# MA612L-Partial Differential Equations

## Lecture 25: Laplace Equation - Green's Function

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# Recap

# Recap

**Definition 1 (Test Function)**
We say $\phi$ is a test function if $\phi$ is an infinitely differentiable function with a compact support.

$$D(\Omega) = C_c^\infty(\Omega)$$

**Definition 2 (Distribution)**
We say $F : \mathcal{D} \to \mathbb{R}$ is a distribution if $F$ is a continuous and linear functional. (That is, it assigns a real number for every test function $\phi \in \mathcal{D}$). Let us denote the real number associated with this distribution as $(F, \phi)$.

**Definition 3 (Derivative of the Distribution)**
Let $F : \mathcal{D} \to \mathbb{R}$ be a distribution. We define the derivative of the distribution $F$ as the distribution $G : \mathcal{D} \to \mathbb{R}$ such that $(G, \phi) = -(F, \phi') \;\; \forall \phi \in \mathcal{D}$.

**Definition 4 (Weak Convergence)**
Let $F_n : \mathcal{D} \to \mathbb{R}$ be a sequence of distributions. We say $F_n$ converges weakly to $F$ if $(F_n, \phi) \to (F, \phi) \;\; \forall \phi \in \mathcal{D}$.

# A few Theorems (without Proof)

### Theorem 1

1. Let $F$ be a continuous function satisfying $(F, \phi) = 0$ for all $\phi \in \mathcal{D}$. Then $F \equiv 0$.
2. Suppose $u \in \mathcal{D}'$, then there exists a sequence $F_n$ in $\mathcal{D}$ such that $F_n \xrightarrow{w} u$ in $\mathcal{D}'$. (Density Theorem)
3. If $g \in L^1_{loc}(\Omega)$,then the function $f\phi$ is integrable for any $\phi \in \mathcal{D}(\Omega)$ and $F_g \in \mathcal{D}'$.
4. If $g \in C^1(\mathbb{R})$, then $F_g' = F_{g'}$
5. If $F \in \mathcal{D}'(\Omega)$, there exists $G \in \mathcal{D}'(\Omega)$ such that $G' = F$.

# Green's Function

# By Parts (Recall)

Remember! The following results in 1D

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du = \int_a^b d(uv) - \int_a^b v\,du$$

$$\int_\Omega u\,dv = \int_\Omega d(uv) - \int_\Omega v\,du = [uv]_{\partial\Omega} - \int_a^b v\,du$$

$$\int_\Omega u d^2 v = \int_\Omega u d(dv) = \int_\Omega d(udv) - \int_\Omega du dv$$

$$= \int_\Omega d(udv) - \int_\Omega d(vdu) + \int_\Omega d^2 u v$$

# By Parts (Recall)

Remember! The following result in bounded $\Omega \subset \mathbb{R}$

$$\int_{\Omega} [ud^2v - vd^2u] = \int_{\Omega} d(udv - vdu)$$

Remember! The following result in bounded $\Omega \subset \mathbb{R}^n$

$$\int_{\Omega} [u\nabla^2 v - v\nabla^2 u] d\mathbf{x} = \int_{\Omega} \nabla.(u\nabla v - v\nabla u) d\mathbf{x} = \int_{\partial\Omega} \nu.(u\nabla v - v\nabla u) dS$$

The last expression is due to Gauss's Divergence Theorem

$$\int_{\Omega} \nabla.F d\mathbf{x} = \int_{\partial\Omega} F.\nu dS$$

# By Parts (Recall)

Remember! The following results in bounded $\Omega \subset \mathbb{R}^n$

$$\int\limits_{\Omega} [u\nabla^2 v - v\nabla^2 u] d\mathbf{x} = \int\limits_{\partial\Omega} \left( u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS \tag{1}$$

$$\int\limits_{\Omega} u\nabla^2 v d\mathbf{x} = \int\limits_{\Omega} v\nabla^2 u d\mathbf{x} + \int\limits_{\partial\Omega} \left( u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS \tag{2}$$

$$-\int\limits_{\Omega} v\nabla^2 u d\mathbf{x} = -\int\limits_{\Omega} u\nabla^2 v d\mathbf{x} + \int\limits_{\partial\Omega} \left( u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS \tag{3}$$

