MA612L-Partial Differential Equations

Lecture 20 : Laplace Equation - Fundamental Solution

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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भारतीय प्रौद्योगिकी संस्थान तिरुपति
TIRUPATI

Fundamental Solution

Compact Support

Definition 1 (Compact Support)
Let $f : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$) be a function. The *support* of f is defined as

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

We say that f has *compact support* if $\operatorname{supp}(f)$ is a compact set in $\mathbb{R}^n$, i.e., it is closed and bounded.

A classical smooth function with compact support is

$$f(x) = \begin{cases} e^{-1/(1-x^2)}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$

Here, f is smooth (C^∞) and zero outside $[-1, 1]$, so it has compact support.

Compact Support

The function

$$g(x) = \begin{cases} 1, & |x| \leq 2, \\ 0, & |x| > 2 \end{cases}$$

also has compact support, with $\operatorname{supp}(g) = \overline{B_2(0)}$, the closed ball of radius 2.
The function

$$h(x) = e^{-x^2}, \quad x \in \mathbb{R},$$

is smooth but ***not* compactly supported,** since $h(x) \neq 0$ for all $x \in \mathbb{R}$

Fundamental Solution

The Laplace equation appears in both the Heat and Wave equations when we assume that u is independent of time.

$$\Delta u = 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^n \tag{1}$$

Prove that Laplace's equation is invariant under rotations.

We know that when

$$u(\mathbf{x}) = v(r)$$

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r)$$

Fundamental Solution

Now for Laplace equation, we have

$$\Delta u = 0$$

Hence

$$v''(r) + \frac{n-1}{r} v'(r) = 0$$

If $v' \neq 0$, then

$$\log(v')' = \frac{v''}{v'} = \frac{1-n}{r}$$

$$\implies v'(r) = \frac{a}{r^{n-1}}$$

where a is a constant.

Fundamental Solution

Now, when $r > 0$, we have

$$v(r) = \begin{cases} b \log r + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3) \end{cases} \tag{2}$$

where b and c are constants.

Definition 1

The function

$$\Phi(\mathbf{x}) := \begin{cases} -\frac{1}{2\pi} \log |\mathbf{x}| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)|\mathbf{x}|^{n-2}} & (n \geq 3) \end{cases} \tag{3}$$

defined for $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$ is the fundamental solution of Laplace equation.

Abuse of notations: $\Phi(x) = \Phi(|\mathbf{x}|)$.Why these particular constants??

Observations

$$|D\Phi(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|^{n-1}}, |D^2\Phi(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|^{n}}, (\mathbf{x} \neq 0)$$

for some constant C.
Φ is harmonic for $\mathbf{x} \neq 0$ from its construction. If we shift the origin to a new point $\mathbf{y}$, the PDE (1) is unchanged. Therefore, $\Phi(x-y)$ is also harmonic as a function of $\mathbf{x}$ as $\mathbf{x} \neq \mathbf{y}$.
Let $f : \mathbb{R}^n \to \mathbb{R}$ and note that the mapping $\mathbf{x} \to \Phi(\mathbf{x}-\mathbf{y})f(\mathbf{x}), \ \ (\mathbf{x} \neq y)$ is harmonic for each point $\mathbf{y} \in \mathbb{R}^n$.

Observations

We can also prove that

$$u_m(\mathbf{x}) = \sum_{i=1}^{m} \Phi(\mathbf{x} - \mathbf{y}_i) f(\mathbf{y}_i)$$

is also harmonic where $\mathbf{x} \notin \{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_m\}$

$$u_m(\mathbf{x}) = \begin{cases} \dfrac{-1}{2\pi} \displaystyle\sum_{i=1}^{m} \log(|\mathbf{x} - \mathbf{y}_i|) f(\mathbf{y}_i) & (n = 2) \\ \dfrac{1}{n(n-2)\alpha(n)} \displaystyle\sum_{i=1}^{m} \dfrac{f(\mathbf{y}_i)}{|\mathbf{x} - \mathbf{y}_i|^{n-2}} & (n \geq 3) \end{cases} \tag{4}$$

(4) solves (1). What will happen if $m \to \infty$?

Observations

By extending further, we obtain the convolution

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y}$$

$$u(\mathbf{x}) = \begin{cases} \dfrac{-1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log(|\mathbf{x}-\mathbf{y}|) f(\mathbf{y}) d\mathbf{y} & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)} \displaystyle\int_{\mathbb{R}^n} \dfrac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2}} d\mathbf{y} & (n \geq 3) \end{cases} \tag{5}$$

Do you expect that (5) solves (1)?

