MA612L-Partial Differential Equations

Lecture 27: Heat Equation - Fundamental Solution

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Heat Equation: Fundamental Solution

Recap

So far, we have studied

- Transport Equation
- Burger's Equation
- Wave Equation - d'Alembert's formula - Kirchhoff's formula - Poisson's formula
- Laplace Equation - Fundamental Solution - Mean Value Formula - Green's Function - Poisson's formula - Dirichlet's Principle

Now, let us study the heat equation with appropriate initial and boundary conditions

$$u_t - \Delta u = 0$$

and the inhomogeneous heat equation

$$u_t - \Delta u = f$$

Fundamental Solution

Theorem 1
If u is smooth and solve $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$, then $u_\lambda(\mathbf{x}, t) = u(\lambda \mathbf{x}, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$

Proof:

$$\frac{\partial u_\lambda(\mathbf{x}, t)}{\partial t} = \frac{\partial u}{\partial t} u(\lambda \mathbf{x}, \lambda^2 t) = \lambda^2 u_t$$

$$\frac{\partial^2 u_\lambda(\mathbf{x}, t)}{\partial x_i^2} = \frac{\partial u}{\partial t} u(\lambda \mathbf{x}, \lambda^2 t) = \lambda^2 u_{x_i x_i}$$

$$(u_\lambda)_t - \Delta u_\lambda = 0$$

Fundamental Solution

Corollary 1

$v(\mathbf{x},t) := \mathbf{x}.Du(\mathbf{x},t) + 2tu_t(\mathbf{x},t)$ solves the heat equation as well.

Proof: Differentiate u_λ w.r.to λ, we have

$$\frac{\partial u_\lambda}{\partial \lambda} = \mathbf{x}.Du(\lambda\mathbf{x},\lambda^2 t) + 2t\lambda u_t(\lambda\mathbf{x},\lambda^2 t)$$

If we set $\lambda = 1$, then

$$\frac{\partial u_\lambda}{\partial \lambda} = \mathbf{x}.Du(\mathbf{x},t) + 2tu_t(\mathbf{x},t) = v(\mathbf{x},t)$$

Since u_λ solves the heat equation

$$(u_\lambda)_t - \Delta u_\lambda = 0,$$

by differentiating it w.r.to λ proves the corollary. (Fill the missing argument!)

Fundamental Solution

Missing Argument:
Because u is smooth we may interchange ∂_λ with ∂_t and Δ. Denote

$$w_\lambda(\mathbf{x}, t) := \partial_\lambda u_\lambda(\mathbf{x}, t).$$

Then

$$\partial_\lambda \big[(\partial_t - \Delta) u_\lambda\big] = (\partial_t - \Delta)\big(\partial_\lambda u_\lambda\big) = (\partial_t - \Delta) w_\lambda = 0.$$

Thus w_λ satisfies the heat equation for every $\lambda > 0$.

Fundamental Solution

Theorem 2

If u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$, then $u_\lambda(\mathbf{x}, t) = \lambda^\alpha u(\lambda^\beta \mathbf{x}, \lambda t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$ for $\beta = \frac{1}{2}$.

Proof:

$$\frac{\partial u}{\partial t} u_\lambda(\mathbf{x}, t) = \lambda^{\alpha+1} u_t$$

$$\frac{\partial^2 u}{\partial x_i^2} u_\lambda(\mathbf{x}, t) = \lambda^{(\alpha+2\beta)} u_{x_i x_i}$$

$$(u_\lambda)_t - \Delta u_\lambda = 0$$

only if $\alpha + 2\beta = \alpha + 1 \implies \beta = \frac{1}{2}$

Fundamental Solution

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Theorem 3

Let $n = 1$ and $u(x,t) = v(\frac{x^2}{t})$. Then

$$u_t = u_{xx}$$

if and only if

$$4zv''(z) + (2+z)v'(z) = 0, z > 0 \tag{1}$$

Further, the general solution of (1) is

$$v(z) = c\int_0^z e^{-s/4}s^{-1/2}ds + d \tag{2}$$

Fundamental Solution

Proof:

$$u(x,t) = v\left(\frac{x^2}{t}\right) \implies u_t = -\frac{x^2}{t^2}v', u_x = \frac{2x}{t}v', u_{xx} = \frac{2}{t}v' + \frac{4x^2}{t^2}v''$$

