MA612L-Partial Differential Equations

Lecture 24: Laplace Equation - Harnack's Inequality and Distribution Theory

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Harnack's Inequality

Harnack's Inequality

Theorem 1 (Harnack's Inequality)
For each connected open set $\Omega_1 \subset\subset \Omega$, there exists a positive constant C depending on Ω_1 such that

$$\sup_{\Omega_1} u \le C \inf_{\Omega_1} u$$

for all nonnegative harmonic function u in Ω.

Proof:
Let $r := \frac{1}{4} dist(\Omega_1, \partial\Omega)$. Choose $\mathbf{x}, \mathbf{y} \in \Omega_1$ such that $|\mathbf{x} - \mathbf{y}| \le r$. Then

$$u(\mathbf{x}) = \fint_{B(\mathbf{x},2r)} u d\mathbf{z} = \frac{1}{\alpha(n)2^n r^n} \int_{B(\mathbf{x},2r)} u d\mathbf{z} \ge \frac{1}{\alpha(n)2^n r^n} \int_{B(\mathbf{y},r)} u d\mathbf{z} = \frac{1}{2^n} \fint_{B(\mathbf{y},r)} u d\mathbf{z}$$

Harnack's Inequality

Proof (Continued):

$$\implies \frac{1}{2^n}u(\mathbf{y}) \leq u(\mathbf{x})$$

if $\mathbf{x}, \mathbf{y} \in \Omega_1$ and $|\mathbf{x} - \mathbf{y}| \leq r$. On the other hand,

$$u(\mathbf{x}) = \fint_{B(\mathbf{x},2r)} u d\mathbf{z} = \frac{1}{\alpha(n)2^n r^n} \int_{B(\mathbf{x},2r)} u d\mathbf{z} \leq 2^n \frac{1}{\alpha(n) r^n} \int_{B(\mathbf{y},r)} u d\mathbf{z} = 2^n \fint_{B(\mathbf{y},r)} u d\mathbf{z}$$

$$\implies u(\mathbf{x}) \leq 2^n u(\mathbf{y})$$

if $\mathbf{x}, \mathbf{y} \in \Omega_1$ and $|\mathbf{x} - \mathbf{y}| \leq r$. Hence

$$\implies \frac{1}{2^n}u(\mathbf{y}) \leq u(\mathbf{x}) \leq 2^n u(\mathbf{y})$$

if $\mathbf{x}, \mathbf{y} \in \Omega_1$ and $|\mathbf{x} - \mathbf{y}| \leq r$.

Harnack's Inequality

Proof (Continued): It is given that Ω_1 is connected and $\Omega_1 \subset\subset \Omega$. Since $\overline{\Omega}_1$ is compact, we can cover $\overline{\Omega}_1$ by a chain of finitely many balls $\{B_i\}_{i=1}^{N}$ each of which has radius r and $B_i \cap B_{i-1} \neq \phi$ for $i = 2, 3, \cdots, N$. Hence

$$\Longrightarrow \frac{1}{2^{nN}} u(\mathbf{y}) \leq u(\mathbf{x})$$

for all $\mathbf{x}, \mathbf{y} \in \Omega_1$. Applying the infimum and supremum definition, the proof follows.

Harnack's Inequality

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Remarks

1. We have that

$$\Longrightarrow \frac{1}{2^n}u(\mathbf{y}) \le u(\mathbf{x}) \le 2^n u(\mathbf{y})$$

2. In particular

$$\Longrightarrow \frac{1}{C}u(\mathbf{y}) \le u(\mathbf{x}) \le Cu(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \Omega_1$$

3. The value of a nonnegative harmonic function within Ω_1 is all comparable.
4. The value of u can't be very small or very large at any point of Ω_1 unless u is very small or very large everywhere in Ω_1.

Harnack's Inequality

Remarks

1. Harnack's inequality says that a positive harmonic function cannot oscillate wildly — it has a form of "internal smoothness" or balance.
2. If u represents a steady-state temperature field in a metal plate (harmonic because it satisfies Laplace's equation), then inside any region, the hottest and coldest points are not arbitrarily far apart — their ratio is controlled.
3. Provides control between max and min values — implies local regularity.
4. Imagine zooming into any small ball inside the domain where u is harmonic. The graph of u within that ball looks "flat-ish" compared to arbitrary continuous functions, that is, no sudden spike or dip in values.

