MA612L-Partial Differential Equations

Lecture 28: Heat Equation - Inhomogeneous IVP

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Heat Equation: Inhomogeneous IVP

Inhomogeneous Problem

Consider the following IVP

$$\begin{cases} u_t - \Delta u = h & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0 & \text{in } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{1}$$

Question

How can we produce a formula for the solution?

From the fundamental solution, we can observe that $\Phi(\mathbf{x}-\mathbf{y}, t-s)$ is a solution of the heat equation for $\mathbf{x} \in \mathbb{R}^n, 0 < s < t$.

Inhomogeneous Problem

For fixed s, the function

$$u = u(\mathbf{x}, t; s) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t - s) h(\mathbf{y}, s) d\mathbf{y} \tag{2}$$

solves

$$\begin{cases} u_t(.; s) - \Delta u(.; s) = 0 & \text{in } \mathbb{R}^n \times (s, \infty) \\ u(.; s) = h(., s) & \text{in } \mathbb{R}^n \times \{t = s\} \end{cases} \tag{3}$$

It is the same as IVP with starting time $t = s$ and $f = h(., s)$. Therefore, $u(.; s)$ is definitely not a solution of (1).

Question

Can we construct a solution for (1) using (3)?

Inhomogeneous Problem

Answer: Yes. Duhamel's principle. We have already seen this for the Wave equation. Define

$$u(\mathbf{x},t) = \int_0^t u(\mathbf{x},t;s)ds \tag{4}$$

That is,

$$u(\mathbf{x},t) = \int_0^t u(\mathbf{x},t;s)ds = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y}, t-s)h(\mathbf{y},s)d\mathbf{y}ds \tag{5}$$

$$= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-s)}} h(\mathbf{y},s)d\mathbf{y}ds \tag{6}$$

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So,

$$u(\mathbf{x},t) = \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-s)}} h(\mathbf{y},s)\,d\mathbf{y}\,ds \tag{7}$$

Theorem 1

Assume $h \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and h has a compact support. Define u by (7). Then

1. $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$
2. $u_t(\mathbf{x},t) - \Delta u(\mathbf{x},t) = h(\mathbf{x},t)$ in $\mathbb{R}^n \times (0,\infty)$
3. $\displaystyle\lim_{\substack{(\mathbf{x},t)\to(\mathbf{x}^0,0) \\ \mathbf{x}\in\mathbb{R}^n, t>0}} u(\mathbf{x},t) = 0$

In (3) for each point $\mathbf{x}^0 \in \mathbb{R}^n$

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Proof of Part 1: The proof is similar to what we have done in the last theorem and the work done in the Wave equation. Since Φ has a singularity at $(\mathbf{0}, 0)$, we can't find u_t and Δu easily. So, change the variables to write it as

$$u(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y}, s) h(\mathbf{x} - \mathbf{y}, t - s) d\mathbf{y} ds$$

As $h \in C_1^2(\mathbb{R}^n \times [0, \infty))$, h has a compact support and Φ is smooth near $t = s$.

$$u_t(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y}, s) h_t(\mathbf{x} - \mathbf{y}, t - s) d\mathbf{y} ds + \int_{\mathbb{R}^n} \Phi(\mathbf{y}, t) h(\mathbf{x} - \mathbf{y}, 0) d\mathbf{y} \tag{8}$$

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Proof of Part 1 (continued): Similarly

$$u_{x_i x_i}(\mathbf{x}, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y}, s) h_{x_i x_i}(\mathbf{x} - \mathbf{y}, t - s) d\mathbf{y} ds \tag{9}$$

Hence $u_t, D_x^2 u, u, D_x u \in C(\mathbb{R}^n \times [0, \infty))$. Hence (1) follows.