$$u_t = u_{xx} \iff -\frac{x^2}{t^2}v' = \frac{2}{t}v' + \frac{4x^2}{t^2}v''$$

$$z = \frac{x^2}{t} \implies \frac{1}{t}(4zv''(z) + (2+z)v') = 0$$

Hence

$$u_t = u_{xx} \iff 4zv''(z) + (2+z)v'(z) = 0, z > 0$$

Now, let us solve $4zv''(z) + (2+z)v'(z) = 0$

$$\frac{v''}{v'} = -\frac{1}{2z} - \frac{1}{4} \implies \log(v') = -\log\sqrt{z} - \frac{z}{4} + C$$

Fundamental Solution

Proof:

$$v' = ce^{-z/4}\frac{1}{\sqrt{z}} \implies v(z) = c\int_0^z \frac{e^{-s/4}}{\sqrt{s}}ds + d$$

Hence the proof. From the above theorem, we have

$$v\left(\frac{x^2}{t}\right) = c\int_0^{\frac{x^2}{t}} e^{-s/4}s^{-1/2}ds + d$$

and

$$v'\left(\frac{x^2}{t}\right) = v'(z).dz = ce^{-x^2/4t}\frac{\sqrt{t}}{x}.\frac{2x}{t} = \frac{2c}{\sqrt{t}}e^{-x^2/4t}$$

Fundamental Solution

Now, let us choose c such that

$$\frac{2c}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-x^2/4t} dx = 1$$

Let $y = \frac{x}{2\sqrt{t}}$, then $dy = \frac{dx}{2\sqrt{t}}$. Using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we get

$$4c \int_{-\infty}^{\infty} e^{-y^2} dy = 1 \implies c = \frac{1}{4\sqrt{\pi}} \implies v'\left(\frac{x^2}{t}\right) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}$$

We define the fundamental solution as

$$\Phi(x,t) = v'\left(\frac{x^2}{t}\right) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}$$

Fundamental Solution

Fundamental Solution

It is easy to verify that

$$\Phi_t = \Phi_{xx}$$

Definition 1

For n-dimensional case, the function

$$\Phi(\mathbf{x}, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}} & x \in \mathbb{R}^n, t > 0 \\ 0 & x \in \mathbb{R}^n, t < 0 \end{cases}$$

is called the fundamental solution of the heat equation.

Derivation: Define

$$u(\mathbf{x}, t) = \frac{1}{t^\alpha} v\left(\frac{\mathbf{x}}{t^\beta}\right), \quad (\mathbf{x} \in \mathbb{R}^n, t > 0)$$

where we need to find constants α, β are constants and the function $v : \mathbb{R}^n \to \mathbb{R}$

Fundamental Solution

If we find a solution u of the heat equation invariant under dilation scaling, that is,

$$u(\mathbf{x}, t) = \lambda^{\alpha} u(\lambda^{\beta}\mathbf{x}, \lambda t)$$

for all $\mathbf{x} \in \mathbb{R}^n, t > 0, \lambda > 0$. If we set $\lambda = t^{-1}$, then

$$u(\mathbf{x}, t) = \frac{1}{t^{\alpha}} u\left(\frac{\mathbf{x}}{t^{\beta}}, 1\right)$$

Hence

$$\mathbf{y} = \frac{\mathbf{x}}{t^{\beta}} \implies v(\mathbf{y}) = u(\mathbf{y}, 1) \implies u(\mathbf{x}, t) = \frac{1}{t^{\alpha}} v(\mathbf{y})$$

Exercise: Prove that when $\beta = \frac{1}{2}, v(y)$ satisfies

$$\alpha v(\mathbf{y}) + \frac{1}{2}\mathbf{y}.Dv(\mathbf{y}) + \Delta v(\mathbf{y}) = 0$$

Fundamental Solution

Again assume that v to be radial, and $v(\mathbf{y}) = w(|\mathbf{y}|) = w(r)$. Hence we get

