# MA612L-Partial Differential Equations

## Lecture 2 : Introduction: What is PDE?

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MA612L PARTIAL DIFFERENTIAL EQUATIONS

# Recap

# Examples

Let us revisit the ODE example again

**Examples**

$$f(x) = 4x^2 \implies f'(x) = 8x \overset{?}{\implies} f(x) = 4x^2$$

$$y = e^{5x} \implies \frac{dy}{dx} = 5e^{5x} \overset{?}{\implies} y = e^{5x}$$

$$y = \sin x + x^2 \implies \frac{dy}{dx} = \cos x + 2x \overset{?}{\implies} y = \sin x + x^2$$

# Examples

Let us revisit the ODE example again

**Examples**

$$f(x) = 4x^2 \implies f'(x) = 8x \implies f(x) = 4x^2 + C_1$$

$$y = e^{5x} \implies \frac{dy}{dx} = 5e^{5x} \implies y = e^{5x} + C_2$$

$$y = \sin x + x^2 \implies \frac{dy}{dx} = \cos x + 2x \implies y = \sin x + x^2 + C_3$$

What do these $C_1, C_2$ and $C_3$ represent? Note that, the derivative measures the slope of tangent lines of a given curve. However, when you know a tangent line, you end up with a family of curves or no curve.

# Examples

Let us revisit the PDE example again

### Examples

$$u = e^{x-y} \implies u_x + u_y = 0 \overset{?}{\implies} u = e^{x-y}$$

$$u = e^{x-y} \implies u_{xx} + u_{yy} = 2u \overset{?}{\implies} u = e^{x-y}$$

Will you get a unique answer?

# Examples

In fact, what we will see in our next class

**Examples**

$$u_x + u_y = 0 \implies u = f(x-y)^1$$

$$u_{xx} + u_{yy} = 2u \quad \text{Helmholtz equation* with } k^2 < 0$$

Will you get a unique answer? [1] Using method of characteristics. *Separation of variable can be used to find solution, however, we need more assumptions and BCs to solve this problem.

# From ODE Course

# ODE

**Definition 1 (ODE)**
Ordinary Differential Equation (ODE) is a differential equation that contains only one independent variable so that all the derivative occurring in it are ordinary derivatives.

**Definition 2 (ODE (Mathematical way))**
Let $\Omega \subset \mathbb{R}, m \in \mathbb{N}$ and $F : \Omega \times \mathbb{R}^m \to \mathbb{R}$ be a function. Then an ODE of order $m$ is defined by the equation

$$F\left(x, y(x), \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}, \cdots, \frac{d^m y(x)}{dx^m}\right) = 0 \tag{1}$$

# ODE

Let $x$ be an independent variable and $\mathbf{y} = (y_1, y_2, \cdots, y_p) \in \mathbb{R}^p$ be the dependent variable depending on $x$.

**Definition 3 (System of ODE)**

Let $\Omega \subset \mathbb{R}, m \in N$ and

$$F : \Omega \overbrace{\times \mathbb{R}^p \times \mathbb{R}^p \cdots \times \mathbb{R}^p}^{mtimes} \to \mathbb{R}^q$$

be a function. Then the system of ODE of order $m$ is defined by the equation

$$F\left(x, \mathbf{y}(x), \frac{d\mathbf{y}(x)}{dx}, \frac{d^2\mathbf{y}(x)}{dx^2}, \cdots, \frac{d^m\mathbf{y}(x)}{dx^m}\right) = \mathbf{0} \tag{2}$$

# Remarks on ODE

$$F\left(x, y(x), \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}, \cdots, \frac{d^ny(x)}{dx^n}\right) = 0 \tag{3}$$

- There is only one independent variable (here $x$)
- All derivatives are ordinary derivatives of the unknown function $y(x)$
- Order of ODE = The highest derivative in the equation
- Equation (4) is linear if $F$ is linear in $y, \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}, \cdots, \frac{d^ny(x)}{dx^n}$ with the coefficients depending on the independent variable $x$.

# Examples

## ODE EXample

$$\frac{dS(t)}{dt} = -k(S(t) - S_i) \quad \text{Survivability with AIDS}$$

$$\frac{dN(t)}{dt} = bN(t) - nN(t) \quad \text{Population Growth}$$

$$m\ddot{x} = mg + b(x) - \beta(\dot{x}) \quad \text{Simple Harmonic Equation}$$

$$\frac{dT}{dt} = k(T - T_m), t > 0, \quad \text{Newton's Law of Cooling}$$

# Calculus

Consider the ODE:

$$\frac{dy}{dx} = r(x)$$

**Theorem 4 (Fundamental Theorem of Calculus)**
If $f$ is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

If $F$ is any antiderivative of $f$ on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

# Existence

**Theorem 5 (Peano's Existence Theorem)**
Consider the ODE:

$$y' = f(x, y), y(x_0) = y_0 \tag{4}$$

Suppose $f(x, y)$ is continuous for all points $(x, y)$ in some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded in $R$, that is there exists a number $K$ such that

$$|f(x, y)| \leq K, \text{for all} (x, y) \in R \tag{5}$$

Then the initial value problem has at least one solution $y(x)$. This solution exists at least for all $x$ in the subinterval $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$, here $\alpha = \min\{a, b/K\}$.