Heat Equation Solution nonhomogeneous IVP

Proof of Part 2 (continued): Now, let us claim that $u_t(\mathbf{x},t) - \Delta u(\mathbf{x},t) = h(\mathbf{x},t)$. From (8) and (9), we obtain that

$$u_t(\mathbf{x},t) - \Delta u(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y},s) h_t(\mathbf{x}-\mathbf{y},t-s)\,d\mathbf{y}\,ds$$

$$+ \int_{\mathbb{R}^n} \Phi(\mathbf{y},t) h(\mathbf{x}-\mathbf{y},0)\,d\mathbf{y}$$

$$- \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y},s) \Delta h(\mathbf{x}-\mathbf{y},t-s)\,d\mathbf{y}\,ds$$

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Proof of Part 2 (continued):

$$\underbrace{u_t(\mathbf{x},t) - \Delta u(\mathbf{x},t)}_{L} = \underbrace{\int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{y},s)\left(\frac{\partial}{\partial t} - \Delta_x\right) h(\mathbf{x}-\mathbf{y},t-s)\,d\mathbf{y}\,ds}_{I}$$

$$+ \underbrace{\int_{\mathbb{R}^n} \Phi(\mathbf{y},t) h(\mathbf{x}-\mathbf{y},0)\,d\mathbf{y}}_{K}$$

Therefore, $L = I + K = I_\epsilon + J_\epsilon + K$

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Proof of Part 2 (continued):

$$I = \underbrace{\int_0^{\epsilon} \int_{\mathbb{R}^n} \Phi(\mathbf{y}, s) \left(\frac{\partial}{\partial s} - \Delta_y \right) h(\mathbf{x} - \mathbf{y}, t - s) d\mathbf{y} ds}_{I_\epsilon}$$

$$+ \underbrace{\int_{\epsilon}^{t} \int_{\mathbb{R}^n} \Phi(\mathbf{y}, s) \left(\frac{\partial}{\partial s} - \Delta_y \right) h(\mathbf{x} - \mathbf{y}, t - s) d\mathbf{y} ds}_{J_\epsilon}$$

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Proof of Part 2 (continued): <u>**Claim:**</u> $|I_\epsilon| < \epsilon C$

$$|I_\epsilon| = \left| \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(\mathbf{y}, s) \left(\frac{\partial}{\partial s} - \Delta_y \right) h(\mathbf{x} - \mathbf{y}, t - s) d\mathbf{y} ds \right|$$

$$\leq \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(\mathbf{y}, s) \left| \left(\frac{\partial}{\partial s} - \Delta_y \right) h(\mathbf{x} - \mathbf{y}, t - s) \right| d\mathbf{y} ds$$

$$\leq \left(\|h_t\|_{L^\infty} + \|D^2 h\|_{L^\infty} \right) \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(\mathbf{y}, s) d\mathbf{y} ds$$

$$\leq \left(\|h_t\|_{L^\infty} + \|D^2 h\|_{L^\infty} \right) \int_0^\epsilon ds \leq \epsilon C$$

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Proof of Part 2 (continued): **Claim:** $J_\epsilon + K = \int_{\mathbb{R}^n} \Phi(\mathbf{y},\epsilon)h(\mathbf{x}-\mathbf{y},t-\epsilon)d\mathbf{y}$

$$J_\epsilon = \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(\mathbf{y},s)\left(\frac{\partial}{\partial s} - \Delta_y\right) h(\mathbf{x}-\mathbf{y},t-s)d\mathbf{y}ds$$

$$= \overbrace{\int_\epsilon^t \int_{\mathbb{R}^n} \left[\left(\frac{\partial}{\partial s} - \Delta_y\right)\Phi(\mathbf{y},s)\right] h(\mathbf{x}-\mathbf{y},t-s)d\mathbf{y}ds}^{0}$$

$$+ \int_{\mathbb{R}^n} \Phi(\mathbf{y},\epsilon)h(\mathbf{x}-\mathbf{y},t-\epsilon)d\mathbf{y} - \overbrace{\int_{\mathbb{R}^n} \Phi(\mathbf{y},t)h(\mathbf{x}-\mathbf{y},0)d\mathbf{y}}^{K}$$

Hence the claim.