Distribution

Green's Function

The Laplace Equation solution method is yet to be covered with Green's Function. However, as we have seen in the remarks, we have used a Dirac measure. In a line of proof, we have mentioned that

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} -\Delta_x \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^n} \delta_{\mathbf{x}} f(\mathbf{y}) d\mathbf{y}$$

which arises from the distribution theory. A similar line of proof is required in Green's Function, for which a basic knowledge of distribution is required. Let us see the distribution function and then have a look at it. Also, the knowledge of distribution helps you with the weak derivative in the later part of the course.

Test Functions

Definition 1 (Support)
Let $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, we define the support as

$$\text{support}(\phi) = \overline{\{\mathbf{x} \in \Omega : \phi(\mathbf{x}) \neq 0\}}$$

Here closure is w.r.to Ω

Definition 2 (Compact Support)
We say a function $\phi : \mathbb{R}^n \to \mathbb{R}$ has a compact support if $\phi \equiv 0$ outside a closed and bounded set in $\mathbb{R}^n$.

Definition 3 (Test Function)
We say ϕ is a test function if ϕ is an infinitely differentiable function with a compact support.

Distribution

Let $\mathcal{D}$ denote the set of all test functions. It is also denoted as $C_c^\infty(\mathbb{R}^n)$. Also

$$\mathcal{D}(\Omega) = C_c^\infty(\Omega)$$

Definition 4 (Distribution)
We say $F : \mathcal{D} \to \mathbb{R}$ is a distribution if F is a continuous and linear functional. (That is, it assigns a real number for every test function $\phi \in \mathcal{D}$). Let us denote the real number associated with this distribution as (F, ϕ).

Distribution

Example 2

Let $g : \mathbb{R} \to \mathbb{R}$ be any bounded function. We define the distribution associated with g as the map $F_g : \mathcal{D} \to \mathbb{R}$. F_g assigns a test function ϕ the real number

$$(F_g, \phi) = \int_{-\infty}^{\infty} g(x)\phi(x)dx$$

Distribution

Example 3

The Heaviside function is defined as

$$H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The distribution associated with g as the map $F_H : \mathcal{D} \to \mathbb{R}$. F_g assigns a test ϕ the real number

$$(F_H, \phi) = \int_0^\infty \phi(x)dx$$

Dirac Delta Function

There are many ways to define the Dirac Delta function

1. A generalized function from the limit of a class of delta sequences
2. Derivative of the Heaviside step function
$$\frac{d}{dx}[H(x)] = \delta(x)$$
3. A function on the real line which is zero except at the origin, where it is infinite
4. But, it is not a function in the traditional sense
$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{1}$$

such that

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Distribution

Example 4

The delta function δ_0 (not a function!) is the distribution $\delta_0 : \mathcal{D} \to \mathbb{R}$ assigns a test function ϕ the real number $\phi(0)$

$$(\delta_0, \phi) = \phi(0)$$

Sometimes we write

$$\int_{\mathbb{R}^n} \delta_0(\mathbf{x})\phi(\mathbf{x})d\mathbf{x} = \phi(\mathbf{0})$$

However, it is rather informal and not accurate, as $\delta_0(x)$ is not a function. Purely notational.

Distribution

We can also define the delta function centered at a point other than 0.

$$\delta_x(y) = \delta(x - y) = \begin{cases} +\infty & x = y \\ 0 & x \neq y \end{cases} \tag{2}$$

For a fixed $\mathbf{x} \in \mathbb{R}^n$, we define $\delta_{\mathbf{x}} : \mathcal{D} \to \mathbb{R}$ to be the distribution which assigns to a test function the real number $\phi(\mathbf{x})$

$$(\delta_{\mathbf{x}}, \phi) = \phi(\mathbf{x})$$

Dirac Delta Function Properties

There are three main properties for the Dirac Delta function

1. $\delta_x = \delta(y - x) = 0, x \neq y$
2. $\displaystyle\int_{x-\epsilon}^{x+\epsilon} \delta(y - x)dy = 1$
3. $\displaystyle\int_{x-\epsilon}^{x+\epsilon} f(y)\delta(y - x)dy = f(x)$

Derivatives of Distributions

Definition 5 (Derivative of the Distribution)
Let $F : \mathcal{D} \to \mathbb{R}$ be a distribution. We define the derivative of the distribution F as the distribution $G : \mathcal{D} \to \mathbb{R}$ such that

$$(G, \phi) = -(F, \phi')$$

for all $\phi \in \mathcal{D}$. If we denote $F' = G$, then

$$(F', \phi) = -(F, \phi')$$

for all $\phi \in \mathcal{D}$.