# Picard's Theorem

**Theorem 6 (Uniqueness Theorem )**
In addition to the condition (5), if $f$ satisfy the Lipschitz condition with respect to $y \in R$, that is, there exists a number $M$ such that

$$|f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1| \quad \text{for all} (x, y_1), (x, y_2) \in R$$

Then the IVP (4) has at most one solution $y(x)$.

# Keywords

- Separable ODE, Exact ODE
- Bernoulli, Euler-Cauchy, Euler-Lagrange, Legendre, Bessel equations
- Superposition Principle, Wronskian
- Constant and Variable Coefficients
- Variation of Parameters, Green’s Function, BVP

# Exercise

1. Solve the following ODEs

$$y'' + y = 0, y(0) = y(2) = 0, 0 \leq x \leq 2$$
$$y'' + 4y = 8x, y(0) = A, y(L) = B, 0 \leq x \leq L$$

2. Find all eigenvalues and eigenfunctions for the following:

$$y'' + \lambda y = 0, y = y(x), 0 < x < 1, y(0) = y(1) = 0$$
$$y'' + \lambda y = 0, y = y(x), 0 < x < 3, y(0) = y'(3) = 0$$
$$y'' + \lambda y = 0, y = y(x), 0 < x < 1, y(0) = y(1) + y'(1) = 0$$
$$x^2 y'' + xy' - \lambda y = 0, y(1) = y(e) = 0, 0 < x < e$$

# PDE-Preliminaries

# PDE

**Definition 7 (PDE)**

Partial Differential Equation (PDE) is a differential equation involving an unknown function (possibly a vector-valued) of two or more variables and a finite number of partial derivatives.

# PDE-Notations

**Independent Variables**: Let us denote the independent variable by $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n, n \geq 2$

**Dependent Variables**: Let us denote the dependent variable or unknown function by $\mathbf{u} = (u_1, u_2, \cdots, u_p) \in \mathbb{R}^p, p \geq 1$

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{Z}_+^n$ and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

Then $D^\alpha u$ denotes

$$D^\alpha u = \frac{\partial^\alpha u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n}$$

# PDE

**Definition 8 (PDE-Formal Definition)**
Let $\Omega \subset \mathbb{R}^n, m \in \mathbb{N}$ and

$$F : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \mathbb{R}^{n^2 p} \times \cdots \times \mathbb{R}^{n^m p} \to \mathbb{R}^q$$

A system of partial differential equations of order $m$ is defined by the equation

$$F(\mathbf{x}, \mathbf{u}, D\mathbf{u}, D^2\mathbf{u}, \cdots, D^m\mathbf{u}) = \mathbf{0} \quad \text{or} \quad F(\mathbf{x}, (\partial^\alpha \mathbf{u})_{|\alpha|} \leq m) = \mathbf{0}$$

Here some $m^{th}$ order derivative of the function $\mathbf{u}$ appears in the system of equations.

# PDE

If $u = (u_1)$ is the only dependent variable and $F$ is real-valued, then the above definition can be written as

**Definition 9 (PDE-Formal Definition, $p = 1, q = 1$)**
Let $\Omega \subset \mathbb{R}^n, m \in \mathbb{N}$ and

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^m} \to \mathbb{R}$$

The partial differential equation of order $m$ is defined by the equation

$$F(\mathbf{x}, u, Du, D^2u, \cdots, D^m u) = 0$$

Here some $m^{th}$ order derivative of the function $u$ appears in the equation.

# PDE

### PDE-2D and 3D

$$F(x, y, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{yx}) = 0$$

$$F(x, y, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, u_{yx}, u_{xxx}, u_{yyy}, \cdots, u_{xxx\cdots x}) = 0$$

$$F(x, y, z, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xz}, u_{yx}, u_{yz}, u_{zy}, u_{zx}) = 0$$

$$F(x, y, z, u_x, u_y, u_z, u_{xx}, \cdots, u_{zzz\cdots z}) = 0$$

Here $u_x = \frac{\partial u}{\partial x}, u_y = \frac{\partial u}{\partial y}, u_z = \frac{\partial u}{\partial z}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$, etc.