Heat Equation Solution nonhomogeneous IVP

Proof of Part 2 (continued): **Claim:**$L = h(\mathbf{x}, t)$. Since $|I_\epsilon| \leq \epsilon C$,

$$\lim_{\epsilon \to 0} I_\epsilon = 0$$

Since

$$J_\epsilon + K = \int_{\mathbb{R}^n} \Phi(\mathbf{y}, \epsilon) h(\mathbf{x} - \mathbf{y}, t - \epsilon) d\mathbf{y}$$

$$\lim_{\epsilon \to 0} J_\epsilon + K = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \Phi(\mathbf{y}, \epsilon) h(\mathbf{x} - \mathbf{y}, t - \epsilon) d\mathbf{y} = h(\mathbf{x}, t) \text{ [How?]}$$

Hence the claim.

Heat Equation Solution nonhomogeneous IVP

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Fix $\delta > 0$. By continuity of h at $(\mathbf{x}, t)$ there exists $\eta > 0$ such that

$$|h(\mathbf{x}-\mathbf{y}, t-\epsilon) - h(\mathbf{x}, t)| < \delta \qquad \text{whenever } |\mathbf{y}| < \eta,\ 0 < \epsilon < \eta.$$

$$J_\epsilon + K - h(\mathbf{x}, t) = \int_{|\mathbf{y}|<\eta} \Phi(\mathbf{y}, \epsilon)\big(h(\mathbf{x}-\mathbf{y}, t-\epsilon) - h(\mathbf{x}, t)\big)\, d\mathbf{y} + \int_{|\mathbf{y}|\geq\eta} \Phi(\mathbf{y}, \epsilon)\big(h(\mathbf{x}-\mathbf{y}, t-\epsilon) - h(\mathbf{x}, t)\big)\, d\mathbf{y}.$$

For $0 < \epsilon < \eta$, we have

$$\left| \int_{|\mathbf{y}|<\eta} \Phi(\mathbf{y}, \epsilon)\big(h(\mathbf{x}-\mathbf{y}, t-\epsilon) - h(\mathbf{x}, t)\big)\, d\mathbf{y} \right| \leq \delta \int_{|\mathbf{y}|<\eta} \Phi(\mathbf{y}, \epsilon)\, d\mathbf{y} \leq \delta.$$

Since $h(\cdot, t-\epsilon)$ vanishes outside the fixed compact set K, the integrand in the far part is nonzero only when $\mathbf{x} - \mathbf{y} \in K$, i.e., only for $\mathbf{y} \in \mathbf{x} - K$. Hence

$$\int_{|\mathbf{y}|\geq\eta} \Phi(\mathbf{y}, \epsilon)\big(h(\mathbf{x}-\mathbf{y}, t-\epsilon) - h(\mathbf{x}, t)\big)\, d\mathbf{y} = \int_{\substack{|\mathbf{y}|\geq\eta \\ \mathbf{y}\in\mathbf{x}-K}} \Phi(\mathbf{y}, \epsilon)\big(h(\mathbf{x}-\mathbf{y}, t-\epsilon) - h(\mathbf{x}, t)\big)\, d\mathbf{y}.$$

Heat Equation Solution nonhomogeneous IVP

Since K is fixed and h is continuous on the compact set $K \times [0, t]$, h is uniformly bounded there; let M be such a bound. Therefore

$$\left| \int_{|\mathbf{y}|\geq\eta} \Phi(\mathbf{y}, \epsilon)\big(h(\mathbf{x}-\mathbf{y}, t-\epsilon) - h(\mathbf{x}, t)\big)\, d\mathbf{y} \right| \leq 2M \int_{\substack{|\mathbf{y}|\geq\eta \\ \mathbf{y}\in\mathbf{x}-K}} \Phi(\mathbf{y}, \epsilon)\, d\mathbf{y} \leq 2M \int_{|\mathbf{y}|\geq\eta} \Phi(\mathbf{y}, \epsilon)\, d\mathbf{y}.$$

$$\int_{|\mathbf{y}|\geq\eta} \Phi(\mathbf{y}, \epsilon)\, d\mathbf{y} = \int_{|\mathbf{z}|\geq\eta/\sqrt{\epsilon}} (4\pi)^{-n/2} e^{-|\mathbf{z}|^2/4}\, d\mathbf{z}, \text{ [use } \mathbf{z} = \mathbf{y}/\sqrt{\epsilon}\text{]}$$

which tends to 0 as $\epsilon \to 0$ [How?]. Hence there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$,

$$2M \int_{|\mathbf{y}|\geq\eta} \Phi(\mathbf{y}, \epsilon)\, d\mathbf{y} < \delta.$$