Observations

Do you expect that (5) solves (1)?? No. We can't compute

$$\Delta u(\mathbf{x}) = \int\limits_{\mathbb{R}^n} \Delta_x \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y} = 0$$

$$|D^2\Phi(\mathbf{x} - \mathbf{y})|$$

is not summable near the singularity at $\mathbf{y} = \mathbf{x}$. The differentiation under the integral sign above is incorrect.
Now, let us assume that $f \in C_c^2(\mathbb{R}^n)$; that is f is continuously differentialble (twice) with compact support.

Poisson Equation

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Theorem 2

Define u by

$$u(\mathbf{x}) = \begin{cases} \dfrac{-1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log(|\mathbf{x}-\mathbf{y}|) f(\mathbf{y}) d\mathbf{y} & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)} \displaystyle\int_{\mathbb{R}^n} \dfrac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2}} d\mathbf{y} & (n \geq 3) \end{cases} \tag{6}$$

Then

1. $u \in C^2(\mathbb{R}^n)$
2. $-\Delta u = f \quad \text{in} \quad \mathbb{R}^n$

Proof

We have

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^n} \Phi(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d\mathbf{y}$$

Now,

$$\frac{u(\mathbf{x}+he_i)-u(\mathbf{x})}{h} = \int_{\mathbb{R}^n} \Phi(\mathbf{y}) \left[\frac{f(\mathbf{x}+he_i-\mathbf{y})-f(\mathbf{x}-\mathbf{y})}{h}\right] d\mathbf{y}$$

As $h \to 0$, we have

$$\frac{f(\mathbf{x}+he_i-\mathbf{y})-f(\mathbf{x}-\mathbf{y})}{h} \to \frac{\partial f}{\partial x_i}(\mathbf{x}-\mathbf{y})$$

uniformly on $\mathbb{R}^n$

Proof

Therefore, we have for $i = 1, 2, \cdots, n$

$$\frac{\partial u}{\partial x_i}(\mathbf{x}) = \int\limits_{\mathbb{R}^n} \Phi(\mathbf{y}) \frac{\partial f}{\partial x_i}(\mathbf{x} - \mathbf{y}) d\mathbf{y}$$

Similarly,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(\mathbf{x}) = \int\limits_{\mathbb{R}^n} \Phi(\mathbf{y}) \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x} - \mathbf{y}) d\mathbf{y}$$

Since f is twice continuously differentiable, and Φ is also continuous, we have $u \in C^2(\mathbb{R}^n)$.

Recall from Lebesgue Measure Theory:

Definition 3 (Lebesgue Outer Measure)
Let $E \subset \mathbb{R}^n$. The *Lebesgue outer measure* of E is defined by

$$m^*(E) := \inf \left\{ \sum_{k=1}^{\infty} \mathrm{vol}(I_k) : E \subset \bigcup_{k=1}^{\infty} I_k,\ I_k \text{ are rectangles in } \mathbb{R}^n \right\},$$

where $\mathrm{vol}(I_k)$ denotes the usual n-dimensional volume of I_k, and the infimum is taken over all countable collections of rectangles covering E.

Recall from Lebesgue Measure Theory:

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Example 4 (Interval)
For $E = [a, b] \subset \mathbb{R}$, we have

$$m^*([a, b]) = b - a.$$

Example 5 (Finite Set)
For a finite set $E = \{x_1, x_2, \ldots, x_n\}$, we have $m^*(E) = 0$.

Example 6 (Countable Set)
For $E = \mathbb{Q} \cap [0, 1]$, we also have $m^*(E) = 0$.

Recall from Lebesgue Measure Theory:

Definition 7 (Lebesgue Measurable Set)

A set $E \subset \mathbb{R}^n$ is called *Lebesgue measurable* if for every set $A \subset \mathbb{R}^n$, the Lebesgue outer measure m^* satisfies

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

where E^c is the complement of E and m^* is the Lebesgue outer measure.
Intuitively, E is measurable if its size (measure) behaves well with respect to any other set A, so that cutting A into pieces inside and outside E exactly adds up in measure.

