MA612L-Partial Differential Equations

Lecture 26: Laplace Equation - Green's Function for a Ball

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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Poisson's Kernel Formula

Green's Function for a Half-space

Now, let us build the Green's function for the half-space $\mathbb{R}^n_+$ and later for the unit ball $B(\mathbf{0}, 1)$. This depends on our corrector function in this region by clever geometric reflection tricks. Consider the half-space

$$\mathbb{R}^n_+ = \{\mathbf{x} = (x_1, x_2, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n : x_n > 0\}$$

Definition 1 (Reflection in the Plane)
If $\mathbf{x} = (x_1, x_2, \cdots, x_{n-1}, x_n) \in \mathbb{R}^n_+$, then its reflection in the plane $\partial\mathbb{R}^n_+$ is the point $\tilde{\mathbf{x}} = (x_1, x_2, \cdots, x_{n-1}, -x_n)$

Let us solve

$$\begin{cases} -\Delta_{\mathbf{y}} \phi^{\mathbf{x}} = 0 & \mathbf{y} \in \mathbb{R}^n_+ \\ \phi^{\mathbf{x}}(\mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x}) & \mathbf{y} \in \partial\mathbb{R}^n_+ \end{cases} \tag{1}$$

Green's Function for a Half-space

Set

$$\phi^{\mathbf{x}}(\mathbf{y}) = \Phi(\mathbf{y} - \tilde{\mathbf{x}}) = \Phi(y_1 - x_1, y_2 - x_2, \cdots y_{n-1} - x_{n-1}, y_n + x_n)$$

Observe that if $y \in \partial\mathbb{R}^n_+$ then

$$\phi^{\mathbf{x}}(\mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x})$$

Definition 2 (Green's Functions for the Half-Space)
The Green's function for the half-space $\mathbb{R}^n_+$ is given by

$$G(\mathbf{x}, \mathbf{y}) := \Phi(\mathbf{y} - \mathbf{x}) - \Phi(\mathbf{y} - \tilde{\mathbf{x}}) \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+, \mathbf{x} \neq \mathbf{y} \tag{2}$$

Green's Function for a Half-space

Case $n \geq 3$: We have

$$\Phi(\mathbf{r}) = \frac{1}{n(n-2)\alpha(n)}|\mathbf{r}|^{2-n}, \quad \text{where } \mathbf{r} = \mathbf{y} - \mathbf{x}.$$

Then

$$\begin{aligned}
\frac{\partial}{\partial y_n}\Phi(\mathbf{y}-\mathbf{x}) &= \frac{1}{n(n-2)\alpha(n)}(2-n)|\mathbf{y}-\mathbf{x}|^{1-n}\frac{\partial|\mathbf{y}-\mathbf{x}|}{\partial y_n} \\
&= \frac{1}{n(n-2)\alpha(n)}(2-n)|\mathbf{y}-\mathbf{x}|^{1-n}\frac{y_n - x_n}{|\mathbf{y}-\mathbf{x}|} \\
&= -\frac{1}{n\alpha(n)}\,\frac{y_n - x_n}{|\mathbf{y}-\mathbf{x}|^n}.
\end{aligned}$$

Green's Function for a Half-space

Similarly,

$$\frac{\partial}{\partial y_n}\Phi(\mathbf{y}-\tilde{\mathbf{x}}) = -\frac{1}{n\alpha(n)}\frac{y_n + x_n}{|\mathbf{y}-\tilde{\mathbf{x}}|^n},$$

where $\tilde{\mathbf{x}} = (x_1, \ldots, x_{n-1}, -x_n)$.
Hence,

$$\boxed{\frac{\partial G}{\partial y_n}(\mathbf{x},\mathbf{y}) = \frac{\partial \Phi(\mathbf{y}-\mathbf{x})}{\partial y_n} - \frac{\partial \Phi(\mathbf{y}-\tilde{\mathbf{x}})}{\partial y_n} = -\frac{1}{n\alpha(n)}\left[\frac{y_n - x_n}{|\mathbf{y}-\mathbf{x}|^n} - \frac{y_n + x_n}{|\mathbf{y}-\tilde{\mathbf{x}}|^n}\right].}$$