For $0 < \epsilon < \min\{\eta, \epsilon_0\}$ we have

$$|J_\epsilon + K - h(\mathbf{x}, t)| \leq \delta + \delta = 2\delta.$$

Since $\delta > 0$ was arbitrary, the claim follows.

Heat Equation Solution nonhomogeneous IVP

Proof of Part 3:

$$u(\mathbf{x},t)=\int_0^t\int_{\mathbb{R}^n}\Phi(\mathbf{y},s)h(\mathbf{x}-\mathbf{y},t-s)d\mathbf{y}ds$$

$$\|u(\mathbf{x},t)\|_{L^\infty}=\left|\int_0^t\int_{\mathbb{R}^n}\Phi(\mathbf{y},s)h(\mathbf{x}-\mathbf{y},t-s)d\mathbf{y}ds\right|_{L^\infty}$$

$$\leq\|h(\mathbf{x},t)\|_{L^\infty}\int_0^t\int_{\mathbb{R}^n}\Phi(\mathbf{y},s)d\mathbf{y}ds\leq\|h(\mathbf{x},t)\|_{L^\infty}\int_0^t ds$$

$$=t\|h(\mathbf{x},t)\|_{L^\infty}\to 0\ \text{ as }\ t\to 0$$

Heat Equation Solution nonhomogeneous IVP

Now, by combining both results, we can obtain the solution to the following IVP.

$$\begin{cases} u_t - \Delta u = h & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = f & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{10}$$

From, above discussion, we can conclude that

$$u(\mathbf{x},t) = \int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t-s)f(\mathbf{y})d\mathbf{y}ds + \int\limits_0^t \int\limits_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t-s)h(\mathbf{y},s)d\mathbf{y}ds \tag{11}$$

Heat Equation: Mean Value Property

Parabolic Cylinder

Definition 1 (Parabolic Cylinder)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded and fix a time $T > 0$. The parabolic cylinder is defined as

$$\Omega_T = \Omega \times (0, T] \tag{12}$$

Definition 2 (Parabolic Boundary)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded and fix a time $T > 0$. The parabolic boundary is defined as

$$\Gamma_T = \overline{\Omega}_T \setminus \Omega_T \tag{13}$$

Parabolic Cylinder

Remarks

- Ω_T is the interior of $\overline{\Omega} \times [0, T]$.
- Ω_T includes the top $\Omega \times \{t = T\}$.
- Therefore, Γ_T includes the bottom and vertical sides of $\Omega \times [0, T]$.
- However, Γ_T does not include the top side of $\Omega \times [0, T]$.

Next, let us derive the mean-value property as we did for the Laplace equation. However, it is not straightforward.

Introductory Remarks

- For fixed x the spheres $\partial B(\mathbf{x}, r)$ are level sets of the fundamental solution $\Phi(\mathbf{x} - \mathbf{y})$ for Laplace's equation.
- For fixed (x, t), there is a relation between the level sets of the fundamental solution $\Phi(\mathbf{x} - \mathbf{y}, t - s)$ and the heat equation.