Example 8 (Intervals in $\mathbb{R}$)

Any interval $[a, b], (a, b), [a, b), (a, b] \subset \mathbb{R}$ is Lebesgue measurable, with measure equal to its length:

$$m([a, b]) = b - a.$$

Proof (Recall from Lebesgue Measure Theory):

Example 9 (Open and Closed Sets)
All open or closed sets in $\mathbb{R}^n$ are Lebesgue measurable.

Example 10 (Null Sets)
Any set of Lebesgue measure zero (e.g., a finite or countable set of points in $\mathbb{R}^n$) is Lebesgue measurable.

Example 11 (Non-Measurable Sets)
There exist sets in $\mathbb{R}$ (like the Vitali set) which are not Lebesgue measurable. These sets are constructed using the axiom of choice and cannot have a well-defined Lebesgue measure.

Recall from Lebesgue Measure Theory:

Definition 12 (σ-Algebra)

Let X be a set. A collection $\mathcal{F} \subset 2^X$ of subsets of X is called a σ-*algebra* if it satisfies the following three properties:

1. (**Non-empty**) $X \in \mathcal{F}$.
2. (**Closed under complement**) If $A \in \mathcal{F}$, then $A^c = X \setminus A \in \mathcal{F}$.
3. (**Closed under countable unions**) If $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$, then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}.$$

From these properties, it also follows that $\mathcal{F}$ is closed under countable intersections:

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}, \quad \text{for any } A_n \in \mathcal{F}.$$

Recall from Lebesgue Measure Theory:

Example 13 (Trivial σ-algebra)
For any set X, $\mathcal{F} = \{\emptyset, X\}$ is a σ-algebra.

Example 14 (Power Set)
The power set $\mathcal{F} = 2^X$ is a σ-algebra.

Example 15 (Borel σ-algebra)
For $X = \mathbb{R}$, the collection of all Borel sets (generated from open intervals by countable unions, intersections, and complements) forms a σ-algebra.

Example 16 (Lebesgue measurable sets σ-algebra)
Lebesgue-measurable sets form a σ-algebra

Proof (Recall from Lebesgue Measure Theory):

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Definition 17 (Lebesgue Measure on $\mathbb{R}^n$)

Lebesgue measure m is a set function $m : \mathcal{L} \subset 2^{\mathbb{R}^n} \to [0, \infty]$, defined on the σ-algebra $\mathcal{L}$ of *Lebesgue measurable sets*, satisfying:

1. (**Non-negativity**) $m(E) \geq 0$ for all $E \in \mathcal{L}$.
2. (**Null empty set**) $m(\emptyset) = 0$.
3. (**Countable additivity**) If $\{E_i\}_{i=1}^{\infty}$ are disjoint sets in $\mathcal{L}$, then

$$m\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} m(E_i).$$

4. (**Translation invariance**) For any $E \in \mathcal{L}$ and $x_0 \in \mathbb{R}^n$,

$$m(E + x_0) = m(E),$$

   where $E + x_0 := \{x + x_0 : x \in E\}$.

Proof (Recall from Lebesgue Measure Theory):

- Lebesgue measure generalizes length/area/volume to very irregular sets.
- It allows integration of more functions (not just continuous ones) and forms the basis of Lebesgue integration in analysis.

Definition 18 (Lebesgue Measurable Function)
A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be ***Lebesgue measurable*** if for every $\alpha \in \mathbb{R}$, the set

$$\{x \in \mathbb{R}^n : f(x) > \alpha\}$$

is Lebesgue measurable.

Proof (Recall from Lebesgue Measure Theory):

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Example 19 (Continuous Functions)
Every continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable because the set $\{x : f(x) > \alpha\}$ is open, hence measurable.

Example 20 (Characteristic Function)
Let $E \subset \mathbb{R}^n$ be Lebesgue measurable. Then its characteristic function

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E \end{cases}$$

is Lebesgue measurable.

Proof (Recall from Lebesgue Measure Theory):

$$\mathcal{L}^p(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is Lebesgue measurable}, \|f\|_{L^p(\Omega)} < \infty\}$$

Here,

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p d\mathbf{x} \right)^{\frac{1}{p}} \quad (1 \leq p \leq \infty)$$

$$\mathcal{L}^\infty(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is Lebesgue measurable}, \|f\|_{L^\infty(\Omega)} < \infty\}$$

Here,

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f|$$

Proof(continued)

