MA612L-Partial Differential Equations

Lecture 21: Laplace Equation - Mean Value, Maximum-Minimum Principle and Liouville's Theorem

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Mean-Value Formulas

Mean-value Formulas

Now, we are looking again at the average integral, which we have discussed for the wave equation.

Theorem 1

$u \in C^2(\Omega)$ is harmonic, if and only if,

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} u dS = \fint_{B(\mathbf{x},r)} u dy \tag{1}$$

for each ball $B(\mathbf{x}, r) \subset \Omega$.

Proof:

As we have seen in the Wave equation, we have

$$\phi(r) := \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y})dS(\mathbf{y}) = \fint_{\partial B(\mathbf{0},1)} u(\mathbf{x}+r\mathbf{z})dS(\mathbf{z})$$

$$\begin{aligned}\implies \phi'(r) &= \fint_{\partial B(\mathbf{0},1)} Du(\mathbf{x}+r\mathbf{z}).\mathbf{z}dS(\mathbf{z}) \\ &= \fint_{\partial B(\mathbf{x},r)} Du(\mathbf{y}).\frac{\mathbf{y}-\mathbf{x}}{r}dS(\mathbf{y}) \\ &= \fint_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu}dS(\mathbf{y})\end{aligned}$$

Proof(continued)

By Green's formula, we have

$$\int_{\Omega} \Delta u d\mathbf{x} = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS$$

Hence

$$\Longrightarrow \ \phi'(r) = \fint_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu} dS(\mathbf{y}) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(\mathbf{x},r)} \frac{\partial u}{\partial \nu} dS(\mathbf{y})$$

$$= \frac{r}{n} \frac{1}{\alpha(n) r^{n}} \int_{B(\mathbf{x},r)} \Delta u d(\mathbf{y}) = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u d(\mathbf{y}) = 0$$

$\Longrightarrow$ ϕ is constant.

Proof(continued)

$$\phi(r) = \lim_{t\to 0} \phi(t) = \lim_{t\to 0} \fint_{\partial B(\mathbf{x},t)} u(\mathbf{y})dS(\mathbf{y}) = u(\mathbf{x})$$

Therefore, we have proved that

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},r)} udS \implies \int_{\partial B(\mathbf{x},r)} udS = n\alpha(n)r^{n-1}u(\mathbf{x})$$

Now, it is sufficient to prove that

$$u(\mathbf{x}) = \fint_{B(\mathbf{x},r)} udy$$

when u is harmonic

Proof(continued)

The following results can be obtained from the Coarea formula (or a curvilinear Fubini Theorem). For continuous integrable functions $f : \mathbb{R}^n \to \mathbb{R}$, the spherical integration formula is given by

$$\int_{\mathbb{R}^n} f \, d\mathbf{x} = \int_0^\infty \left(\int_{\partial B(\mathbf{x},r)} f \, dS \right) dr$$

In particular,

$$\int_{B(\mathbf{x},r)} f \, d\mathbf{x} = \int_0^r \left(\int_{\partial B(\mathbf{x},s)} f \, dS \right) ds$$

"Integrating over a ball is nothing but integrating over each inner sphere starting from 0."

Proof(continued)

$$\fint_{B(\mathbf{x},r)} u dy = \frac{1}{\alpha(n)r^n} \int_{B(\mathbf{x},r)} u dy = \frac{1}{\alpha(n)r^n} \int_0^r \left(\int_{\partial B(\mathbf{x},s)} u dS \right) ds$$

$$= \frac{1}{\alpha(n)r^n} \int_0^r n\alpha(n)s^{n-1} u(\mathbf{x}) ds = u(\mathbf{x})$$

Conversely, suppose, u is not harmonic, then there exists a ball $B(\mathbf{x}, r) \subset \Omega$ such that $\Delta u \not\equiv 0$. Suppose $\Delta u > 0$ within $B(\mathbf{x}, r)$. Then

$$0 = \phi'(r) = \frac{r}{n} \fint_{B(\mathbf{x},r)} \Delta u d(\mathbf{y}) > 0 \Rightarrow\Leftarrow$$

Hence the proof.

Maximum-Minimum Principle

Strong Maximum Principle

Theorem 2

Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic within Ω.

1. If Ω is connected and there exists a point $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u$$

   then u is constant within Ω

2. Further

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$$

Proof:

Suppose there exists a point $x_0 \in \Omega$ with

$$u(x_0) = M := \max_{\overline{\Omega}} u$$

Then for $0 < r < dist(x_0, \partial\Omega)$, the mean value property asserts that

$$M = u(x_0) = \fint_{B(\mathbf{x}_0, r)} u d\mathbf{y} \leq M$$

The equality holds only if $u \equiv M$ within $B(\mathbf{x}_0, r)$. Hence $u(\mathbf{y}) = M$ for all $\mathbf{y} \in B(\mathbf{x}, r)$. Therefore, $\Omega_1 = \{\mathbf{x} \in \Omega : u(\mathbf{x}) = M\}$ is open and relatively closed in Ω. Also $\Omega_1 = \Omega$ if Ω is connected. Hence, u is constant within Ω. The second part follows immediately.

Strong Maximum Principle:

Remarks

1. The first part of the theorem is the strong maximum principle
2. The second part of the theorem is called the maximum principle.
3. If we replace u by $-u$, we obtain strong minimum and minimum principles.
4. If $u = f$ on $\partial\Omega$ where $f \geq 0$, then u is positive everywhere in Ω if f is positive somewhere on $\partial\Omega$
5. We can prove the uniqueness of solutions of boundary value problems for the Poisson equation using this maximum principle.

Uniqueness

Theorem 3 (Uniqueness)
Let $f \in C(\partial\Omega), g \in C(\Omega)$. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases} \tag{2}$$

Proof; Let u_1 and u_2 satisfy (2). Let $w = u_1 - u_2 \implies \Delta w = 0$. Now, we apply the strong maximum principle on this, and we obtain that $w \equiv 0$. Hence $u_1 = u_2$.

Thanks

Doubts and Suggestions

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