Heat Ball

Definition 3 (Heat Ball)

For fixed $\mathbf{x} \in \mathbb{R}^n, t \in \mathbb{R}, r > 0$, define

$$E(\mathbf{x}, t; r) := \left\{(\mathbf{y}, s) \in \mathbb{R}^{n+1} : s < t, \Phi(\mathbf{x} - \mathbf{y}, t - s) \geq \frac{1}{r^n}\right\} \tag{14}$$

Remarks

- $E(\mathbf{x}, t; r)$ is a region in space-time, the boundary of which is a level set of $\Phi(\mathbf{x} - \mathbf{y}, t - s)$.
- The point $(\mathbf{x}, t)$ is at the center of the top.
- We denote $E(r) = E(\mathbf{0}, 0; r) = \left\{(\mathbf{y}, s) \in \mathbb{R}^{n+1} : s < 0, \Phi(\mathbf{y}, -s) \geq \frac{1}{r^n}\right\}$

Heat Ball

Define a function $\psi : E(r) \to \mathbb{R}$ by

$$\psi := -\frac{n}{2}\log(-4\pi s) + \frac{|\mathbf{y}|^2}{4s} + n\log r \tag{15}$$

Observe the following

$$(1D)\psi_y = \frac{y}{2s} \implies y\psi_y = \frac{y^2}{2s} \implies \frac{y^2}{s} = 2y\psi_y \implies (nD)\sum_{i=1}^{n}\frac{|\mathbf{y}|^2}{s} = 2\sum_{i=1}^{n} y_i\psi_{y_i} \tag{16}$$

Heat Ball

Prove the following:

$$\iint\limits_{E(1)} \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds = 4$$

Proof: For $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$, we have

$$\Phi(\mathbf{x}, t) = \frac{1}{(4\pi t)^{n/2}} \mathrm{e}^{\frac{-|\mathbf{x}|^2}{4t}}$$

Therefore, for $\mathbf{y} \in \mathbb{R}^n$ and $s < 0$,

$$\Phi(-\mathbf{y}, -s) \geq 1 \iff \frac{1}{(-4\pi s)^{n/2}} \mathrm{e}^{\frac{|\mathbf{y}|^2}{4s}} \geq 1 \iff (-4\pi s)^{n/2} \leq \mathrm{e}^{\frac{|\mathbf{y}|^2}{4s}}$$

$$\iff (-4\pi s)^{n/2} \leq 1 \iff s \geq -\frac{1}{4\pi}$$

Heat Ball

Proof (continued):

$$\begin{aligned}
\Phi(-\mathbf{y},-s) \geq 1 \quad &\iff \quad \ln\left(\frac{1}{(-4\pi s)^{n/2}} \mathrm{e}^{\frac{|\mathbf{y}|^2}{4s}}\right) \geq 0 \\
&\iff \quad (-n/2)\ln\left(-4\pi s\right) + \frac{|\mathbf{y}|^2}{4s} \geq 0 \\
&\iff \quad |\mathbf{y}|^2 \leq 2ns\ln(-4\pi s)
\end{aligned}$$

This implies that

$$\begin{aligned}
E(0,0;1) &= \{(\mathbf{y},s) \in \mathbb{R}^{n+1} \mid s \leq 0,\ \Phi(-\mathbf{y},-s) \geq 1\} \\
&= \left\{(\mathbf{y},s) \in \mathbb{R}^{n+1} \;\middle|\; -\frac{1}{4\pi} \leq s \leq 0,\ |\mathbf{y}|^2 \leq 2ns\ln(-4\pi s)\right\}
\end{aligned}$$

Heat Ball

Proof (continued): Thus

$$\iint\limits_{E(0,0;1)} \frac{|\mathbf{y}|^2}{s^2}\, dyds = \int\limits_{-\frac{1}{4\pi}}^{0} \int\limits_{|\mathbf{y}|^2 \le 2ns\ln(-4\pi s)} \frac{|\mathbf{y}|^2}{s^2}\, d\mathbf{y}ds$$

$$= \int\limits_{-\frac{1}{4\pi}}^{0} \int\limits_{0}^{\sqrt{2ns\ln(-4\pi s)}} \int\limits_{\partial B(0,r)} \frac{|\mathbf{y}|^2}{s^2}\, dS(\mathbf{y})drds$$

$$= \int\limits_{-\frac{1}{4\pi}}^{0} \int\limits_{0}^{\sqrt{2ns\ln(-4\pi s)}} \int\limits_{\partial B(0,1)} \frac{|rw|^2}{s^2} r^{n-1}\, dS(w)drds$$