Remember that Φ has a singularity near 0. What type of singularity? Let us isolate this singularity as follows: Consider a small ball of radius $\epsilon > 0$. Then

$$\Delta u(\mathbf{x}) = \underbrace{\int_{B(\mathbf{0},\epsilon)} \Delta_x \Phi(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d\mathbf{y}}_{I_\epsilon} + \underbrace{\int_{\mathbb{R}^n - B(\mathbf{0},\epsilon)} \Phi(\mathbf{y}) \Delta_x f(\mathbf{x}-\mathbf{y}) d\mathbf{y}}_{J_\epsilon}$$

Now

$$|I_\epsilon| \leq C \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(\mathbf{0},\epsilon)} |\Phi(\mathbf{y})| d\mathbf{y}$$

From volume of ball $B(\mathbf{0},\epsilon)$ and (3)

$$|I_\epsilon| \leq \begin{cases} C\epsilon^2 |\log \epsilon| & (n=2) \\ C\epsilon^2 & (n \geq 3) \end{cases} \tag{7}$$

Preliminaries

Theorem 21 (Gauss-Green Theorem)
Suppose $u \in C^1(\overline{\Omega})$. Then

$$\int\limits_{\Omega} u_{x_i} dx = \int\limits_{\partial\Omega} u\nu^i dS \tag{8}$$

where $i = 1, 2, \cdots, n$.

Theorem 22 (Integration by Parts)
Suppose $u, v \in C^1(\overline{\Omega})$. Then

$$\int\limits_{\Omega} u_{x_i} v d\mathbf{x} = -\int\limits_{\Omega} u v_{x_i} d\mathbf{x} + \int\limits_{\partial\Omega} uv\nu^i dS \tag{9}$$

where $i = 1, 2, \cdots, n$.

Proof(continued)

An integration by parts, we obtain that

$$|J_\epsilon| = \int_{\mathbb{R}^n - B(\mathbf{0},\epsilon)} \Phi(\mathbf{y})\Delta_x f(\mathbf{x}-\mathbf{y})d\mathbf{y}$$

$$= \underbrace{-\int_{\mathbb{R}^n - B(\mathbf{0},\epsilon)} D\Phi(\mathbf{y}).\Delta_y f(\mathbf{x}-\mathbf{y})d\mathbf{y}}_{K_\epsilon} + \underbrace{\int_{\partial B(\mathbf{0},\epsilon)} \Phi(\mathbf{y})\frac{\partial f}{\partial \nu}(\mathbf{x}-\mathbf{y})dS(\mathbf{y})}_{L_\epsilon}$$

where ν denotes the inward pointing unit normal along $\partial B(\mathbf{0},\epsilon)$

Proof(continued)

Now

$$|L_\epsilon| \leq \|Df\|_{L^\infty(\mathbb{R}^n)} \int\limits_{\partial B(\mathbf{0},\epsilon)} |\Phi(\mathbf{y})| dS\mathbf{y}$$

From volume of sphere $\partial B(\mathbf{0}, \epsilon)$ and (3)

$$|L_\epsilon| \leq \begin{cases} C\epsilon|\log \epsilon| & (n = 2) \\ C\epsilon & (n \geq 3) \end{cases} \tag{10}$$

Proof(continued)

An integration by parts for K_ϵ, we obtain that

$$|K_\epsilon| = \int\limits_{\mathbb{R}^n - B(\mathbf{0},\epsilon)} \Delta\Phi(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) d\mathbf{y} - \int\limits_{\partial B(\mathbf{0},\epsilon)} \frac{\partial \Phi(\mathbf{y})}{\partial \nu} f(\mathbf{x}-\mathbf{y}) dS(\mathbf{y})$$

$$= - \int\limits_{\partial B(\mathbf{0},\epsilon)} \frac{\partial \Phi(\mathbf{y})}{\partial \nu} f(\mathbf{x}-\mathbf{y}) dS(\mathbf{y})$$

Since Φ is harmonic in the region $\mathbb{R}^n - B(\mathbf{0},\epsilon)$

$$D\Phi(\mathbf{y}) = -\frac{1}{n\alpha(n)} \frac{\mathbf{y}}{|\mathbf{y}|^n}, \quad (\mathbf{y} \neq 0)$$

$$\nu = \frac{-\mathbf{y}}{|\mathbf{y}|} = -\frac{|\mathbf{y}|}{\epsilon}$$

on $\partial B(\mathbf{0},\epsilon)$.

