MA612L-Partial Differential Equations

Lecture 22: Laplace Equation - Mollifiers, Regularity and Liouville's Theorem

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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Mollifiers

Locally Integrable

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say $\Omega_1 \subset\subset \Omega$ if Ω_1 is compactly contained in Ω. That is, $\Omega_1 \subset \overline{\Omega_1} \subset \Omega$ and $\overline{\Omega_1}$ is compact.

Definition 1 (Locally Integrable)

$$L^p_{loc}(\Omega) := \{f : \Omega \to \mathbb{R} : f \in L^p(\Omega_1) \text{ for each } \Omega_1 \subset\subset \Omega\}$$

Let $f : \Omega \to \mathbb{R}$ is measurable. We say $f \in L^1_{loc}$ iff

$$\int_K |f(\mathbf{x})| d\mathbf{x} < \infty$$

for all compact sets $K \subset \Omega$.

Locally Integrable

Example 1

1. Constant functions defined on the real line are locally integrable but not globally integrable, as the real line has infinite measure
2. Continuous functions
3. Integrable functions
4. $f(x) = 1/x, x \in (0,1)$ is locally integrable, but not globally integrable. [Since any compact set $K \subset (0,1)$ has positive distance from 0 and 1 and f is bounded on K]
5. $1/x \in L^1_{loc}(\mathbb{R} \setminus \{0\})$
6. $f(x) = 1/x$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$ is not locally integrable in $x = 0$.

Read more details about this in the Distribution Theory courses.

Locally Integrable

Theorem 1

1. Locally integral functions form a linear space
2. L^p_{loc} is a complete metrizable space.
3. $f \in L_p(\Omega)$ is locally integrable

Lebesgue Differentiation Theorem

Theorem 2 (Lebesgue Differentiation Theorem)
Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally integrable. Then for a.e point $x_0 \in \mathbb{R}^n$,

1. $$\fint_{B(\mathbf{x}_0,r)} f\,d\mathbf{x} \to f(\mathbf{x}_0) \quad \text{as} \quad r \to 0$$

2. $$\fint_{B(\mathbf{x}_0,r)} |f(\mathbf{x}) - f(\mathbf{x}_0)|\,d\mathbf{x} \to 0 \quad \text{as} \quad r \to 0$$

   The point at which (2) holds is called a Lebesgue point of f.

A function is the derivative of its own integral (or measure), recovered through local averaging.

Mollifiers

Definition 2 (Mollifier)
Define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(\mathbf{x}) := \begin{cases} C exp\left(\frac{1}{|\mathbf{x}|^2 - 1}\right) & \text{if } |\mathbf{x}| < 1 \\ 0 & \text{if } |\mathbf{x}| \geq 1 \end{cases} \tag{1}$$

the constant $C > 0$ selected so that

$$\int_{\mathbb{R}^n} \eta d\mathbf{x} = 1$$

We call η the standard mollifier.

Mollifiers

Definition 3 (Mollifier)
For each $\epsilon > 0$, set

$$\eta_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^n}\eta\left(\frac{\mathbf{x}}{\epsilon}\right)$$

We can prove that $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$ and satisfy

$$\int\limits_{\mathbb{R}^n} \eta_\epsilon d\mathbf{x} = 1$$

and

$$\text{support}(\eta_\epsilon) \subset B(\mathbf{0}, \epsilon)$$

Here

$$\text{support}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$$

Mollifiers

Definition 4 (Mollification)
If $f : \Omega \to \mathbb{R}$ is locally integrable, define its mollification

$$f^\epsilon := \eta_\epsilon * f \quad \text{in} \quad \Omega_\epsilon$$

That is,

$$f^\epsilon(\mathbf{x}) = \int_\Omega \eta_\epsilon(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y} = \int_{B(\mathbf{0},\epsilon)} \eta_\epsilon(\mathbf{y})f(\mathbf{x}-\mathbf{y})d\mathbf{y} \;\; \text{for} \;\; x \in \Omega_\epsilon$$

Mollifiers

Theorem 3 (Properties of Mollifiers)

1. $f^\epsilon \in C^\infty(\Omega_\epsilon)$
2. $f^\epsilon \to f$ almost everywhere as $\epsilon \to 0$
3. If $f \in C(\Omega)$, then $f^\epsilon \to f$ uniformly on compact subsets of Ω
4. If $1 \leq p \leq \infty$ and $f \in L^p_{loc}(\Omega)$, then $f^\epsilon \to f$ in $L^p_{loc}(\Omega)$

Proof: Fix $\mathbf{x} \in \Omega_\epsilon, i \in \{1, 2, \cdots, n\}$ and h is so small that $\mathbf{x} + h\mathbf{e}_i \in \Omega_\epsilon$. Then

$$\frac{f^\epsilon(\mathbf{x} + h\mathbf{e}_i) - f^\epsilon(\mathbf{x})}{h} = \frac{1}{\epsilon^n} \int_\Omega \frac{1}{h} \left[\eta\left(\frac{\mathbf{x} + h\mathbf{e}_i - \mathbf{y}}{\epsilon} \right) - \eta\left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon} \right) \right] f(\mathbf{y}) d\mathbf{y}$$

$$= \frac{1}{\epsilon^n} \int_{\Omega_1} \frac{1}{h} \left[\eta\left(\frac{\mathbf{x} + h\mathbf{e}_i - \mathbf{y}}{\epsilon} \right) - \eta\left(\frac{\mathbf{x} - \mathbf{y}}{\epsilon} \right) \right] f(\mathbf{y}) d\mathbf{y}$$

