \in C^2(\mathbb{R}^n \times [0,\infty))$ by theorem 2. Now, by Leibniz rule,

$$u_t(\mathbf{x},t) = u(\mathbf{x},t;t) + \int_0^t u_t(\mathbf{x},t;s)ds = \int_0^t u_t(\mathbf{x},t;s)ds$$

$$u_{tt}(\mathbf{x},t) = u_t(\mathbf{x},t;t) + \int_0^t u_{tt}(\mathbf{x},t;s)ds = h(\mathbf{x},t) + \int_0^t u_{tt}(\mathbf{x},t;s)ds$$

Solution of inhomogeneous wave equation

Proof (continued): Also,

$$\Delta u(\mathbf{x},t) = \int_0^t \Delta u(\mathbf{x},t;s)ds = \int_0^t u_{tt}(\mathbf{x},t;s)ds = u_{tt} - h(\mathbf{x},t)$$

$$\implies u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) = h(\mathbf{x},t), \quad (x \in \mathbb{R}^n, t > 0)$$

Also, $u(\mathbf{x},0) = u_t(\mathbf{x},0) = 0$. Hence the theorem.
The solution of the inhomogeneous problem is the sum of d'Alembert's formula or Kirchhoff's formula or Poisson's formula and (5).

Solution of inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = h \text{ in } \mathbb{R} \times (0, \infty) \\ u = 0 \text{ on } \mathbb{R} \times \{t = 0\} \\ u_t = 0 \text{ on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{6}$$

Then by d'Alembert's formula

$$u(x, t; s) = \frac{1}{2} \int_{x-t+s}^{x+t+s} h(y, s) dy, \quad u(x, t) = \frac{1}{2} \int_0^t \int_{x-t+s}^{x+t+s} h(y, s) dy ds$$

$$u(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} h(y, t - s) dy ds$$

Solution of inhomogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = h \text{ in } \mathbb{R}^3 \times (0, \infty) \\ u = 0 \text{ on } \mathbb{R}^3 \times \{t = 0\} \\ u_t = 0 \text{ on } \mathbb{R}^3 \times \{t = 0\} \end{cases} \tag{7}$$

By Kirchhoff's formula

$$u(\mathbf{x}, t; s) = (t - s) \fint_{\partial B(\mathbf{x}, t-s)} h(\mathbf{y}, s) dS$$

$$u(x, t) = \int_0^t (t - s) \left(\fint_{\partial B(\mathbf{x}, t-s)} h(\mathbf{y}, s) dS \right) ds$$

Solution of inhomogeneous wave equation

$$\implies u(x,t) = \frac{1}{4\pi}\int_0^t \int_{\partial B(\mathbf{x},t-s)} \frac{h(\mathbf{y},s)}{t-s}\,dS\,ds$$

$$\implies u(x,t) = \frac{1}{4\pi}\int_0^t \int_{\partial B(\mathbf{x},r)} \frac{h(\mathbf{y},t-r)}{r}\,dS\,dr$$

$$\implies u(x,t) = \frac{1}{4\pi}\int_{\partial B(\mathbf{x},t)} \frac{h(\mathbf{y},t-|\mathbf{y}-\mathbf{x}|)}{|\mathbf{y}-\mathbf{x}|}\,d\mathbf{y}$$

Exercises

Exercise 1: Compact Support

Let u solve

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^3 \times (0, \infty) \\ u = f \text{ on } \mathbb{R}^3 \times \{t = 0\} \\ u_t = g \text{ on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

where g, h are smooth and have compact support. Show there exists constants C such that

$$|u(x,t)| \leq \frac{C}{t} \quad (x \in \mathbb{R}^3, t > 0)$$

Energy Methods

Solution of inhomogeneous wave equation

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with a smooth boundary $\partial\Omega$ and set $\Omega_T = \Omega \times (0, T], \Gamma_T = \overline{\Omega}_T - \Omega_T$ where $T > 0$. Let us consider the following IVP/BVP

$$\begin{cases} u_{tt} - \Delta u = h \text{ in } \Omega_T \\ u = f \text{ on } \Gamma_T \\ u_t = g \text{ on } \Omega \times \{t = 0\} \end{cases} \tag{8}$$

Theorem 2 (Uniqueness for Wave Equation)
There exists at most one function $u \in C^2(\overline{\Omega}_T)$ solving (8).