Case $n = 2$: We have

$$\Phi(\mathbf{r}) = -\frac{1}{2\pi}\log|\mathbf{r}|, \quad \text{where } \mathbf{r} = \mathbf{y}-\mathbf{x}.$$

Green's Function for a Half-space

Then

$$\begin{aligned}\frac{\partial}{\partial y_n}\Phi(\mathbf{y}-\mathbf{x}) &= -\frac{1}{2\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|}\frac{\partial|\mathbf{y}-\mathbf{x}|}{\partial y_n}\\ &= -\frac{1}{2\pi}\frac{1}{|\mathbf{y}-\mathbf{x}|}\frac{y_n-x_n}{|\mathbf{y}-\mathbf{x}|}\\ &= -\frac{1}{2\pi}\frac{y_n-x_n}{|\mathbf{y}-\mathbf{x}|^2}.\end{aligned}$$

Since $\alpha(2)=\pi$, we can write this as

$$\frac{\partial}{\partial y_n}\Phi(\mathbf{y}-\mathbf{x}) = -\frac{1}{n\alpha(n)}\frac{y_n-x_n}{|\mathbf{y}-\mathbf{x}|^n} \quad \text{for } n=2.$$

Green's Function for a Half-space

Similarly,

$$\frac{\partial}{\partial y_n}\Phi(\mathbf{y}-\tilde{\mathbf{x}}) = -\frac{1}{2\pi}\frac{y_n + x_n}{|\mathbf{y}-\tilde{\mathbf{x}}|^2},$$

and therefore

$$\boxed{\frac{\partial G}{\partial y_n}(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\left[\frac{y_n - x_n}{|\mathbf{y}-\mathbf{x}|^2} - \frac{y_n + x_n}{|\mathbf{y}-\tilde{\mathbf{x}}|^2}\right] = -\frac{1}{n\alpha(n)}\left[\frac{y_n - x_n}{|\mathbf{y}-\mathbf{x}|^n} - \frac{y_n + x_n}{|\mathbf{y}-\tilde{\mathbf{x}}|^n}\right].}$$

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Green's Function for a Half-space

Therefore, we obtain that

$$\frac{\partial G}{\partial y_n}(\mathbf{x},\mathbf{y}) = \frac{\partial \Phi(\mathbf{y}-\mathbf{x})}{\partial y_n} - \frac{\partial \Phi(\mathbf{y}-\tilde{\mathbf{x}})}{\partial y_n}$$
$$= \frac{-1}{n\alpha(n)}\left[\frac{y_n - x_n}{|\mathbf{y}-\mathbf{x}|^n} - \frac{y_n + x_n}{|\mathbf{y}-\tilde{\mathbf{x}}|^n}\right]$$

If $\mathbf{y} \in \partial\mathbb{R}^n_+$, then

$$\frac{\partial G}{\partial \nu}(\mathbf{x},\mathbf{y}) = -\frac{\partial G}{\partial y_n}(\mathbf{x},\mathbf{y}) = -\frac{2x_n}{n\alpha(n)}\frac{1}{|\mathbf{x}-\mathbf{y}|^n}$$

If u solves the boundary value problem

$$\begin{cases} -\Delta u = 0 & \mathbf{y} \in \mathbb{R}^n_+ \\ u = f & \mathbf{y} \in \partial\mathbb{R}^n_+ \end{cases} \tag{3}$$

Green's Function for a Half-space

Then the representation formula

$$u(x) = \int_{\Omega} h(\mathbf{y})G(\mathbf{x},\mathbf{y})d\mathbf{y} - \int_{\partial\Omega} f(\mathbf{y})\frac{\partial G}{\partial \nu}(\mathbf{x},\mathbf{y})dS(\mathbf{y}) \tag{4}$$

becomes

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^n} dS(\mathbf{y}) \tag{5}$$