Derivatives of Distributions

Example 5

Let $g : \mathbb{R} \to \mathbb{R}$ be any bounded function and $g \in C^1(\mathbb{R})$. $F_g : \mathcal{D} \to \mathbb{R}$.

$$(F_g, \phi) = \int_{\infty}^{\infty} g(x)\phi(x)dx$$

Therefore, by integration by parts,

$$(F_g, \phi') = \int_{-\infty}^{\infty} g(x)\phi'(x)dx = -\int_{-\infty}^{\infty} g'(x)\phi(x)dx = -(F_g', \phi)$$

$$\implies (F_g', \phi) = -(F_g, \phi') = \int_{-\infty}^{\infty} g'(x)\phi(x)dx$$

Derivatives of Distributions

Example 6

For the Heaviside Function also, the derivative of F_H, denoted F_H', must satisfy

$$
\begin{aligned}
(F_H', \phi) &= -(F_H, \phi') \\
&= -\int_0^\infty \phi'(x)dx = -\lim_{b\to\infty}\int_0^b \phi'(x)dx \\
&= -\lim_{b\to\infty}[\phi(x)]_{x=0}^{x=b} \\
&= \phi(0) = (\delta_0, \phi)
\end{aligned}
$$

The derivative of the distribution associated with the Heaviside function is the delta function.

Derivatives of Distributions

Example 7

For Dirac Delta Function also, the derivative of δ_0, denoted δ_0' must satisfy

$$(\delta_0', \phi) = -(\delta_0, \phi') = -\phi'(0)$$

$$\phi(x) = x \implies (\delta_0', \phi) = -1$$

$$\phi(x) = 4x^2 - 1 \implies (\delta_3', \phi) = -\phi'(3) = -24$$

$$\phi(x) = (x-a)^n \implies (\delta_a^{(n)}, \phi) = (-1)^n \phi^{(n)}(a) = (-1)^n n!$$

Convergence of Distributions

Definition 6 (Weak Convergence)

Let $F_n : \mathcal{D} \to \mathbb{R}$ be a sequence of distributions. We say F_n converges weakly to F if

$$(F_n, \phi) \to (F, \phi)$$

for all $\phi \in \mathcal{D}$.

Example 8

$$\eta_\epsilon(\mathbf{x}) = \frac{1}{\epsilon}^n \eta\left(\frac{|\mathbf{x}|}{\epsilon}\right)$$

Then $\eta_\epsilon \xrightarrow{w} \delta_0$

Convergence of Distributions

Example 9

$$F_n(x) = \sin(nx) \xrightarrow{w} 0$$

We denote the set of all distributions $F : \mathcal{D} \to \mathbb{R}$ as $\mathcal{D}'$. Suppose $u' = 0$ in $\mathcal{D}'(\mathbb{R})$. Then $u =$ constant in $\mathcal{D}'(\mathbb{R})$

A few Theorems (without Proof)

Theorem 10

1. Let F be a continuous function satisfying $(F, \phi) = 0$ for all $\phi \in \mathcal{D}$. Then $F \equiv 0$.
2. Suppose $u \in \mathcal{D}'$, then there exists a sequence F_n in $\mathcal{D}$ such that $F_n \xrightarrow{w} u$ in $\mathcal{D}'$. (Density Theorem)
3. If $g \in L^1_{loc}(\Omega)$,then the function $f\phi$ is integrable for any $\phi \in \mathcal{D}(\Omega)$ and $F_g \in \mathcal{D}'$.
4. If $g \in C^1(\mathbb{R})$, then $F_g' = F_{g'}$
5. If $F \in \mathcal{D}'(\Omega)$, there exists $G \in \mathcal{D}'(\Omega)$ such that $G' = F$.

Thanks

Doubts and Suggestions

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