Heat Ball

Proof (continued): Thus

$$\iint\limits_{E(0,0;1)} \frac{|\mathbf{y}|^2}{s^2}\, dyds = \int\limits_{-\frac{1}{4\pi}}^{0} \int\limits_{0}^{\sqrt{2ns\ln(-4\pi s)}} \int\limits_{\partial B(0,1)} \frac{r^{n+1}}{s^2}\, dS(w)drds$$

$$= \int\limits_{-\frac{1}{4\pi}}^{0} \int\limits_{0}^{\sqrt{2ns\ln(-4\pi s)}} \frac{r^{n+1}}{s^2} drds \overbrace{\int\limits_{\partial B(0,1)} dS(w)}^{n\alpha(n)}$$

$$= \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int\limits_{-\frac{1}{4\pi}}^{0} \int\limits_{0}^{\sqrt{2ns\ln(-4\pi s)}} \frac{r^{n+1}}{s^2} drds$$

Heat Ball

Proof (continued): By calculating this last integral, we get

$$\iint_{E(0,0;1)} \frac{|\mathbf{y}|^2}{s^2}\,dyds = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_{-\frac{1}{4\pi}}^{0} \frac{(2ns\ln(-4\pi s))^{(n+2)/2}}{(n+2)s^2}ds$$

$$= \frac{(2n)^{(n+2)/2}}{(n+2)}\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_{-\frac{1}{4\pi}}^{0} \frac{(s\ln(-4\pi s))^{(n+2)/2}}{s^2}ds$$

Heat Ball

Proof (continued): By calculating this last integral, we get

$$\int_{-\frac{1}{4\pi}}^{0} \frac{(s\ln(-4\pi s))^{(n+2)/2}}{s^2} ds = \int_{1}^{0} \frac{\left((-\frac{x}{4\pi})\ln(x)\right)^{(n+2)/2}}{(-\frac{x}{4\pi})^2} \left(-\frac{1}{4\pi}\right) dx \quad (x = -4\pi s)$$

$$= \int_{0}^{\infty} \frac{\left(\frac{e^{-z}}{4\pi} z\right)^{(n+2)/2}}{(-\frac{e^{-z}}{4\pi})^2} \left(-\frac{1}{4\pi}\right) (-e^{-z})\, dz \quad (z = -\ln x)$$

$$= \frac{1}{(4\pi)^{n/2}} \int_{0}^{\infty} e^{-nz/2} z^{(n+2)/2}\, dz$$

$$= \frac{1}{(4\pi)^{n/2}} \int_{0}^{\infty} e^{-w} \left(\frac{2w}{n}\right)^{(n+2)/2} \frac{2}{n}\, dw \quad (w = nz/2)$$

Heat Ball

Proof (continued):

$$\int_{-\frac{1}{4\pi}}^{0} \frac{(s\ln(-4\pi s))^{(n+2)/2}}{s^2} ds = \frac{1}{(4\pi)^{n/2}}\left(\frac{2}{n}\right)^{n/2+2}\int_0^\infty \mathrm{e}^{-w} w^{n/2+1}\, dw$$

$$= \frac{1}{2^n \pi^{n/2} 2^{-n/2-2} n^{n/2+2}} \int_0^\infty \mathrm{e}^{-w} w^{n/2+2-1}\, dw$$

$$= \frac{1}{\pi^{n/2} 2^{n/2-2} n^{n/2+2}} \Gamma\left(2+\frac{n}{2}\right)$$

Heat Ball

Proof (continued): So,

$$\iint\limits_{E(0,0;1)} \frac{|\mathbf{y}|^2}{s^2}\, d\mathbf{y}ds = \frac{(2n)^{(n+2)/2}}{(n+2)} \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{\pi^{n/2}2^{n/2-2}n^{n/2+2}} \Gamma\left(2+\frac{n}{2}\right)$$

