

MA612L-Partial Differential Equations

Lecture 24: Laplace Equation - Analyticity and Harnack's Inequality

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Recap

Recap



$$\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^n \quad (1)$$

Definition 1 (Fundamental Solution)

The function

$$\Phi(\mathbf{x}) := \begin{cases} -\frac{1}{2\pi} \log |\mathbf{x}| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)|\mathbf{x}|^{n-2}} & (n \geq 3) \end{cases} \quad (2)$$

defined for $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$ is the fundamental solution of Laplace equation.

Theorem 1

Define u by

$$u(\mathbf{x}) = \begin{cases} \frac{-1}{2\pi} \int_{\mathbb{R}^2} \log(|\mathbf{x} - \mathbf{y}|) f(\mathbf{y}) d\mathbf{y} & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^{n-2}} d\mathbf{y} & (n \geq 3) \end{cases} \quad (3)$$

Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$ in $\mathbb{R}^n$

Mean-value Formulas



Theorem 2

$u \in C^2(\Omega)$ is harmonic, if and only if,

$$u(\mathbf{x}) = \oint_{\partial B(\mathbf{x}, r)} u dS = \int_{B(\mathbf{x}, r)} u dy \quad \forall B(\mathbf{x}, r) \subset \Omega \quad (4)$$

Theorem 3

Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic within Ω . If Ω is connected and there exists a point $x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u$ then u is constant within Ω . Further $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$

Theorem 4 (Smoothness)

If $u \in C(\Omega)$ satisfies the mean value property for each ball $B(\mathbf{x}, r)$, then $u \in C^\infty(\Omega)$



Liouville's Theorem

Liouville's Theorem



Theorem 5 (Liouville's Theorem)

If $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is harmonic and bounded. Then u is constant.

Proof: Fix $x_0 \in \mathbb{R}^n, r > 0$.

Claim 1:

$$|Du(x_0)| \leq \frac{C}{r^{n+1}} \|u\|_{L^1(B(\mathbf{x}_0, r))}$$

Claim 2:

$$\frac{C}{r^{n+1}} \|u\|_{L^1(B(\mathbf{x}_0, r))} \leq \frac{C\alpha(n)}{r} \|u\|_{L^\infty(\mathbb{R}^n)}$$

If claims (1) and (2) are true, then, as $r \rightarrow \infty$, $Du(\mathbf{x}_0) \rightarrow 0$. Hence $Du \equiv 0$ and hence u is constant.

Liouville's Theorem



Claim 1:

$$|Du(x_0)| \leq \frac{C}{r^{n+1}} \|u\|_{L^1(B(\mathbf{x}_0, r))}$$

Proof of Claim 1: We can immediately show that

$$|u(x_0)| \leq \frac{C}{r^n} \|u\|_{L^1(B(\mathbf{x}_0, r))}$$

How? By the Mean value property

$$u(\mathbf{x}) = \int_{B(\mathbf{x}, r)} u dy = \frac{1}{\alpha(n)r^n} \int u dy \quad \forall B(\mathbf{x}, r) \subset \Omega$$

$$|u(\mathbf{x})| \leq \frac{1}{\alpha(n)r^n} \int |u| dy = \frac{1}{\alpha(n)r^n} \|u\|_{L^1(B(\mathbf{x}_0, r))}$$

Liouville's Theorem



Since u is harmonic, differentiating the Laplace equation, we can prove that u_{x_i} is also harmonic. Again, by the mean-value property,

$$u(\mathbf{x}) = \oint_{B(\mathbf{x}, r)} u dy \quad \forall B(\mathbf{x}, r) \subset \Omega$$

In particular, it is true for $B(\mathbf{x}_0, \frac{r}{2})$

$$\begin{aligned} |u_{x_i}(\mathbf{x}_0)| &= \left| \oint_{B(\mathbf{x}_0, \frac{r}{2})} u_{x_i} d\mathbf{x} \right| = \left| \frac{1}{\alpha(n)(\frac{r}{2})^n} \int_{\partial B(\mathbf{x}_0, \frac{r}{2})} u \nu^i dS \right| \\ &\leq \frac{2n}{r} \|u\|_{L^\infty(\partial B(\mathbf{x}_0, \frac{r}{2}))} \end{aligned}$$