(5) is called Poisson's formula, and the function

$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^n} \quad \mathbf{x} \in \mathbb{R}^n_+, y \in \partial\mathbb{R}^n_+ \tag{6}$$

is called Poisson's Kernel for $\mathbb{R}^n_+$

Poisson's formula Half-space

Theorem 1 (Poisson's formula Half-space)
Assume $f \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and defined u by

$$u(x) = \frac{2x_n}{n\alpha(n)} \int\limits_{\partial\mathbb{R}^n_+} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^n} dS(\mathbf{y}) \tag{7}$$

Then

1. $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(R^n_+)$
2. $\Delta u = 0$ in $\mathbb{R}^n_+$
3. $\lim\limits_{\substack{\mathbf{x}\to\mathbf{x}^0 \\ \mathbf{x}\in\mathbb{R}^n_+}} u(\mathbf{x}) = f(\mathbf{x}^0)$

for each point $\mathbf{x}^0 \in \partial\mathbb{R}^n_+$.

Proof: Exercise

Green’s Function for a ball

Green's Function for a ball

Definition 3 (Point Dual)
If $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$, then

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2}$$

is called the point dual to $\mathbf{x}$ w.r.to $\partial B(\mathbf{0}, 1)$. The mapping $\mathbf{x} \to \tilde{\mathbf{x}}$ is inversion through the unit sphere $\partial B(\mathbf{0}, 1)$.

Fix $x \in B^{\mathbf{0}}(\mathbf{0}, 1)$. We need to find a corrector function to satisfy

$$\begin{cases} -\Delta_{\mathbf{y}} \phi^{\mathbf{x}} = 0 & \mathbf{y} \in B^{\mathbf{0}}(\mathbf{0}, 1) \\ \phi^{\mathbf{x}}(\mathbf{y}) = \Phi(\mathbf{y} - \mathbf{x}) & \mathbf{y} \in \partial B(\mathbf{0}, 1) \end{cases} \tag{8}$$

The corrector function is defined by

$$\phi^{\mathbf{x}}(\mathbf{y}) := \Phi(|\mathbf{x}|(\mathbf{y} - \tilde{\mathbf{x}}))$$

Green's Function for the unit ball

Therefore, we obtain that

Definition 4 (Green's Functions for the unit ball)
The Green's function for the unit ball is given by

$$G(\mathbf{x},\mathbf{y}) := \Phi(\mathbf{y}-\mathbf{x}) - \Phi(|\mathbf{x}|(\mathbf{y}-\tilde{\mathbf{x}})) \quad \mathbf{x},\mathbf{y} \in B(\mathbf{0},1), \mathbf{x} \neq \mathbf{y} \tag{9}$$

$$\begin{aligned}\frac{\partial G}{\partial y_i}(\mathbf{x},\mathbf{y}) &= \frac{\partial \Phi(\mathbf{y}-\mathbf{x})}{\partial y_i} - \frac{\partial \Phi(\mathbf{y}-\tilde{\mathbf{x}})}{\partial y_i} \\ &= \frac{1}{n\alpha(n)}\left[\frac{x_i - y_i}{|\mathbf{y}-\mathbf{x}|^n} + \frac{y_i|\mathbf{x}|^2 - x_i}{|\mathbf{y}-x|^n}\right]\end{aligned}$$

if $\mathbf{y} \in \partial B(\mathbf{0},1)$. Then

$$\frac{\partial G}{\partial \nu}(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} y_i \frac{\partial G}{\partial y_i}(\mathbf{x},\mathbf{y}) = -\frac{1}{n\alpha(n)}\frac{1-|\mathbf{x}|^2}{|\mathbf{x}-\mathbf{y}|^n}$$