$$= \frac{(2n)^{(n+2)/2}}{(n+2)} \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{\pi^{n/2}2^{n/2-2}n^{n/2+2}} \left(1+\frac{n}{2}\right)\frac{n}{2}\Gamma\left(\frac{n}{2}\right)$$

$$= 4$$

Heat Ball

Exercise 1: Heat Ball

Prove the following

1. $\Phi(\mathbf{y}, -s) = \dfrac{1}{r^n}$ on $\partial E(r)$
2. $\psi = 0$ on $\partial E(r)$
3. $$\frac{1}{r^n} \iint\limits_{E(r)} \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds = \iint\limits_{E(1)} \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds$$
4. $$\frac{1}{r^n} \iint\limits_{E(r)} u(\mathbf{y}, s) \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds = \iint\limits_{E(1)} u(r\mathbf{y}, r^2 s) \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds$$

Mean Value Property for the Heat Equation

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Theorem 2 (Mean Value Property for the Heat Equation)
Let $u \in C_1^2(\Omega_T)$ solve the heat equation. Then

$$u(\mathbf{x}, t) = \frac{1}{4r^n} \iint\limits_{E(\mathbf{x},t;r)} u(\mathbf{y}, s) \frac{|\mathbf{x}-\mathbf{y}|^2}{(t-s)^2} d\mathbf{y} ds \tag{17}$$

for each $E(\mathbf{x}, t; r) \subset \Omega_T$.

Proof: Now, let us translate the space and time coordinates to $\mathbf{x} = \mathbf{0}$ and $t = 0$, and from the exercise

$$\phi(r; t) = \frac{1}{r^n} \iint\limits_{E(r)} u(\mathbf{y}, s) \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds = \iint\limits_{E(1)} u(r\mathbf{y}, r^2 s) \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds$$

Mean Value Property for the Heat Equation

Now,

$$\phi'(r;t) = \iint\limits_{E(1)} \sum_{i=1}^{n} \left[u_{y_i}(r\mathbf{y}, r^2 s) y_i \frac{|\mathbf{y}|^2}{s^2} + 2r u_s(r\mathbf{y}, r^2 s) \frac{|\mathbf{y}|^2}{s} \right] d\mathbf{y} ds$$

$$= \overbrace{\frac{1}{r^{n+1}} \iint\limits_{E(r)} \sum_{i=1}^{n} u_{y_i}(\mathbf{y}, s) y_i \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds}^{A} + \overbrace{\frac{1}{r^{n+1}} \iint\limits_{E(r)} \sum_{i=1}^{n} 2u_s(\mathbf{y}, s) \frac{|\mathbf{y}|^2}{s} d\mathbf{y} ds}^{B}$$

By the definition of ψ and (16), we can write B as follows

$$B = \frac{1}{r^{n+1}} \iint\limits_{E(r)} \sum_{i=1}^{n} 2u_s \frac{|\mathbf{y}|^2}{s} d\mathbf{y} ds = \iint\limits_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i} d\mathbf{y} ds$$

Mean Value Property for the Heat Equation

By integration by parts, we obtain that

$$B = \frac{1}{r^{n+1}}\left[4u_s\sum_{i=1}^{n} y_i\psi\right]_{\partial E(r)} - \frac{1}{r^{n+1}}\iint\limits_{E(r)} 4\sum_{i=1}^{n}\psi(u_s + y_i(u_s)_{y_i})d\mathbf{y}ds$$

$$= -\frac{1}{r^{n+1}}\iint\limits_{E(r)} 4nu_s\psi d\mathbf{y}ds - \frac{1}{r^{n+1}}\iint\limits_{E(r)} 4\sum_{i=1}^{n} y_i\psi(u_s)_{y_i}d\mathbf{y}ds$$

$$= -\frac{1}{r^{n+1}}\iint\limits_{E(r)} 4nu_s\psi d\mathbf{y}ds - \frac{1}{r^{n+1}}\iint\limits_{E(r)} 4\sum_{i=1}^{n} y_i\psi(u_{y_i})_s d\mathbf{y}ds$$

$$= -\frac{1}{r^{n+1}}\iint\limits_{E(r)} 4nu_s\psi d\mathbf{y}ds + \frac{1}{r^{n+1}}\iint\limits_{E(r)} 4\sum_{i=1}^{n} y_i\psi_s u_{y_i} d\mathbf{y}ds$$