Liouville's Theorem



Since ν is outward unit normal, $|\nu^i| \leq 1$. Hence

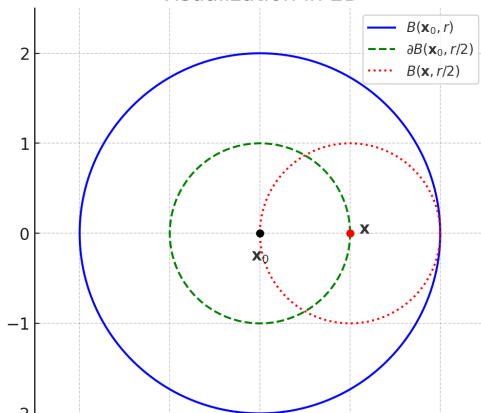
$$\begin{aligned} \left| \frac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{\partial B(\mathbf{x}_0, \frac{r}{2})} u \nu^i dS \right| &\leq \frac{1}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{\partial B(\mathbf{x}_0, \frac{r}{2})} |u| |\nu^i| dS \\ &\leq \frac{\|u\|_{L^\infty(\partial B(\mathbf{x}_0, \frac{r}{2}))}}{\alpha(n)\left(\frac{r}{2}\right)^n} \int_{\partial B(\mathbf{x}_0, \frac{r}{2})} 1 dS \\ &\leq \frac{2n}{r} \|u\|_{L^\infty(\partial B(\mathbf{x}_0, \frac{r}{2}))} \end{aligned}$$

Liouville's Theorem



$$\mathbf{x} \in \partial B\left(\mathbf{x}_0, \frac{r}{2}\right) \implies \mathbf{x} \in B\left(\mathbf{x}, \frac{r}{2}\right) \implies \mathbf{x} \in B(\mathbf{x}_0, r) \subset \Omega$$

Visualization in 2D



Liouville's Theorem



$$|u(x_0)| \leq \frac{C}{r^n} \|u\|_{L^1(B(\mathbf{x}_0, r))} \implies |u(\mathbf{x})| \leq \left(\frac{2}{r}\right)^n \frac{1}{\alpha(n)} \|u\|_{L^1(B(\mathbf{x}_0, r))}$$

Therefore,

$$|Du(\mathbf{x}_0)| \leq \left(\frac{2}{r}\right)^{n+1} \frac{1}{\alpha(n)} \|u\|_{L^1(B(\mathbf{x}_0, r))}$$

Hence, the claim 1.

Liouville's Theorem



Exercise 1: Exercises

1. Prove that

$$|D^\alpha u(\mathbf{x}_0)| \leq \left(\frac{2^{n+1}nk}{r} \right)^k \frac{1}{r^n \alpha(n)} \|u\|_{L^1(B(\mathbf{x}_0, r))}$$

for $k = |\alpha|$ and for each ball $B(\mathbf{x}_0, r) \subset \Omega$

Hint: Use

$$\mathbf{x} \in \partial B(\mathbf{x}_0, \frac{r}{k}) \implies \mathbf{x} \in B\left(\mathbf{x}, \frac{k-1}{k}r\right) \subset B(\mathbf{x}_0, r) \subset \Omega$$

2. Prove Claim 2.



Representation Formula

Representation Formula



Theorem 6 (Representation Formula)

If $h \in C_c^2(\mathbb{R}^n)$, $n \geq 3$, then any bounded solution of

$$-\Delta u = h \text{ in } \mathbb{R}^n$$

has the form

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}) h(\mathbf{y}) d\mathbf{y} + C \quad (\mathbf{x} \in \mathbb{R}^n)$$

for some constant C .

Proof: From the construction of $\Phi(\mathbf{x})$, it is immediate that $\Phi(\mathbf{x}) \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$ for $n \geq 3$.

Representation Formula



Proof (continued):

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y})h(\mathbf{y})d\mathbf{y}$$

is a bounded solution of $-\Delta u = h$ in $\mathbb{R}^n$. Suppose u_1 is another solution, then $w := u_1 - u$ is constant as per Liouville's theorem. Hence proved.

Remark

If $n = 2$,

$$\Phi(\mathbf{x}) = \frac{-1}{2\pi} \log |\mathbf{x}|$$

is unbounded as $|\mathbf{x}| \rightarrow \infty$ and hence

$$u(\mathbf{x}) = \int_{\mathbb{R}^2} \Phi(\mathbf{x} - \mathbf{y})h(\mathbf{y})d\mathbf{y}$$

may be unbounded



Analyticity

Analyticity



Theorem 7 (Analyticity)

Assume u is harmonic in Ω . Then u is analytic in Ω .

Proof: Fix any point $\mathbf{x}_0 \in \Omega$.

Claim: u can be represented by a convergent powers series in some neighbourhood of $\mathbf{x}_0$.

Let $r := \frac{1}{4} \text{dist}(\mathbf{x}_0, \partial\Omega)$. Then $M := \frac{1}{\alpha(n)r^n} \|u\|_{L^1(B(\mathbf{x}_0, 2r))} < \infty$

Now, $B(\mathbf{x}, r) \subset B(\mathbf{x}_0, 2r) \subset \Omega$ for each $x \in B(\mathbf{x}_0, r)$.