# Green's Function

$$y'' + p(x)y' + q(x)y = r(x), y'(x_0) = y_0, x \geq x_0 \tag{4}$$

$$y_p(x) = \int\limits_{x_0}^{x} G(x,t)r(t)dt \tag{5}$$

where the function $G(x,t)$ is called the Green's function.
The Green's function for an initial value problem is given by

$$\boxed{G(x,t) = \frac{y_1(t)y_2(x) - y_2(t)y_1(x)}{W(t)}} \tag{6}$$

# Green's Function ODE

$$y'' + p(x)y' + q(x)y = r(x), y(a) = y_0, y(b) = y_1, x \in [a, b] \tag{7}$$

The following Green's function works for the boundary value problem.

$$y_p(x) = \int_a^b G(x,t)r(t)dt \tag{8}$$

$$\boxed{G(x,t) = \begin{cases} \dfrac{y_1(t)y_2(x)}{W(t)} & a \leq t \leq x \\ \dfrac{y_2(t)y_1(x)}{W(t)} & x \leq t \leq b \end{cases}} \tag{9}$$

# Motivation

Now, let us find a general representation formula for the solution of Poisson's equation

$$\begin{cases} -\Delta u = h & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega \end{cases} \tag{10}$$

Let $u$ be a solution of (10). Then

$$u(\mathbf{x}) = \int_{\Omega} \delta_{\mathbf{x}} u(\mathbf{y}) d\mathbf{y}$$

# Green's Function Motivation

Fix $\mathbf{x} \in \Omega$. Suppose we can solve the problem

$$\begin{cases} -\Delta_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x}} & \mathbf{y} \in \Omega \\ G(\mathbf{x}, \mathbf{y}) = 0 & \mathbf{y} \in \partial\Omega \end{cases} \tag{11}$$

**Theorem 2 (Representation Formula using Green's Function)**
If $u \in C^2(\overline{\Omega})$ solves the problem (10), then

$$u(x) = \int\limits_{\Omega} h(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\mathbf{y} - \int\limits_{\partial\Omega} f(\mathbf{y}) \frac{\partial G}{\partial \nu}(\mathbf{x}, \mathbf{y}) dS(\mathbf{y}) \tag{12}$$

where $G$ is called Green's function.

Let us obtain $G$.

# Green's Function Motivation

**Proof:**

$$
\begin{aligned}
u(\mathbf{x}) &= \int\limits_{\Omega} u(\mathbf{y})\delta_{\mathbf{x}} d\mathbf{y} \\
&= -\int\limits_{\Omega} \Delta_{\mathbf{y}} G(\mathbf{x},\mathbf{y}) u(\mathbf{y}) d\mathbf{y} \quad \text{(Use (2) with } u = G, v = u) \\
&= -\int\limits_{\Omega} G(\mathbf{x},\mathbf{y}) \overbrace{\Delta_{\mathbf{y}} u(\mathbf{y})}^{h(\mathbf{y})} d\mathbf{y} + \int\limits_{\partial\Omega} \overbrace{G(\mathbf{x},\mathbf{y})}^{0} \frac{\partial u}{\partial \nu}(\mathbf{y}) dS(\mathbf{y}) - \int\limits_{\partial\Omega} \frac{\partial G}{\partial \nu}(\mathbf{x},\mathbf{y}) u(\mathbf{y}) dS(\mathbf{y}) \\
&= \int\limits_{\Omega} G(\mathbf{x},\mathbf{y}) h(\mathbf{y}) d\mathbf{y} - \int\limits_{\partial\Omega} \frac{\partial G}{\partial \nu}(\mathbf{x},\mathbf{y}) f(\mathbf{y}) dS(\mathbf{y})
\end{aligned}
$$

# Corrector Function

**Definition 5 (Corrector Function)**
Let $\phi^{\mathbf{x}} = \phi^{\mathbf{x}}(\mathbf{y})$ be the solution of the following Laplace equation

$$\begin{cases} -\Delta_{\mathbf{y}} \phi^{\mathbf{x}} = 0 & \mathbf{y} \in \Omega \\ \phi^{\mathbf{x}}(\mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x}) & \mathbf{y} \in \partial\Omega \end{cases} \tag{13}$$

The function $\phi^{\mathbf{x}}$ is called the corrector function.