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0$$

Let $\alpha = \frac{n}{2}$, then it becomes

$$\frac{n}{2} w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0$$

Multiplying by r^{n-1}, we get

$$r^{n-1} \frac{n}{2} w + \frac{r^n}{2} w' + r^{n-1} w'' + r^{n-2}(n-1) w' = 0$$

$$\frac{1}{2}[n r^{n-1} w + r^n w'] + [r^{n-1} w'' + r^{n-2}(n-1) w'] = 0$$

Fundamental Solution

$$\frac{1}{2}[r^n w]' + [r^{n-1} w']' = 0 \implies \left[\frac{1}{2} r^n w + r^{n-1} w'\right]' = 0$$

$$\implies \frac{1}{2} r^n w + r^{n-1} w' = c$$

Assuming $\lim_{r\to\infty} w = 0$, $\lim_{r\to\infty} w' = 0$, we conclude that $c = 0$.

$$\implies w' = -\frac{r}{2} w \implies w = k e^{-r^2/4}$$

Hence

$$u(\mathbf{x}, t) = \frac{k}{t^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}$$

solves the heat equation.

Heat Equation: Homogeneous IVP

Fundamental Solution

Theorem 4

If we choose $k = \frac{1}{(4\pi)^{n/2}}$, then

$$\int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}, t) = 1$$

for each time $t > 0$

For,

$$\int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}, t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}|^2}{4t}} d\mathbf{x} = \frac{1}{(\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-|\mathbf{z}|^2} d\mathbf{z} = \frac{1}{(\pi)^{n/2}} \prod_{i=1}^{n} \int\limits_{\mathbb{R}} e^{-z_i^2} dz_i = 1$$

Fundamental Solution

Consider the following IVP

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = f & \text{in } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Then the solution of this equation is given by

$$u(\mathbf{x}, t) = \int\limits_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t) f(\mathbf{y}) d\mathbf{y}$$

$$u(\mathbf{x}, t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} f(\mathbf{y}) d\mathbf{y} \tag{3}$$

Heat Equation Solution IVP

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Theorem 5

Assume $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define u by (3). Then

1. $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$
2. $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$
3. $\displaystyle\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0)\\ \mathbf{x}\in\mathbb{R}^n,t>0}} u(\mathbf{x},t) = f(\mathbf{x}^0)$

In (3) for each point $\mathbf{x}^0 \in \mathbb{R}^n$

Heat Equation Solution IVP

Proof: Since the function $\frac{1}{(4\pi t)^{n/2}}e^{-\frac{|\mathbf{x}|^2}{4t}}$ is infinitely differentiable, with uniformly bounded derivatives of all orders on $\mathbb{R}^n \times (0,\infty)$. Hence, (1) Follows. Since Φ solves the heat equation, we have

$$u_t - \Delta u = \int\limits_{\mathbb{R}^n} \Phi_t - \Delta_x \Phi(\mathbf{x}-\mathbf{y},t) f(\mathbf{y}) d\mathbf{y}$$
$$= 0$$

Hence (2) follows.
Claim: $|u(\mathbf{x},t) - f(\mathbf{x}^0)| < \epsilon, \forall \epsilon > 0, \mathbf{x}^0 \in \mathbb{R}^n$ whenever $|\mathbf{x}-\mathbf{x}^0| < \delta, t > 0.$
Fix $\mathbf{x}^0 \in \mathbb{R}^n, \epsilon > 0$. Choose $\delta > 0$ such that

$$|g(\mathbf{y}) - g(\mathbf{x}^0)| < \frac{\epsilon}{2} \text{ whenever } |\mathbf{y}-\mathbf{x}^0| < \delta, \mathbf{y} \in \mathbb{R}^n$$

Heat Equation Solution IVP

Proof(continued):

$$
\begin{aligned}
u(\mathbf{x},t) - f(\mathbf{x}^0) &= \int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t) f(\mathbf{y})d\mathbf{y} - f(\mathbf{x}^0).1 \\
&= \int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t) f(\mathbf{y})d\mathbf{y} - f(\mathbf{x}^0)\int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t)d\mathbf{y} \\
&= \int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t)[f(\mathbf{y}) - f(\mathbf{x}^0)]d\mathbf{y}
\end{aligned}
$$

$$
\Longrightarrow \quad |u(\mathbf{x},t) - f(\mathbf{x}^0)| = \left|\int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t)[f(\mathbf{y}) - f(\mathbf{x}^0)]d\mathbf{y}\right|
$$