for some open set $\Omega_1 \subset\subset \Omega$

Properties of Mollifiers

Since

$$\frac{1}{h}\left[\eta\left(\frac{\mathbf{x}+h\mathbf{e}_i-\mathbf{y}}{\epsilon}\right)-\eta\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon}\right)\right]\to\frac{1}{\epsilon}\frac{\partial\eta}{\partial x_i}\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon}\right)$$

uniformly on Ω_1, $\frac{\partial f^\epsilon}{\partial x_i}(\mathbf{x})$ exists. Also,

$$\frac{\partial f^\epsilon}{\partial x_i}(\mathbf{x})=\int_\Omega\frac{\partial\eta}{\partial x_i}\left(\mathbf{x}-\mathbf{y}\right)f(\mathbf{y})d\mathbf{y}$$

Extending this further, we can prove that

$$D^\alpha f^\epsilon(\mathbf{x})=\int_\Omega D^\alpha\eta_\epsilon\left(\mathbf{x}-\mathbf{y}\right)f(\mathbf{y})d\mathbf{y}$$

$$\implies f^\epsilon\in C^\infty(\Omega_\epsilon)$$

Properties of Mollifiers

Using the Lebesgue differentiation theorem, we have

$$\lim_{r\to 0} \fint_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mathbf{y} = 0$$

for a.e. $\mathbf{x} \in \Omega$. Fix such a point $\mathbf{x}$. Then

$$\begin{aligned}
|f^\epsilon(\mathbf{x}) - f(\mathbf{x})| &= \left| \int_{B(\mathbf{x},\epsilon)} \eta_\epsilon(\mathbf{x}-\mathbf{y})[f(\mathbf{y}) - f(\mathbf{x})] d\mathbf{y} \right| \\
&\le \frac{1}{\epsilon^n} \int_{B(\mathbf{x},\epsilon)} \eta\left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon}\right) [f(\mathbf{y}) - f(\mathbf{x})] d\mathbf{y} \\
&\le C \fint_{B(\mathbf{x},\epsilon)} [f(\mathbf{y}) - f(\mathbf{x})] d\mathbf{y} \to 0 \quad \text{as} \quad \epsilon \to 0
\end{aligned}$$

Properties of Mollifiers

Since

$$\lim_{r\to 0} \fint_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mathbf{y} = 0$$

for a.e. $\mathbf{x} \in \Omega$.

$$|f^{\epsilon}(\mathbf{x}) - f(\mathbf{x})| \to 0$$

a.e as $\epsilon \to 0$. Suppose $f \in C(\Omega)$. Given $\Omega_1 \subset\subset \Omega$, let us choose $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and note that f is uniformly continuous on Ω_2. Therefore,

$$|f^{\epsilon}(\mathbf{x}) - f(\mathbf{x})| \to 0$$

uniformly for $\mathbf{x} \in \Omega_1$. Hence $f^{\epsilon} \to f$ uniformly on Ω_1.
Exercise: The proof of part(4)

Regularity

Regularity

In this section, let us prove that if $u \in C^2(\Omega)$ is harmonic, then necessarily $u \in C^\infty(\Omega)$.

- The harmonic functions are automatically infinitely differentiable
- It is called the regularity theorem
- Algebraic structure of Laplace equation

$$\Delta u = 0$$

leads to an analytic deduction that all the partial derivatives of u exist, although it is a second-order PDE.

Smoothness

Theorem 4 (Smoothness)
If $u \in C(\Omega)$ satisfies the mean value property for each ball $B(\mathbf{x}, r)$, then

$$u \in C^{\infty}(\Omega)$$

Proof: Let η be a standard mollifier and remember that η is a radial function. Now define

$$u^{\epsilon} := \eta_{\epsilon} * u \text{ in } \Omega_{\epsilon} = \{\mathbf{x} \in \Omega : dist(\mathbf{x}, \partial\Omega) > \epsilon\}$$

By the properties of mollifiers, we can show that $u^{\epsilon} \in C^{\infty}(\Omega_{\epsilon})$.

Smoothness

Claim: $u \equiv u^\epsilon$ on Ω_ϵ

$$u^\epsilon(\mathbf{x}) = \int\limits_{\Omega} \eta_\epsilon(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y}$$

$$= \frac{1}{\epsilon^n} \int\limits_{B(\mathbf{x},\epsilon)} \eta\left(\frac{|\mathbf{x}-\mathbf{y}|}{\epsilon}\right) u(\mathbf{y})d\mathbf{y}$$

Claim (continued)

$$
\begin{aligned}
u^\epsilon(\mathbf{x}) &= \frac{1}{\epsilon^n}\int\limits_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)\left(\int\limits_{\partial B(\mathbf{x},r)} u(\mathbf{y})dS\mathbf{y}\right)dr \\
&= \frac{1}{\epsilon^n}u(\mathbf{x})\int\limits_0^\epsilon \eta\left(\frac{r}{\epsilon}\right)n\alpha(n)r^{n-1}dr \\
&= u(\mathbf{x})\int\limits_{\partial B(\mathbf{0},\epsilon)} \eta_\epsilon dS\mathbf{y} \\
&= u(\mathbf{x})
\end{aligned}
$$

Hence $u^\epsilon \equiv u$ and $u \in C^\infty(\Omega_\epsilon)$ for each $\epsilon > 0$. Hence the claim and the proof.

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in