Then using (2) with $u = \phi^{\mathbf{x}}$ and $v = u$

$$\int_{\Omega} \phi^{\mathbf{x}} \Delta_{\mathbf{y}} u(\mathbf{y}) d\mathbf{y} = \overbrace{\int_{\Omega} \Delta_{\mathbf{y}} \phi^{\mathbf{x}}(\mathbf{y}) u(\mathbf{y}) d\mathbf{y}}^{0} - \int_{\partial\Omega} \left[ \frac{\partial \phi^{\mathbf{x}}}{\partial \nu} u - \phi^{\mathbf{x}} \frac{\partial u}{\partial \nu} \right] dS(\mathbf{y})$$

# Corrector Function

Hence, we obtain that

$$0 = -\int_{\Omega} \phi^{\mathbf{x}} \Delta_{\mathbf{y}} u(\mathbf{y}) d\mathbf{y} + \int_{\partial\Omega} \left[ \Phi(\mathbf{y}-\mathbf{x}) \frac{\partial u}{\partial \nu} - \frac{\partial \phi^{\mathbf{x}}}{\partial \nu} u(\mathbf{y}) \right] dS(\mathbf{y}) \tag{14}$$

Now, let us evaluate the tricky part of the following integral

$$\int_{\Omega} \Phi(\mathbf{x}-\mathbf{y}) \Delta_{\mathbf{y}} u(\mathbf{y}) d\mathbf{y}$$

Since $\Phi(\mathbf{x}-\mathbf{y})$ has a singularity at $\mathbf{y} \neq \mathbf{x}$, we need to be careful while integrating it.

# Green's Function Motivation

Let $u \in C^2(\overline{\Omega})$ be an arbitrary function. Fix $x \in \Omega$, choose $\epsilon > 0$ such that $B(\mathbf{x}, \epsilon) \subset \Omega$. Consider the region $\Omega_\epsilon := \Omega - B(\mathbf{x}, \epsilon)$. Now, consider $u = \Phi(\mathbf{y} - \mathbf{x})$ and $v = u$ in (2), then

$$\int\limits_{\Omega_\epsilon} \Phi(\mathbf{y}-\mathbf{x}) \Delta_{\mathbf{y}} u(\mathbf{y}) d\mathbf{y} = \int\limits_{\Omega_\epsilon} \overbrace{\Delta_{\mathbf{y}} \Phi(\mathbf{y}-\mathbf{x})}^{0} u(\mathbf{y}) d\mathbf{y} - \int\limits_{\partial\Omega_\epsilon} \frac{\partial \Phi}{\partial \nu}(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) dS(\mathbf{y})$$

$$+ \int\limits_{\partial\Omega_\epsilon} \Phi(\mathbf{y}-\mathbf{x}) \frac{\partial u}{\partial \nu} dS(\mathbf{y})$$

$$\int\limits_{\Omega_\epsilon} \Phi(\mathbf{y}-\mathbf{x}) \Delta_{\mathbf{y}} u(\mathbf{y}) d\mathbf{y} = \int\limits_{\partial\Omega_\epsilon} \left[ \Phi(\mathbf{y}-\mathbf{x}) \frac{\partial u}{\partial \nu} - \frac{\partial \Phi}{\partial \nu}(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) \right] dS(\mathbf{y})$$

# Green's Function Motivation

Claim 1: (Follows immediately. )

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \Phi(\mathbf{y} - \mathbf{x}) \Delta_{\mathbf{y}} u(\mathbf{y}) d\mathbf{y} = \int_{\Omega} \Phi(\mathbf{y} - \mathbf{x}) \Delta_{\mathbf{y}} u(\mathbf{y}) d\mathbf{y}$$

Claim 2:

$$\lim_{\epsilon \to 0} \left[ - \int_{\partial \Omega_\epsilon} \frac{\partial \Phi}{\partial \nu} (\mathbf{y} - \mathbf{x}) u(y) dS(\mathbf{y}) \right] = - \int_{\partial \Omega} \frac{\partial \Phi}{\partial \nu} (\mathbf{y} - \mathbf{x}) u(\mathbf{y}) dS(\mathbf{y}) - u(\mathbf{x})$$

Claim 3:

$$\lim_{\epsilon \to 0} \int_{\partial \Omega_\epsilon} \Phi(\mathbf{y} - \mathbf{x}) \frac{\partial u}{\partial \nu} dS(\mathbf{y}) = \int_{\partial \Omega} \Phi(\mathbf{y} - \mathbf{x}) \frac{\partial u}{\partial \nu} dS(\mathbf{y})$$