From the exercise problem, we have that

$$|D^\alpha u(\mathbf{x}_0)| \leq \left(\frac{2^{n+1}nk}{r} \right)^k \frac{1}{r^n \alpha(n)} \|u\|_{L^1(B(\mathbf{x}_0, r))}$$

Analyticity

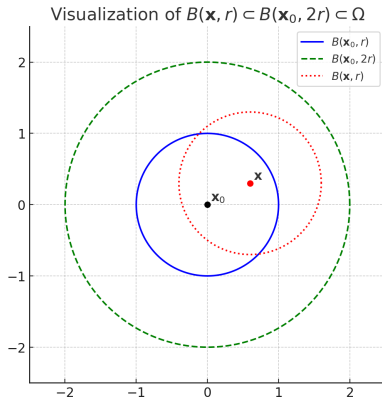


Figure 2: 2D Illustration

Analyticity



Proof (continued):

$$|D^\alpha u(\mathbf{x}_0)| \leq \left(\frac{2^{n+1}nk}{r} \right)^k M \leq M \left(\frac{2^{n+1}n}{r} \right)^{|\alpha|} |\alpha|^{|\alpha|}$$

Stirling's Formula:

$$\begin{aligned} \lim_{k \rightarrow \infty} e^{-k} \frac{k^{k+1/2}}{k!} &= \frac{1}{\sqrt{2\pi}} \\ \Rightarrow \lim_{|\alpha| \rightarrow \infty} e^{-|\alpha|} \frac{|\alpha|^{|\alpha|+1/2}}{|\alpha|!} &= \frac{1}{\sqrt{2\pi}} \Rightarrow e^{-|\alpha|} \frac{|\alpha|^{|\alpha|+1/2}}{|\alpha|!} \leq C_1 \\ \Rightarrow |\alpha|^{|\alpha|+1/2} &\leq C_1 e^{|\alpha|} |\alpha|! \Rightarrow |\alpha|^{|\alpha|} \leq C e^{|\alpha|} |\alpha|! \end{aligned}$$

for some constant C .

Analyticity



Proof (continued): Also

$$n^k = \underbrace{(1 + 1 + 1 + \cdots + 1)}_{n \text{ times}} = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!}$$

$$\implies |\alpha|! \leq n^{|\alpha|} \alpha! \implies |\alpha|^{|\alpha|} \leq C(ne)^{|\alpha|} |\alpha|!$$

Hence

$$|D^\alpha u(\mathbf{x}_0)| \leq M \left(\frac{2^{n+1}n}{r} \right)^{|\alpha|} |\alpha|^{|\alpha|} \leq CM \left(\frac{2^{n+1}n^2e}{r} \right)^{|\alpha|} \alpha!$$

$$\implies \frac{|D^\alpha u(\mathbf{x}_0)|}{\alpha!} \leq CM \left(\frac{2^{n+1}n^2e}{r} \right)^{|\alpha|}$$

Analyticity



Proof (continued): The Taylor series at for u at x_0 is given by

$$\sum_{|\alpha|} \frac{D^\alpha u(\mathbf{x}_0)}{\alpha!} (\mathbf{x} - \mathbf{x}_0)^\alpha$$

To prove that the Taylor series converges, we need to prove that the remainder term $|R_N|$ converges to 0 as $N \rightarrow \infty$ for some interval of convergence.

$$\sum_{|\alpha|} \frac{D^\alpha u(\mathbf{x}_0)}{\alpha!} (\mathbf{x} - \mathbf{x}_0)^\alpha$$

Now,

$$R_N(\mathbf{x}) := u(\mathbf{x}) - \sum_{k=0}^{N-1} \sum_{|\alpha|} \frac{D^\alpha u(\mathbf{x}_0)}{\alpha!} (\mathbf{x} - \mathbf{x}_0)^\alpha$$

Analyticity



Proof (continued):

$$\implies R_N(\mathbf{x}) = \sum_{|\alpha|=N} \frac{D^\alpha u(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0))}{\alpha!} (\mathbf{x} - \mathbf{x}_0)^\alpha \quad (\text{How??})$$

for some $t(\mathbf{x}) \in [0, 1]$

$$\implies |R_N(\mathbf{x})| \leq \sum_{|\alpha|=N} \frac{|D^\alpha u(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0))|}{\alpha!} |\mathbf{x} - \mathbf{x}_0|^\alpha$$

$$\implies |R_N(\mathbf{x})| \leq \sum_{|\alpha|=N} CM \left(\frac{2^{n+1} n^2 e}{r} \right)^N |\mathbf{x} - \mathbf{x}_0|^N$$

Analyticity



Proof (continued):

$$\implies |R_N(\mathbf{x})| \leq CM \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2e}{r} \right)^N |\mathbf{x} - \mathbf{x}_0|^N$$