# Four Important Linear PDEs

# Four Important Linear PDEs

- Transport Equation

$$u_t + au_x = 0$$

- Heat Equation

$$u_t = \alpha^2 u_{xx}$$

- Wave Equation

$$u_{tt} = c^2 u_{xx}$$

- Laplace Equation

$$u_{xx} + u_{yy} = 0$$

In all problems in this presentation, valid domain for time, space and function space are assumed.

# Transport Equation

Transport equation is the simplest PDE with constant coefficients. It is also called Convection or Advection equation. It has applications in different fields of science and engineering streams

- Chemistry: Purification and Crystallization
- Mechanical Engineer: Convection-Diffusion Equation
- Civil engineers : Pollutant Transport
- Electrical Engineers: Phonon and Dispersion
- Financial Mathematics: Dynamics of financial derivatives

For mathematicians, it is simply a first-order PDE.

# Chemistry: Purification and Crystallization

$$\frac{\partial C}{\partial t} + v\frac{\partial C}{\partial x} = 0$$

- $C(x,t)$: Concentration of solute
- $v$: Constant fluid velocity
- $x$: Spatial position
- $t$: Time

# Mechanical Engineering: Convection–Diffusion

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

- $T(x,t)$: Temperature
- $v$: Flow velocity
- $\alpha$: Thermal diffusivity (assume $\alpha = 0$)
- $x$: Position
- $t$: Time

# Civil Engineering: Pollutant Transport

$$\frac{\partial C}{\partial t} + v\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2}$$

- $C(x, t)$: Pollutant concentration
- $v$: Groundwater flow velocity
- $D$: Dispersion coefficient (assume $D = 0$)
- $x$: Distance along flow
- $t$: Time

# Electrical Engineering: Phonon Transport

$$\frac{\partial u}{\partial t} + v_g \frac{\partial u}{\partial x} = 0$$

- $u(x,t)$: Energy density of phonons
- $v_g$: Group velocity of phonons
- $x$: Position in material
- $t$: Time

# Financial Mathematics: Option Pricing Drift

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} - rV = 0$$

- $V(S,t)$: Price of a financial derivative (e.g., an option)
- $S$: Underlying asset price
- $r$: Risk-free interest rate
- $t$: Time

# Mathematics: First-order Linear PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

- $u(x, t)$: Transported quantity
- $a$: Constant wave or transport speed
- $x$: Spatial variable
- $t$: Time

# Heat Equation

# Heat Equation

Heat equation has different names in different engineering and science stream.

- Thermodynamics: Heat Equation
- Mechanical Engineer: Diffusion Equation
- Civil engineers (Terzaghi’s theory): Consolidation equation for drilling
- Electrical Engineers: Telegraph equation
- Financial Mathematics: Black-Scholes equation

For mathematicians, it is simply a second order PDE or parabolic PDE

# Heat Equation

Heat Equation with heat source, convection term are applied in the field of

- Cancer Treatment: Bioheat Equation

$$\rho C \frac{\partial T}{\partial t} = k \Delta T + \omega_b \rho_b C_b (T_a - T) + Q_r$$

- Density: $\rho$
- Specific heat of tissue/blood: $C$, $C_b$
- Temperature: $T$
- Thermal conductivity: $k$
- Blood Perfusion: $\omega_b$
- Blood Density: $\rho_b$
- Arterial Temperature: $T_a$
- Heat Source Term: $Q_r$

# Heat Equation

Heat Equation is also applied in

- Medical imaging: Image smoothing

$$\nabla^2 I = \frac{\phi''(S/\epsilon)}{\epsilon^2} \nabla S \otimes \nabla S + \frac{\phi'(S/\epsilon)}{\epsilon} \nabla^2 S \tag{6}$$

- Inverse Problem: In DCIS-Breast Cancer

$$c\frac{\partial \sigma}{\partial t} = \frac{\partial^2 \sigma}{\partial x^2} - \lambda(x)\sigma \tag{7}$$

# Diffusion Equation

Atmospheric Diffusion Equation: The dispersion of pollutant concentration from multiple point sources

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = k_x\frac{\partial^2 c}{\partial x^2} + k_y\frac{\partial^2 c}{\partial y^2} + k_z\frac{\partial^2 c}{\partial z^2} + s \tag{8}$$

In Chemical Engineering for concentration around the point $(x, y, z)$

$$\frac{\partial c}{\partial t} = k_x\frac{\partial^2 c}{\partial x^2} + k_y\frac{\partial^2 c}{\partial y^2} + k_z\frac{\partial^2 c}{\partial z^2} \tag{9}$$

# Mechanical Engineering: Diffusion Equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$

- $C(x,t)$: Concentration of diffusing species (e.g., gas or vapor)
- $D$: Diffusion coefficient
- $x$: Position
- $t$: Time

# Civil Engineering: Terzaghi's Consolidation Equation

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2}$$

- $u(z, t)$: Excess pore water pressure
- $c_v$: Coefficient of consolidation
- $z$: Depth
- $t$: Time