Green's Function for the unit ball

Proof for:

$$\boxed{\frac{\partial G}{\partial y_i}(\mathbf{x},\mathbf{y}) = \frac{1}{n\alpha(n)}\left[\frac{x_i - y_i}{|\mathbf{y}-\mathbf{x}|^n} + \frac{y_i|\mathbf{x}|^2 - x_i}{|\mathbf{y}-\tilde{\mathbf{x}}|^n}\right].}$$

$$\Phi(\mathbf{r}) = \begin{cases} -\dfrac{1}{2\pi}\log|\mathbf{r}|, & n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)}|\mathbf{r}|^{2-n}, & n \geq 3, \end{cases} \qquad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|^2},$$

and

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{y}-\mathbf{x}) - \Phi\big(|\mathbf{x}|(\mathbf{y}-\tilde{\mathbf{x}})\big), \qquad \mathbf{x},\mathbf{y} \in B(\mathbf{0},1),\ \mathbf{x} \neq \mathbf{y}.$$

We compute the partial derivative w.r.t y_i. First set $\mathbf{r} = \mathbf{y} - \mathbf{x}$ and $\mathbf{s} = |\mathbf{x}|(\mathbf{y}-\tilde{\mathbf{x}})$.

Green's Function for the unit ball

By the chain rule,

$$\frac{\partial}{\partial y_i}\Phi(\mathbf{y}-\mathbf{x}) = \big(\nabla\Phi(\mathbf{r})\big)_i, \qquad \frac{\partial}{\partial y_i}\Phi\big(|\mathbf{x}|(\mathbf{y}-\tilde{\mathbf{x}})\big) = \big(\nabla\Phi(\mathbf{s})\big)\cdot\frac{\partial \mathbf{s}}{\partial y_i}.$$

Using the well-known gradient of the fundamental solution (valid for both cases),

$$\nabla\Phi(\mathbf{z}) = -\frac{1}{n\alpha(n)}\frac{\mathbf{z}}{|\mathbf{z}|^n}, \qquad \text{(with } \alpha(2)=\pi\text{)},$$

we get

$$\frac{\partial}{\partial y_i}\Phi(\mathbf{y}-\mathbf{x}) = -\frac{1}{n\alpha(n)}\frac{y_i - x_i}{|\mathbf{y}-\mathbf{x}|^n}.$$

Green's Function for the unit ball

For the second term note $\mathbf{s} = |\mathbf{x}|(\mathbf{y} - \tilde{\mathbf{x}})$, hence $\dfrac{\partial \mathbf{s}}{\partial y_i} = |\mathbf{x}|\,\mathbf{e}_i$. Thus

$$
\begin{aligned}
\frac{\partial}{\partial y_i}\Phi\big(|\mathbf{x}|(\mathbf{y} - \tilde{\mathbf{x}})\big) &= \big(\nabla\Phi(\mathbf{s})\big)_j \left(|\mathbf{x}|\,\delta_{ji}\right) = |\mathbf{x}|\big(\nabla\Phi(\mathbf{s})\big)_i \\
&= |\mathbf{x}| \left(-\frac{1}{n\alpha(n)} \frac{s_i}{|\mathbf{s}|^n} \right) = -\frac{1}{n\alpha(n)}\,|\mathbf{x}|\,\frac{|\mathbf{x}|(y_i - \tilde{x}_i)}{\big(|\mathbf{x}|\,|\mathbf{y} - \tilde{\mathbf{x}}|\big)^n} \\
&= -\frac{1}{n\alpha(n)}\,|\mathbf{x}|^{2-n}\,\frac{y_i - \tilde{x}_i}{|\mathbf{y} - \tilde{\mathbf{x}}|^n}.
\end{aligned}
$$

Green's Function for the unit ball

Subtracting the two contributions,

$$\frac{\partial G}{\partial y_i}(\mathbf{x},\mathbf{y}) = -\frac{1}{n\alpha(n)}\frac{y_i - x_i}{|\mathbf{y}-\mathbf{x}|^n} - \left(-\frac{1}{n\alpha(n)}|\mathbf{x}|^{2-n}\frac{y_i - \tilde{x}_i}{|\mathbf{y}-\tilde{\mathbf{x}}|^n}\right)$$

$$= \frac{1}{n\alpha(n)}\left[\frac{x_i - y_i}{|\mathbf{y}-\mathbf{x}|^n} + |\mathbf{x}|^{2-n}\frac{y_i - \tilde{x}_i}{|\mathbf{y}-\tilde{\mathbf{x}}|^n}\right].$$