Proof(continued)

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$$\frac{\partial \phi}{\partial \nu}(\mathbf{y}) = \nu . D\Phi(\mathbf{y}) = \frac{1}{n\alpha(n)\epsilon^{n-1}}$$

on $\partial B(\mathbf{0}, \epsilon)$.

$$|K_\epsilon| = -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(\mathbf{0},\epsilon)} f(\mathbf{x} - \mathbf{y}) dS(\mathbf{y})$$

$$= -⨍_{\partial B(\mathbf{x},\epsilon)} f(y) dS(y)$$

$$\to -f(\mathbf{x}) \quad (\text{as} \quad \epsilon \to 0)$$

Hence $-\Delta u(\mathbf{x}) = f(\mathbf{x})$ as $\epsilon \to 0$. Hence the proof.

Remarks

For a few problems, you need to solve

$$-\Delta\Phi = \delta_0 \quad \text{in} \quad \mathbb{R}^n$$

where δ_0 denotes the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0. Here, we have

$$\begin{aligned} -\Delta u(\mathbf{x}) &= \int_{\mathbb{R}^n} -\Delta_x \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \delta_{\mathbf{x}} f(\mathbf{y}) d\mathbf{y} \\ &= f(\mathbf{x}) \quad (\mathbf{x} \in \mathbb{R}^n) \end{aligned}$$

Mean-Value Formulas

Mean-value Formulas

Now, we are looking again the average integral, which we have discussed for wave equation.

Theorem 23

$u \in C^2(\Omega)$ is harmonic, if and only if,

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u dS = \fint_{B(\mathbf{x},r)} u dy \tag{11}$$

for each ball $B(\mathbf{x}, r) \subset \Omega$.

Proof:

As we have seen in the Wave equation, we have

$$\phi(r) := \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y})dS(\mathbf{y}) = \fint_{\partial B(\mathbf{0},1)} u(\mathbf{x}+r\mathbf{z})dS(\mathbf{z})$$

$$\implies \phi'(r) = \fint_{\partial B(\mathbf{0},1)} Du(\mathbf{x}+r\mathbf{z}).\mathbf{z}dS(\mathbf{z})$$

$$= \fint_{\partial B(\mathbf{x},r)} Du(\mathbf{y}).\frac{\mathbf{y}-\mathbf{x}}{r}dS(\mathbf{y})$$

$$= \fint_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu}dS(\mathbf{y})$$

Proof(continued)

By Green's formula, we have

$$\int_{\Omega} \Delta u d\mathbf{x} = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS$$

Hence

$$\begin{aligned}\implies \phi'(r) &= \fint_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu} dS(\mathbf{y}) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu} dS(\mathbf{y}) \\ &= \frac{r}{n} \frac{1}{\alpha(n)r^n} \int_{B(\mathbf{x},r)} \Delta u d(\mathbf{y}) = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u d(\mathbf{y}) = 0\end{aligned}$$

$\implies$ ϕ is constant.

Proof(continued)

$$\phi(r) = \lim_{t\to 0} \phi(t) = \lim_{t\to 0} \fint_{\partial B(\mathbf{x},t)} u(\mathbf{y})dS(\mathbf{y}) = u(\mathbf{x})$$

Therefore, we have proved that

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} udS \implies \int_{\partial B(\mathbf{x},r)} udS = n\alpha(n)r^{n-1}u(\mathbf{x})$$

Now, it is sufficient prove that

$$u(\mathbf{x}) = \fint_{B(\mathbf{x},r)} udy$$

when u is harmonic

Proof(continued)

The following results can be obtained from the Coarea formula (or a curvilinear Fubini Theorem). For continuous integrable functions $f : \mathbb{R}^n \to \mathbb{R}$, the spherical integration formula is given by

$$\int_{\mathbb{R}^n} f\,d\mathbf{x} = \int_0^\infty \left(\int_{\partial B(\mathbf{x},r)} f\,dS \right) dr$$

In particular,

$$\int_{B(\mathbf{x},r)} f\,d\mathbf{x} = \int_0^r \left(\int_{\partial B(\mathbf{x},s)} f\,dS \right) ds$$

Proof(continued)

$$\fint_{B(\mathbf{x},r)} u dy = \frac{1}{\alpha(n) r^n} \int_{B(\mathbf{x},r)} u dy = \frac{1}{\alpha(n) r^n} \int_0^r \left(\int_{\partial B(\mathbf{x},s)} u dS \right) ds$$

$$= \frac{1}{\alpha(n) r^n} \int_0^r n\alpha(n) s^{n-1} u(\mathbf{x}) ds = u(\mathbf{x})$$

Conversely, suppose, u is not harmonic, then there exists a ball $B(\mathbf{x}, r) \subset \Omega$ such that $\Delta u \neq 0$. Suppose $\Delta u > 0$ within $B(\mathbf{x}, r)$. Then

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u d(\mathbf{y}) > 0 \Rightarrow\Leftarrow$$

Hence the proof.