Then

$$\begin{aligned} \frac{a_{N+1}}{a_N} < 1 &\implies \left(\frac{2^{n+1}n^2e}{r} \right)^{N+1} \left(\frac{r}{2^{n+1}n^2e} \right)^N |\mathbf{x} - \mathbf{x}_0| < 1 \\ &\implies |\mathbf{x} - \mathbf{x}_0| < \frac{r}{2^{n+1}n^2e} \end{aligned}$$

For, convergence, this is fine, but we need $|R_N(\mathbf{x})| \rightarrow 0$. So, choose

$$|\mathbf{x} - \mathbf{x}_0| < \frac{r}{2^{n+2}n^3e}$$

Analyticity



Proof (continued): Then

$$\begin{aligned}|R_N(\mathbf{x})| &\leq CM \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2e}{r} \right)^N |\mathbf{x} - \mathbf{x}_0|^N \\ &\leq CM \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2e}{r} \right)^N \left(\frac{r}{2^{n+2}n^3e} \right)^N \\ &\leq CM \sum_{|\alpha|=N} \left(\frac{1}{2n} \right)^N \leq CMn^N \left(\frac{1}{2n} \right)^N \\ &= \frac{CM}{2^N} \rightarrow 0 \text{ as } N \rightarrow \infty\end{aligned}$$

Hence the proof.



Harnack's Inequality

Harnack's Inequality



Theorem 8 (Harnack's Inequality)

For each connected open set $\Omega_1 \subset\subset \Omega$, there exists a positive constant C depending on Ω_1 such that

$$\sup_{\Omega_1} u \leq C \inf_{\Omega_1} u$$

for all nonnegative harmonic function u in Ω .

Proof:

Let $r := \frac{1}{4} \text{dist}(\Omega_1, \partial\Omega)$. Choose $\mathbf{x}, \mathbf{y} \in \Omega_1$ such that $|\mathbf{x} - \mathbf{y}| \leq r$. Then

$$u(\mathbf{x}) = \int_{B(\mathbf{x}, 2r)} u d\mathbf{z} = \frac{1}{\alpha(n)2^n r^n} \int_{B(\mathbf{x}, 2r)} u d\mathbf{z} \geq \frac{1}{\alpha(n)2^n r^n} \int_{B(\mathbf{y}, r)} u d\mathbf{z} = \frac{1}{2^n} \int_{B(\mathbf{y}, r)} u d\mathbf{z}$$

Harnack's Inequality



Proof (Continued):

$$\implies \frac{1}{2^n} u(\mathbf{y}) \leq u(\mathbf{x})$$

if $\mathbf{x}, \mathbf{y} \in \Omega_1$ and $|\mathbf{x} - \mathbf{y}| \leq r$. On the other hand,

$$u(\mathbf{x}) = \int_{B(\mathbf{x}, 2r)} u d\mathbf{z} = \frac{1}{\alpha(n)2^n r^n} \int_{B(\mathbf{x}, 2r)} u d\mathbf{z} \leq 2^n \frac{1}{\alpha(n)r^n} \int_{B(\mathbf{y}, r)} u d\mathbf{z} = 2^n \int_{B(\mathbf{y}, r)} u d\mathbf{z}$$

$$\implies u(\mathbf{x}) \leq 2^n u(\mathbf{y})$$

if $\mathbf{x}, \mathbf{y} \in \Omega_1$ and $|\mathbf{x} - \mathbf{y}| \leq r$. Hence

$$\implies \frac{1}{2^n} u(\mathbf{y}) \leq u(\mathbf{x}) \leq 2^n u(\mathbf{y})$$

if $\mathbf{x}, \mathbf{y} \in \Omega_1$ and $|\mathbf{x} - \mathbf{y}| \leq r$.

Harnack's Inequality



Proof (Continued): It is given that Ω_1 is connected and $\Omega_1 \subset\subset \Omega$. Since $\overline{\Omega}_1$ is compact, we can cover $\overline{\Omega}_1$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$ each of which has radius r and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, 3, \dots, N$. Hence

$$\implies \frac{1}{2^{nN}} u(\mathbf{y}) \leq u(\mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in \Omega_1$. Applying infimum and supremum definition, the proof follows.

Harnack's Inequality



Remarks

1. We have that

$$\implies \frac{1}{2^n} u(\mathbf{y}) \leq u(\mathbf{x}) \leq 2^n u(\mathbf{y})$$

2. In particular

$$\implies \frac{1}{C} u(\mathbf{y}) \leq u(\mathbf{x}) \leq C u(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \Omega_1$$

3. The value of nonnegative harmonic function within Ω_1 are all comparable.
4. The value of u can't be very small or very large at any point of Ω_1 unless u is very small or very large everywhere in Ω_1 .

Thanks

Doubts and Suggestions

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