Finally substitute $\tilde{x}_i = \dfrac{x_i}{|\mathbf{x}|^2}$. Then

$$|\mathbf{x}|^{2-n}(y_i - \tilde{x}_i) = |\mathbf{x}|^{2-n}\left(y_i - \frac{x_i}{|\mathbf{x}|^2}\right) = \frac{y_i|\mathbf{x}|^2 - x_i}{|\mathbf{x}|^n}.$$

Therefore, the formula becomes

$$\boxed{\frac{\partial G}{\partial y_i}(\mathbf{x},\mathbf{y}) = \frac{1}{n\alpha(n)}\left[\frac{x_i - y_i}{|\mathbf{y}-\mathbf{x}|^n} + \frac{y_i|\mathbf{x}|^2 - x_i}{|\mathbf{x}|^n\,|\mathbf{y}-\tilde{\mathbf{x}}|^n}\right].}$$

$$|\mathbf{x}|^2\,|\mathbf{y}-\tilde{\mathbf{x}}|^2 = |\mathbf{x}|^2\left\langle \mathbf{y}-\frac{\mathbf{x}}{|\mathbf{x}|^2},\, \mathbf{y}-\frac{\mathbf{x}}{|\mathbf{x}|^2}\right\rangle = |\mathbf{x}|^2\left(|\mathbf{y}|^2 - 2\frac{\mathbf{y}\cdot\mathbf{x}}{|\mathbf{x}|^2} + \frac{|\mathbf{x}|^2}{|\mathbf{x}|^4}\right)$$

$$= |\mathbf{x}|^2\left(1 - 2\frac{\mathbf{y}\cdot\mathbf{x}}{|\mathbf{x}|^2} + \frac{1}{|\mathbf{x}|^2}\right) = |\mathbf{x}|^2 - 2\,\mathbf{y}\cdot\mathbf{x} + 1.$$

On the other hand,

$$|\mathbf{y}-\mathbf{x}|^2 = |\mathbf{y}|^2 - 2\,\mathbf{y}\cdot\mathbf{x} + |\mathbf{x}|^2 = 1 - 2\,\mathbf{y}\cdot\mathbf{x} + |\mathbf{x}|^2 = |\mathbf{x}|^2\,|\mathbf{y}-\tilde{\mathbf{x}}|^2,$$

$$\Longrightarrow\ |\mathbf{y}-\mathbf{x}| = |\mathbf{x}|\,|\mathbf{y}-\tilde{\mathbf{x}}|.\ \Longrightarrow\ |\mathbf{y}-\mathbf{x}|^n = |\mathbf{x}|^n\,|\mathbf{y}-\tilde{\mathbf{x}}|^n,$$

$$\boxed{\frac{\partial G}{\partial y_i}(\mathbf{x},\mathbf{y}) = \frac{1}{n\alpha(n)}\left[\frac{x_i - y_i}{|\mathbf{y}-\mathbf{x}|^n} + \frac{y_i|\mathbf{x}|^2 - x_i}{|\mathbf{y}-\mathbf{x}|^n}\right].}$$

Poisson's formula for the unit ball

If u solves the boundary value problem

$$\begin{cases} -\Delta u = 0 & \mathbf{y} \in B^{\mathbf{0}}(\mathbf{0}, 1) \\ u = f & \mathbf{y} \in \partial B(\mathbf{0}, 1) \end{cases} \tag{10}$$

Then the representation formula

$$u(x) = \int_{\Omega} h(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\mathbf{y} - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G}{\partial \nu}(\mathbf{x}, \mathbf{y}) dS(\mathbf{y}) \tag{11}$$

becomes

$$u(x) = \frac{1 - |\mathbf{x}|^2}{n\alpha(n)} \int_{\partial B(\mathbf{0},1)} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} dS(\mathbf{y}) \tag{12}$$