# Green's Function Motivation

If we prove claim (2) and claim (3), then

$$\int_{\Omega_\epsilon} \Phi(\mathbf{y}-\mathbf{x})\Delta_{\mathbf{y}} u(\mathbf{y})d\mathbf{y} = -\int_{\partial\Omega_\epsilon} \frac{\partial \Phi}{\partial \nu}(\mathbf{y}-\mathbf{x})u(\mathbf{y})dS(\mathbf{y}) + \int_{\partial\Omega_\epsilon} \Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu}dS(\mathbf{y})$$

$$\int_{\Omega} \Phi(\mathbf{y}-\mathbf{x})\Delta_{\mathbf{y}} u(\mathbf{y})d\mathbf{y} = -\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(\mathbf{y}-\mathbf{x})u(y)dS(\mathbf{y}) - u(\mathbf{x}) + \int_{\partial\Omega} \Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu}dS(\mathbf{y})$$

Consequently

$$u(x) = -\int_{\Omega} \Phi(\mathbf{y}-\mathbf{x})\Delta_{\mathbf{y}} u(\mathbf{y})d\mathbf{y} + \int_{\partial\Omega} \left[\Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu} - \frac{\partial \Phi}{\partial \nu}(\mathbf{y}-\mathbf{x})u(y)\right] dS(\mathbf{y}) \tag{15}$$

# Green's Function

(15)-(14) yields

$$u(x) = -\int\limits_{\Omega}[\Phi(\mathbf{y}-\mathbf{x}) - \phi^{\mathbf{x}}(\mathbf{y})]\Delta_{\mathbf{y}} u(\mathbf{y})d\mathbf{y} + \int\limits_{\partial\Omega}\left[\frac{\partial \phi^{\mathbf{x}}}{\partial \nu} - \frac{\partial \Phi}{\partial \nu}(\mathbf{y}-\mathbf{x})\right] u(\mathbf{y})dS(\mathbf{y})$$
$$= -\int\limits_{\Omega}[\Phi(\mathbf{y}-\mathbf{x}) - \phi^{\mathbf{x}}(\mathbf{y})]\Delta_{\mathbf{y}} u(\mathbf{y})d\mathbf{y} - \int\limits_{\partial\Omega}\frac{\partial[\Phi(\mathbf{y}-\mathbf{x}) - \phi^{\mathbf{x}}]}{\partial \nu} u(\mathbf{y})dS(\mathbf{y})$$

**Definition 6 (Green's Function)**

Define the function $G(\mathbf{x},\mathbf{y})$ for given $\mathbf{x},\mathbf{y} \in \Omega$

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{y}-\mathbf{x}) - \phi^{\mathbf{x}} \tag{16}$$

This is called Green's function for the region $\Omega$

# Proof of Claim 2

Now, we are left with proving claims 2 and 3. For claim 2, reacall the $K_\epsilon$ proof from previous lectures. We need to prove

$$\lim_{\epsilon\to 0}\left[-\int\limits_{\partial\Omega_\epsilon}\frac{\partial\Phi}{\partial\nu}(\mathbf{y}-\mathbf{x})u(\mathbf{y})dS(\mathbf{y})\right] = -\int\limits_{\partial\Omega}\frac{\partial\Phi}{\partial\nu}(\mathbf{y}-\mathbf{x})u(\mathbf{y})dS(\mathbf{y}) - u(\mathbf{x})$$

Since

$$-\int\limits_{\partial\Omega_\epsilon}\frac{\partial\Phi}{\partial\nu}(\mathbf{y}-\mathbf{x})u(\mathbf{y})dS(\mathbf{y}) = -\int\limits_{\partial\Omega}\frac{\partial\Phi}{\partial\nu}(\mathbf{y}-\mathbf{x})u(\mathbf{y})dS(\mathbf{y}) + \int\limits_{\partial B(\mathbf{x},\epsilon)}\frac{\partial\Phi}{\partial\nu}(\mathbf{y}-\mathbf{x})u(\mathbf{y})dS(\mathbf{y}),$$

It is sufficient to prove.

$$\lim_{\epsilon\to 0}\int\limits_{\partial B(\mathbf{x},\epsilon)}\frac{\partial\Phi}{\partial\nu}(\mathbf{y}-\mathbf{x})u(\mathbf{y})dS(\mathbf{y}) = -u(\mathbf{x})$$