Maximum-Minimum Principle

Strong Maximum Principle

Theorem 24

Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic within Ω.

1. If Ω is connected and there exists a point $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u$$

   then u is constant within Ω

2. Further

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

Proof:

Suppose there exists a point $x_0 \in \Omega$ with

$$u(x_0) = M := \max_{\overline{\Omega}} u$$

Then for $0 < r < dist(x_0, \partial\Omega)$, the mean value property asserts that

$$M = u(x_0) = ⨍_{B(\mathbf{x}_0, r)} u d\mathbf{y} \leq M$$

The equality holds only if $u \equiv M$ within $B(\mathbf{x}_0, r)$. Hence $u(\mathbf{y}) = M$ for all $\mathbf{y} \in B(\mathbf{x}, r)$. Therefore, $\Omega_1 = \{\mathbf{x} \in \Omega : u(\mathbf{x}) = M\}$ is open and relatively closed in Ω. Also $\Omega_1 = \Omega$ if Ω is connected. Hence, u is constant within Ω. The second part follows immediately.

Strong Maximum Principle:

Remarks

1. The first part of the theorem is strong maximum principle
2. The second part of the theorem is called maximum principle.
3. If we replace u by $-u$, we obtain strong minimum and minimum principles.
4. If $u = f$ on $\partial\Omega$ where $f \geq 0$, then u is positive everywhere in Ω if f is positive somewhere on $\partial\Omega$
5. We can prove the uniqueness solutions of boundary value problems for Poisson equation using this maximum principle.

Uniqueness

Theorem 25 (Uniqueness)
Let $f \in C(\partial\Omega), h \in C(\Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega \end{cases} \tag{12}$$

Proof; Let u_1 and u_2 satisfy (12). Let $w = u_1 - u_2 \implies \Delta w = 0$. Now, we apply the strong maximum principle on this, we obtain that $w \equiv 0$. Hence $u_1 = u_2$.

Mollifiers

Locally Integrable

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say $\Omega_1 \subset\subset \Omega$ if Ω_1 is compactly contained in Ω. That is, $\Omega_1 \subset \overline{\Omega_1} \subset \Omega$ is compact.

Definition 2 (Locally Integrable)

$$L^p_{loc}(\Omega) := \{f : \Omega \to \mathbb{R} : f \in L^p(\Omega_1) \text{for each } \Omega_1 \subset\subset \Omega\}$$

Let $f : \Omega \to \mathbb{R}$ is measurable. We say $f \in L^1_{loc}$ iff

$$\int_K |f(\mathbf{x})| d\mathbf{x} < \infty$$

for all compact sets $K \subset \Omega$.

Locally Integrable

Example 1

1. Constant functions defined on real line is locally integrable but not globally integrable as the real line has infinite measure
2. Continuous functions
3. Integrable functions
4. $f(x) = 1/x, x \in (0, 1)$ is locally integrable, but not globally integrable. [Since any compact set $K \subset (0, 1)$ has positive distance from 0 and 1 and f is bounded on K]
5. $1/x \in L^1_{loc}(\mathbb{R} \setminus 0)$
6. $f(x) = 1/x$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$ is not locally integrable in $x = 0$.

Read more details about this on Distribution Theory courses.

Locally Integrable

Theorem 26

1. The locally integral functions form a linear space
2. L^p_{loc} is a complete metrizable space.
3. $f \in L_p(\Omega)$ is locally integrable

Lebesgue Differentiation Theorem

Theorem 27 (Lebesgue Differentiation Theorem)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable. Then for a.e point $x_0 \in \mathbb{R}^n$,

1. $$\fint_{B(\mathbf{x}_0,r)} f d\mathbf{x} \to f(\mathbf{x}_0) \quad \text{as } r \to 0$$

2. $$\fint_{B(\mathbf{x}_0,r)} |f(\mathbf{x}) - f(\mathbf{x}_0)| d\mathbf{x} \to 0 \quad \text{as } r \to 0$$

The point at which (2) holds is called a Lebesgue point of f.

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in