Heat Equation Solution IVP

Proof(continued): Let $|\mathbf{x}-\mathbf{x}^0|<\frac{\delta}{2}$, then

$$
\begin{aligned}
|u(\mathbf{x},t)-f(\mathbf{x}^0)| &\le \int\limits_{B(\mathbf{x}^0,\delta)} \Phi(\mathbf{x}-\mathbf{y},t)|f(\mathbf{y})-f(\mathbf{x}^0)|d\mathbf{y} \\
&\quad + \underbrace{\int\limits_{\mathbb{R}^n\setminus B(\mathbf{x}^0,\delta)} \Phi(\mathbf{x}-\mathbf{y},t)|f(\mathbf{y})-f(\mathbf{x}^0)|d\mathbf{y}}_{J} \\
&\le \int\limits_{B(\mathbf{x}^0,\delta)} \Phi(\mathbf{x}-\mathbf{y},t)\epsilon d\mathbf{y} + J = \epsilon \int\limits_{B(\mathbf{x}^0,\delta)} \Phi(\mathbf{x}-\mathbf{y},t)d\mathbf{y} + J \\
&\le \epsilon + J
\end{aligned}
$$

Heat Equation Solution IVP

Proof(continued): It is enough to claim $J \to 0$ as $t \to 0$. If $|\mathbf{y} - \mathbf{x}^0| \geq \delta$, then

$$\begin{aligned} |\mathbf{y} - \mathbf{x}^0| &= |\mathbf{y} - \mathbf{x} + \mathbf{x} - \mathbf{x}^0| \\ &\leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{x}^0| \\ &\leq |\mathbf{y} - \mathbf{x}| + \frac{\delta}{2} \\ &\leq |\mathbf{y} - \mathbf{x}| + \frac{1}{2}|\mathbf{y} - \mathbf{x}^0| \end{aligned}$$

$$\implies \frac{1}{2}|\mathbf{y} - \mathbf{x}^0| \leq |\mathbf{y} - \mathbf{x}|$$

$$J = \int\limits_{\mathbb{R}^n \setminus B(\mathbf{x}^0, \delta)} \Phi(\mathbf{x} - \mathbf{y}, t)|f(\mathbf{y}) - f(\mathbf{x}^0)|d\mathbf{y} \leq 2\|f\|_{L^\infty} \int\limits_{\mathbb{R}^n \setminus B(\mathbf{x}^0, \delta)} \Phi(\mathbf{x} - \mathbf{y}, t)d\mathbf{y}$$

Heat Equation Solution IVP

Proof(continued):

$$J \leq \frac{C}{t^{n/2}} \int\limits_{\mathbb{R}^n \setminus B(\mathbf{x}^0,\delta)} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} d\mathbf{y}$$

$$\leq \frac{C}{t^{n/2}} \int\limits_{\mathbb{R}^n \setminus B(\mathbf{x}^0,\delta)} e^{-\frac{|\mathbf{y}-\mathbf{x}^{\mathbf{0}}|^2}{16t}} d\mathbf{y}$$

$$\leq \frac{C}{t^{n/2}} \int\limits_{\delta}^{\infty} e^{-\frac{r^2}{16t}} r^{n-1} dr \to 0 \text{ as } t \to 0$$

Hence the proof.

Heat Equation Solution IVP

Remarks

- Notice that if f is bounded and continuous and $f \geq 0$, $f \not\equiv 0$, then $u(\mathbf{x}, t)$ is positive for all points $\mathbf{x} \in \mathbb{R}^n, t > 0$, since the integrand is positive

$$u(\mathbf{x}, t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} f(\mathbf{y}) d\mathbf{y} \tag{4}$$

- That is, the heat equation forces infinite propagation speed for disturbances.
- If the initial temperature is positive somewhere, then the temperature at a later time is everywhere positive. (For the wave equation, it is finite speed propagation).

Heat Equation Solution IVP

Remarks

Also, for Φ we can write

$$\begin{cases} \Phi_t - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi = \delta_0 & \text{in } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where δ_0 denotes the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0.

Thanks

Doubts and Suggestions

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