# Proof of Claim 2

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Now,

$$\nabla_{\mathbf{y}}\Phi(\mathbf{y}) = -\frac{1}{n\alpha(n)}\frac{\mathbf{y}}{|\mathbf{y}|^n} \quad \text{and} \quad \nu = \frac{\mathbf{y}-\mathbf{x}}{|\mathbf{y}-\mathbf{x}|}$$

$$\implies \frac{\partial\Phi}{\partial\nu}(\mathbf{y}-\mathbf{x}) = \nabla_{\mathbf{y}}\Phi(\mathbf{y}-\mathbf{x}).\nu = -\frac{1}{n\alpha(n)}\frac{1}{|\mathbf{y}-\mathbf{x}|^{n-1}} = -\frac{1}{n\alpha(n)}\frac{1}{\epsilon^{n-1}}$$

Hence,

$$\lim_{\epsilon\to 0}\int_{\partial B(\mathbf{x},\epsilon)}\frac{\partial\Phi}{\partial\nu}(\mathbf{y}-\mathbf{x})u(\mathbf{y})dS(\mathbf{y}) = -\lim_{\epsilon\to 0}\frac{1}{n\alpha(n)}\frac{1}{\epsilon^{n-1}}\int_{\partial B(\mathbf{x},\epsilon)}u(\mathbf{y})dS(\mathbf{y})$$

$$= -\lim_{\epsilon\to 0}⨍_{\partial B(\mathbf{x},\epsilon)}u(\mathbf{y})dS(\mathbf{y}) = -u(\mathbf{x})$$

# Proof of Claim 3

Now, we need to prove

$$\lim_{\epsilon\to 0}\int\limits_{\partial\Omega_\epsilon} \Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu}dS(\mathbf{y}) = \int\limits_{\partial\Omega} \Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu}dS(\mathbf{y})$$

$$\int\limits_{\partial\Omega_\epsilon} \Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu}dS(\mathbf{y}) = \int\limits_{\partial\Omega} \Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu}dS(\mathbf{y}) - \int\limits_{\partial B(\mathbf{x},\epsilon)} \Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu}dS(\mathbf{y}),$$

It is sufficient to prove that

$$\lim_{\epsilon\to 0}\int\limits_{\partial B(\mathbf{x},\epsilon)} \Phi(\mathbf{y}-\mathbf{x})\frac{\partial u}{\partial \nu}dS(\mathbf{y}) = 0$$

# Proof of Claim 3

This is similar to $L_\epsilon$ in our previous lecture. We have

$$\begin{aligned}
\left| \int_{\partial B(\mathbf{x},\epsilon)} \Phi(\mathbf{y}-\mathbf{x}) \frac{\partial u}{\partial \nu} dS(\mathbf{y}) \right| &\le \int_{\partial B(\mathbf{x},\epsilon)} |\Phi(\mathbf{y}-\mathbf{x})| \left| \frac{\partial u}{\partial \nu} \right| dS(\mathbf{y}) \\
&\le \left| \frac{\partial u}{\partial \nu} \right|_{L^\infty(B(\mathbf{x},\epsilon))} \int_{\partial B(\mathbf{x},\epsilon)} |\Phi(\mathbf{y}-\mathbf{x})| dS(\mathbf{y}) \\
&\le \begin{cases} C\epsilon|\log \epsilon| & (n=2) \\ C\epsilon & n>2 \end{cases}
\end{aligned}$$

Hence

$$\lim_{\epsilon \to 0} \int_{\partial B(\mathbf{x},\epsilon)} \Phi(\mathbf{y}-\mathbf{x}) \frac{\partial u}{\partial \nu} dS(\mathbf{y}) = 0$$

# Representation Formula using Green's Function

Therefore, we proved the following

$$\begin{cases} -\Delta u = h & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega \end{cases} \tag{17}$$

**Theorem 3 (Representation Formula using Green's Function)**
If $u \in C^2(\overline{\Omega})$ solves the problem (20), then

$$u(\mathbf{x}) = \int_{\Omega} h(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\mathbf{y} - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G}{\partial \nu}(\mathbf{x}, \mathbf{y}) dS(\mathbf{y}) \tag{18}$$

where $G$ is called Green's function.