(12) is called Poisson's formula for the unit ball

Poisson's formula the unit ball

If we define $r > 0$ and u solves the boundary value problem

$$\begin{cases} -\Delta u = 0 & \mathbf{y} \in B^{\mathbf{0}}(\mathbf{0}, r) \\ u = f & \mathbf{y} \in \partial B(\mathbf{0}, r) \end{cases} \tag{13}$$

$$u(x) = \frac{r^2 - |\mathbf{x}|^2}{n\alpha(n)} \int\limits_{\partial B(\mathbf{0},r)} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} dS(\mathbf{y}) \tag{14}$$

(14) is called Poisson's formula for the ball $B(\mathbf{0}, r)$ and the function

$$K(x, y) = \frac{r^2 - |\mathbf{x}|^2}{n\alpha(n)} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} \quad \mathbf{x} \in B^{\mathbf{0}}(\mathbf{0}, r), y \in \partial B(\mathbf{0}, r) \tag{15}$$

is called Poisson's Kernel for the ball $B(\mathbf{0}, r)$

Poisson's formula for the ball $B(0, r)$

Theorem 2 (Poisson's formula Half-space)
Assume $f \in C(\partial B(\mathbf{0}, r))$ and defined u by

$$u(x) = \frac{r^2 - |\mathbf{x}|^2}{n\alpha(n)} \int\limits_{\partial B(\mathbf{0},r)} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^n} dS(\mathbf{y}) \tag{16}$$

Then

1. $u \in C^\infty(B^{\mathbf{0}}(\mathbf{0}, r))$
2. $\Delta u = 0$ in $B^{\mathbf{0}}(\mathbf{0}, r)$
3. $\lim\limits_{\substack{\mathbf{x} \to \mathbf{x}^0 \\ \mathbf{x} \in B^{\mathbf{0}}(\mathbf{0},r)}} u(\mathbf{x}) = f(\mathbf{x}^0)$

for each point $\mathbf{x}^0 \in \partial B(\mathbf{0}, r)$.

Proof: Exercise

Energy Methods

Uniqueness

So far, we have discussed the solution of the Laplace equations using fundamental solutions and Green's functions. Now, we show some energy methods in the L^2 norm. Consider the following boundary value problem

$$\begin{cases} -\Delta u = h & \text{in} \quad \Omega \\ u = f & \text{on} \quad \partial\Omega \end{cases} \tag{17}$$

We have already proved the uniqueness of this boundary value problem in our previous lectures with the maximum and minimum principles. Now, let us provide an alternative proof.

Theorem 3 (Uniqueness)
There exist at most one solution $u \in C^2(\overline{\Omega})$ of (17)

Uniqueness

Proof: Let u_1 and u_2 be two solutions of (17). Define $w = u_1 - u_2$. Then $\Delta w = 0 \in \Omega$ Hence

$$0 = -\int_{\Omega} w \Delta w d\mathbf{x} = \int_{\Omega} |Dw|^2 d\mathbf{x}$$

Hence $Dw \equiv 0$ in Ω. Therefore, w is constant in Ω. However, $w = 0$ on $\partial\Omega$. Therefore, $w = 0$ in Ω and hence the result.

Dirichlet's Principle

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Definition 5 (Admissible Set)

A set is said to be an admissible set if

$$\mathcal{A} = \{w \in C^2(\overline{\Omega}) : w = f \text{ on } \partial\Omega\} \tag{18}$$

Definition 6 (Energy Functional)

Define the energy functional as

$$I[w] = \int_{\Omega} \left(\frac{1}{2}|Dw|^2 - wh \right) d\mathbf{x} \tag{19}$$

where $w \in \mathcal{A}$.

Dirichlet's Principle

Theorem 4 (Dirichlet's Principle)
Assume $u \in C^2(\overline{\Omega})$ solve (17). Then

$$I[u] = \min_{w \in \mathcal{A}} I[w] \tag{20}$$

Conversely, if $u \in \mathcal{A}$ satisfies (20), then u solves the boundary value problem (17).