# Electrical Engineering: Telegraph Equation

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + RC\frac{\partial V}{\partial t}$$

- $V(x,t)$: Voltage along the transmission line
- $L$: Inductance per unit length (assume $L = 0$)
- $C$: Capacitance per unit length
- $R$: Resistance per unit length
- $x$: Position along the line
- $t$: Time

# Financial Mathematics: Black–Scholes Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- $V(S,t)$: Option price
- $S$: Underlying asset price
- $\sigma$: Volatility of the asset
- $r$: Risk-free interest rate
- $t$: Time

Use substitution, $x = \ln(S/K)$, $\tau = T - t$, and $V(S,t) = Ke^{-r\tau}u(x,\tau)$.

# Wave Equation

# Wave Equation

Again, wave equation has different names in different engineering and science stream.

- Physics: Vibration of String
- Mechanical Engineer:Vibration of Beams
- Civil engineers (Terzaghi’s theory): Seismic Waves Equation
- Electrical Engineers: Electromagnetic Waves
- Financial Mathematics: Analogous wave models

For mathematicians, it is simply a second order PDE or hyperbolic PDE

# Wave Equation

Lighthill's acoustic wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2\nabla^2\right)\rho = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \tag{10}$$

$$T_{ij} = \rho u_i u_j - \tau_{ij} + [(p - p_0) - c_0^2(\rho - \rho_0)]\delta_{ij}$$

# Physics: Vibrating String (Wave Equation)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- $u(x,t)$: Displacement of the string
- $c$: Wave speed in the medium
- $x$: Position along the string
- $t$: Time

# Mechanical Engineering: Vibrations of Beams or Rods

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

- $u(x,t)$: Displacement of the structure
- $a$: Wave speed determined by material properties
- $x$: Spatial coordinate
- $t$: Time

# Civil Engineering: Seismic and Ground Waves

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$

- $u(x,t)$: Displacement in the ground
- $v$: Propagation speed of seismic wave
- $x$: Depth or lateral position
- $t$: Time

# Electrical Engineering: Electromagnetic Waves

$$\frac{\partial^2 E}{\partial t^2} = c^2 \frac{\partial^2 E}{\partial x^2}$$

- $E(x, t)$: Electric field component of the EM wave
- $c$: Speed of light in the medium
- $x$: Position in space
- $t$: Time

# Financial Mathematics: Analogous Wave Models

$$\frac{\partial^2 V}{\partial t^2} = c^2 \frac{\partial^2 V}{\partial x^2}$$

- $V(x,t)$: Price function (in rare wave-based financial models)
- $c$: Abstract propagation rate of market signal
- $x$: Log-price or other spatially transformed variable
- $t$: Time

# Mathematics: Second-Order Hyperbolic PDE

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$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- $u(x, t)$: General wave profile (displacement, signal, etc.)
- $c$: Constant wave speed
- $x$: Space variable
- $t$: Time

# Laplace/Poisson Equation

# Physics: Electrostatics (Laplace's Equation)

$$\nabla^2 \phi = 0$$

- $\phi(x, y, z)$: Electric potential
- $\nabla^2$: Laplacian operator
- Used when charge density is zero in a region

# Physics: Electrostatics (Poisson's Equation)

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}$$

- $\phi$: Electric potential
- $\rho$: Charge density
- $\varepsilon_0$: Permittivity of free space

# Mechanical Engineering: Steady-State Heat

$$\nabla^2 T = 0 \quad \text{(Laplace)}, \qquad \nabla^2 T = -\frac{q}{k} \quad \text{(Poisson)}$$

- $T(x, y)$: Temperature
- $q$: Internal heat generation
- $k$: Thermal conductivity

# Civil Engineering: Groundwater Flow

$$\nabla^2 h = 0$$

- $h(x, y)$: Hydraulic head
- $\nabla^2$: Spatial Laplacian
- Describes steady, incompressible flow through porous media

# Electrical Engineering: Field Simulation

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

- $V(x, y, z)$: Electric potential
- $\rho$: Charge distribution
- $\varepsilon$: Permittivity of the medium

# Financial Mathematics: Steady-State Option Pricing

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0$$

- $V(S)$: Option value (steady-state)
- $S$: Underlying asset price
- $\sigma$: Volatility
- $r$: Interest rate
- This ODE is elliptic — akin to Laplace in transformed coordinates

# Mathematics: Elliptic PDEs

$$\nabla^2 u = 0 \quad \text{(Laplace)}, \qquad \nabla^2 u = f(x, y) \quad \text{(Poisson)}$$

- $u(x, y)$: Unknown scalar field
- $f(x, y)$: Source term (for Poisson)
- $\nabla^2$: Laplace operator in 2D or 3D