# Representation Formula using Green's Function

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**Definition 7 (Green's Function)**

Define the function $G(\mathbf{x}, \mathbf{y})$ for given $\mathbf{x}, \mathbf{y} \in \Omega$

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x}) - \phi^{\mathbf{x}} \tag{19}$$

This is called Green's function for the region $\Omega$

# Representation Formula using Green's Function

We proved the following

$$\begin{cases} -\Delta u = h & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega \end{cases} \tag{20}$$

**Theorem 4 (Representation Formula using Green's Function)**
If $u \in C^2(\overline{\Omega})$ solves the problem (20), then

$$u(\mathbf{x}) = \int\limits_{\Omega} h(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\mathbf{y} - \int\limits_{\partial\Omega} f(\mathbf{y}) \frac{\partial G}{\partial \nu}(\mathbf{x}, \mathbf{y}) dS(\mathbf{y}) \tag{21}$$

where $G$ is called Green's function.

# Representation Formula using Green's Function

If we have $h \equiv 0$, then

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega \end{cases} \tag{22}$$

As a corollary

**Theorem 5 (Corollary)**
If $u \in C^2(\overline{\Omega})$ solves the problem (22), then

$$u(\mathbf{x}) = -\int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G}{\partial \nu}(\mathbf{x}, \mathbf{y}) dS(\mathbf{y}) \tag{23}$$

where $G$ is called Green's function.

# Symmetry of Green's Function

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**Theorem 6 (Symmetry)**

For all $\mathbf{x}, \mathbf{y} \in \Omega$ and $\mathbf{x} \neq \mathbf{y}$, we ave

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x}) \tag{24}$$

**Proof:** Fix, $\mathbf{x}, \mathbf{y} \in \Omega$ and $\mathbf{x} \neq \mathbf{y}$. Write

$$v(\mathbf{z}) := G(\mathbf{x}, \mathbf{z}), w(\mathbf{z}) := G(\mathbf{y}, \mathbf{z}), \quad \mathbf{z} \in \Omega$$

Then

$$\begin{cases} \Delta v(\mathbf{z}) = 0 & \mathbf{z} \neq \mathbf{x} \in \Omega \\ v = 0 & \mathbf{z} \in \partial\Omega \\ \Delta w(\mathbf{z}) = 0 & \mathbf{z} \neq \mathbf{y} \in \Omega \\ w = 0 & \mathbf{z} \in \partial\Omega \end{cases}$$

# Symmetry of Green's Function

**Proof (continued):** Claim:

$$v(\mathbf{y}) = G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x}) = w(\mathbf{x}), \quad \mathbf{x},\mathbf{y} \in \Omega$$

Fix $\epsilon > 0$ and consider $\Omega_\epsilon = \Omega \setminus B(\mathbf{x},\epsilon) \cup B(\mathbf{y},\epsilon)$. Then, by the identity which we proved earlier, we have

$$\int_{\Omega_\epsilon} [v\Delta w - w\Delta v] d\mathbf{z} = \int_{\partial B(\mathbf{x},\epsilon) \cup B(\mathbf{y},\epsilon)} \left( v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) dS$$

$$0 = \int_{\partial B(\mathbf{x},\epsilon)} \left( v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) dS + \int_{\partial B(\mathbf{y},\epsilon)} \left( v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) dS$$

$$\Longrightarrow \int_{\partial B(\mathbf{x},\epsilon)} \left( v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \right) dS = \int_{\partial B(\mathbf{y},\epsilon)} \left( w\frac{\partial v}{\partial \nu} - v\frac{\partial w}{\partial \nu} \right) dS$$

# Symmetry of Green's Function

**Proof (continued):** Now, consider $v(\mathbf{z}) = \Phi(\mathbf{z} - \mathbf{x}) - \phi^{\mathbf{x}}(\mathbf{z})$, then by Claims 2 and 3 of the previous theorem, we can show that

$$\int\limits_{\partial B(\mathbf{x},\epsilon)} \left( v \frac{\partial w}{\partial \nu} - w \frac{\partial v}{\partial \nu} \right) dS = w(\mathbf{x})$$

Similarly, $w(\mathbf{z}) = \Phi(\mathbf{z} - \mathbf{y}) - \phi^{\mathbf{y}}(\mathbf{z})$, then by Claims 2 and 3 of the previous theorem, we can show that

$$\int\limits_{\partial B(\mathbf{y},\epsilon)} \left( w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu} \right) dS = v(\mathbf{y})$$

Hence the proof.

# Thanks

**Doubts and Suggestions**

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