Proof: Choose $w \in \mathcal{A}$. Then (17) implies

$$0 = \int_\Omega (-\Delta u - h)(u - w)d\mathbf{x} = \int_\Omega (-\Delta u)(u - w)d\mathbf{x} - \int_\Omega h(u - w)d\mathbf{x}$$

By integration by parts, we get

$$0 = \int_\Omega Du.D(u - w)d\mathbf{x} - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(u - w)d\mathbf{x} - \int_\Omega h(u - w)d\mathbf{x}$$

Dirichlet's Principle

Proof (continued): Since $u = f$ on $\partial\Omega$ and $w \in \mathcal{A}$, $u - w = 0$ on $\partial\Omega$. Hence

$$0 = \int_\Omega Du.D(u-w)d\mathbf{x} - \int_\Omega h(u-w)d\mathbf{x}$$

$$0 = \int_\Omega Du.Dud\mathbf{x} - \int_\Omega Du.Dwd\mathbf{x} - \int_\Omega hud\mathbf{x} + \int_\Omega hwd\mathbf{x}$$

$$\implies \int_\Omega |Du|^2 d\mathbf{x} - \int_\Omega hud\mathbf{x} = \int_\Omega Du.Dwd\mathbf{x} - \int_\Omega hwd\mathbf{x}$$

Now, by Cauchy-Schwarz and Cauchy inequalities, we have

$$|Du.Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2$$

Dirichlet's Principle

Proof (continued):

$$\implies \int_\Omega |Du|^2 d\mathbf{x} - \int_\Omega hu d\mathbf{x} \leq \frac{1}{2}\int_\Omega |Du|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega |Dw|^2 d\mathbf{x} - \int_\Omega hu d\mathbf{x}$$

$$\implies \frac{1}{2}\int_\Omega |Du|^2 d\mathbf{x} - \int_\Omega hu d\mathbf{x} \leq \frac{1}{2}\int_\Omega |Dw|^2 d\mathbf{x} - \int_\Omega hu d\mathbf{x}$$

$$\implies I[u] \leq I[w], \quad w \in \mathcal{A}$$

$$u \in \mathcal{A} \implies I[u] = \min_{w \in \mathcal{A}} I[w]$$

Dirichlet's Principle

Proof (continued): Conversely suppose, $u \in \mathcal{A}$ satisfies

$$I[u] = \min_{w \in \mathcal{A}} I[w]$$

Claim: u solves the boundary value problem (17).
Fix any $v \in C_c^\infty(\Omega)$ and define

$$i(\tau) := I[u + \tau v], \quad \tau \in \mathbb{R}$$

Since $u + \tau v \in \mathcal{A}$ for each τ, $i(.)$ has a minimum at zero. That is, if i' exists, then $i'(0) = 0$. Then

$$\begin{aligned} i(\tau) &= \frac{1}{2}\int_\Omega |Du + \tau Dv|^2 d\mathbf{x} - \int_\Omega h(u + \tau v) d\mathbf{x} \\ &= \frac{1}{2}\int_\Omega |Du|^2 d\mathbf{x} + \int_\Omega \tau Dv.Dv d\mathbf{x} + \int_\Omega \frac{\tau^2}{2}|Dv|^2 d\mathbf{x} - \int_\Omega h(u + \tau v) d\mathbf{x} \end{aligned}$$

Dirichlet's Principle

Proof (continued): Now,

$$i'(\tau) = \int_\Omega Dv.Dv d\mathbf{x} + \int_\Omega \tau |Dv|^2 d\mathbf{x} - \int_\Omega hv d\mathbf{x}$$

$$= \int_\Omega (-\Delta u - h) v d\mathbf{x} + \int_\Omega \tau |Dv|^2 d\mathbf{x}$$

$$i'(0) = 0 \implies 0 = \int_\Omega (-\Delta u - h) v d\mathbf{x}$$

Since this is true for all each function $v \in C_c^\infty(\Omega)$, and by denseness of $C_c^\infty(\Omega)$, we have $-\Delta u = h$ in Ω

Dirichlet's Principle

Remarks

- If $u \in \mathcal{A}$, the PDE $-\Delta u = h$ is equivalent to the statement that u minimized the energy $I[.]$
- Dirichlet's principle is an instance of the **calculus of variations** applied to the Laplace equation.