Mean Value Property for the Heat Equation

Since $\psi_s = -\frac{|\mathbf{y}|^2}{4s^2} - \frac{n}{2s}$

$$B = -\frac{1}{r^{n+1}} \iint\limits_{E(r)} 4nu_s\psi d\mathbf{y}ds + \frac{1}{r^{n+1}} \iint\limits_{E(r)} 4\sum_{i=1}^{n} y_i u_{y_i} \left(-\frac{|\mathbf{y}|^2}{4s^2} - \frac{n}{2s}\right) d\mathbf{y}ds$$

$$= -\frac{1}{r^{n+1}} \left[\iint\limits_{E(r)} 4nu_s\psi d\mathbf{y}ds - \overbrace{\iint\limits_{E(r)} 4\sum_{i=1}^{n} y_i u_{y_i} \frac{|\mathbf{y}|^2}{4s^2} d\mathbf{y}ds}^{A} - \iint\limits_{E(r)} \sum_{i=1}^{n} y_i u_{y_i} \frac{2n}{s} d\mathbf{y}ds \right]$$

$$\implies \phi'(r) = A + B = -\frac{1}{r^{n+1}} \left[\iint\limits_{E(r)} 4nu_s\psi d\mathbf{y}ds - \iint\limits_{E(r)} \sum_{i=1}^{n} y_i u_{y_i} \frac{2n}{s} d\mathbf{y}ds \right]$$

Mean Value Property for the Heat Equation

Since $u_s = \Delta u$, we have

$$\phi'(r;t) = -\frac{1}{r^{n+1}}\left[\iint\limits_{E(r)} 4n\Delta u\psi d\mathbf{y}ds - \iint\limits_{E(r)} \sum_{i=1}^{n} y_i u_{y_i}\frac{2n}{s} d\mathbf{y}ds\right]$$

Applying integration by parts again, we obtain that

$$\phi'(r) = \frac{1}{r^{n+1}}\left[\iint\limits_{E(r)} 4n\sum_{i=1}^{n} u_{y_i}\psi_{y_i} d\mathbf{y}ds - \iint\limits_{E(r)} \sum_{i=1}^{n} y_i u_{y_i}\frac{2n}{s} d\mathbf{y}ds\right]$$

$$\phi'(r) = \frac{1}{r^{n+1}}\left[\iint\limits_{E(r)} 4n\sum_{i=1}^{n} u_{y_i}\frac{y_i}{2s} d\mathbf{y}ds - \iint\limits_{E(r)} \sum_{i=1}^{n} y_i u_{y_i}\frac{2n}{s} d\mathbf{y}ds\right] = 0$$

Mean Value Property for the Heat Equation

Therefore, ϕ is constant. Since

$$\phi(r;t) = \frac{1}{r^n} \iint\limits_{E(r)} u(\mathbf{y}, s) \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds$$

$$\phi(r) = \lim_{t \to 0} \phi(r;t) = \lim_{t \to 0} \frac{1}{t^n} \iint\limits_{E(r)} u(\mathbf{0}, s) \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds = u(\mathbf{0}, 0) \lim_{t \to 0} \frac{1}{t^n} \iint\limits_{E(r)} \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds$$

$$\phi(r) = 4u(\mathbf{0}, 0)$$

Hence, the proof follows as

$$u(\mathbf{0}, 0) = \frac{1}{4r^n} \iint\limits_{E(r)} u(\mathbf{y}, s) \frac{|\mathbf{y}|^2}{s^2} d\mathbf{y} ds$$

Thanks

Doubts and Suggestions

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