Proof: Suppose u_1 and u_2 are two solution such that $u_1, u_2 \in C^2(\overline{\Omega}_T)$. Define $w := u_1 - u_2$.

Solution of inhomogeneous wave equation

Proof (continued): Then

$$\begin{cases} w_{tt} - \Delta w = h \text{ in } \Omega_T \\ w = 0 \text{ on } \Gamma_T \\ w_t = 0 \text{ on } \Omega \times \{t = 0\} \end{cases} \tag{9}$$

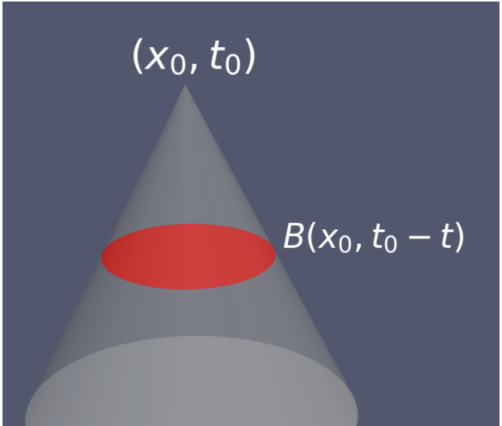
Now, define the energy as

$$E(t) := \frac{1}{2} \int_\Omega w_t^2(\mathbf{x}, t) + |Dw(\mathbf{x}, t)|^2 d\mathbf{x}, \quad (0 \leq t \leq T)$$

$$\frac{dE}{dt} = \int_\Omega w_t w_{tt} + Dw.Dw_t d\mathbf{x} = \int_\Omega w_t (w_{tt} - \Delta w) d\mathbf{x} = 0$$

Solution of inhomogeneous wave equation

Proof (continued): There is no boundary term since $w = 0$ and hence $w_t = 0$ on $\partial\Omega \times [0, T]$. Thus for all $0 \le t \le T$, $E(t) = E(0) = 0$. Therefore, $w_t \equiv 0, Dw \equiv 0$. Since $w \equiv 0$ on $\Omega \times \{t = 0\}$, $u_1 \equiv u_2$ in Ω_T.



Solution of inhomogeneous wave equation

Suppose $u \in C^2$ solves

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty)$$

Let $x_0 \in \mathbb{R}^n, t_0 > 0$, and consider the cone

$$C = \{(x,t) : 0 \le t \le t_0, |\mathbf{x} - \mathbf{x}_0| \le t_0 - t\}$$

Theorem 3 (Finite Propagation Speed)
If $u \equiv u_t \equiv 0$ on $B(\mathbf{x}_0, t_0)$, then $u \equiv 0$ within the cone C.

Proof: Define the energy as

$$E(t) := \frac{1}{2} \int_{\Omega} u_t^2(\mathbf{x},t) + |Du(\mathbf{x},t)|^2 d\mathbf{x}, \quad (0 \le t \le t_0)$$

Solution of inhomogeneous wave equation

Proof (continued):

$$\frac{dE}{dt} = \int_{B(\mathbf{x}_0, t_0 - t)} u_t u_{tt} + Du.Du_t d\mathbf{x} - \int_{\partial B(\mathbf{x}_0, t_0 - t)} u_t^2 + |Du|^2 dS$$

$$= \int_{B(\mathbf{x}_0, t_0 - t)} u_t(u_{tt} - \Delta u) d\mathbf{x} + \int_{\partial B(\mathbf{x}_0, t_0 - t)} \frac{\partial u}{\partial \nu} u_t dS$$

$$- \int_{\partial B(\mathbf{x}_0, t_0 - t)} u_t^2 + |Du|^2 dS$$

$$= \int_{\partial B(\mathbf{x}_0, t_0 - t)} \left(\frac{\partial u}{\partial \nu} u_t - u_t^2 - |Du|^2 \right) dS$$

Solution of inhomogeneous wave equation

Proof (continued): Now, by Cauchy-Schwarz and Cauchy inequalities.

$$\left|\frac{\partial u}{\partial \nu} u_t\right| \leq |u_t||Du| \leq \frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2$$

Now, $\frac{dE}{dt} \leq 0$ and so, $E(t) \leq E(0) = 0$ for all $0 \leq t \leq t_0$. Thus $u_t \equiv 0, Du \equiv 0$ and hence $u \equiv 0$ within the cone C.

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in