MA612L-Partial Differential Equations

Lecture 29-30: Heat Equation: Strong Maximum Principle (Self Study)

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Heat Equation: Strong Maximum Principle

Strong Maximum Principle

Theorem 1 (Strong Maximum Principle)
Assume $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ solve the heat equation in Ω_T. Then

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u \tag{1}$$

Furthermore, if Ω is connected and there exists a point $(\mathbf{x}_0, t_0) \in \Omega_T$ such that

$$u(\mathbf{x}_0, t_0) = \max_{\overline{\Omega}_T} u \tag{2}$$

then u is constant in $\overline{\Omega}_{t_0}$.

Strong Maximum Principle

Proof: Let

$$M := \max_{\overline{\Omega}_T} u$$

Suppose there exists a point $(x_0, t_0) \in \Omega_T$ with $u(\mathbf{x}_0, t_0) = M$. Then for all sufficiently small $r > 0$, $E(\mathbf{x}_0, t_0; r) \subset \Omega_T$. By the mean value property, we obtain that

$$M = u(\mathbf{x}_0, t_0) = \frac{1}{4r^n} \iint\limits_{E(\mathbf{x}_0, t_0; r)} u(\mathbf{y}, s) \frac{|\mathbf{x}_0 - \mathbf{y}|^2}{(t_0 - s)^2} d\mathbf{y} ds$$

However,

$$\frac{1}{4r^n} \iint\limits_{E(\mathbf{x}_0, t_0; r)} \frac{|\mathbf{x}_0 - \mathbf{y}|^2}{(t_0 - s)^2} d\mathbf{y} ds = 1$$

$$\implies M = u(\mathbf{x}_0, t_0) \leq M$$

Strong Maximum Principle

Proof: If u is identically equal to M within $E(\mathbf{x}_0, t_0; r)$, then

$$M = u(\mathbf{x}_0, t_0)$$

Therefore,

$$M = u(\mathbf{y}, s) \;\; \forall (\mathbf{y}, s) \in E(\mathbf{x}_0, t_0; r)$$

Now, draw a line segment L in Ω_T connecting $(\mathbf{x}_0, t_0)$ with some other point $(\mathbf{y}_0, s_0) \in \Omega_T$ with $s_0 < t_0$. Consider

$$r_0 = \min\{s \geq s_0 : u(\mathbf{x}, t) = M \;\; \forall (\mathbf{x}, t) \in L, s \leq t \leq t_0\}$$

Since u is continuous, the minimum is attained. Assume $r_0 > s_0$. Then $\exists (\mathbf{z}_0, r_0)$ such that $u(\mathbf{z}_0, r_0) = M$

Strong Maximum Principle

Proof: Further for all sufficiently small $r > 0$,

$$u \equiv M \text{ on } E(\mathbf{z}_0, r_0; r)$$

Suppose $r_0 \neq s_0$, then there exists a small $\sigma > 0$ such that

$$L \cap \{r_0 - \sigma \leq t \leq r_0\} \subset E(\mathbf{z}_0, r_0; r) \Longrightarrow\Longleftarrow$$

Hence, $r_0 = s_0$ and $u \equiv M$ on L.
If Ω is connected, let $\mathbf{x} \in \Omega$ and any time $0 \leq t \leq t_0$. Then there exists a sequence of points

$$\{\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_m = \mathbf{x}\}$$

such that the line segments in $\mathbb{R}^n$ connecting $\mathbf{x}_{i-1}$ to $\mathbf{x}_i$ lie in Ω for $i = 1, 2, \cdots, m$.

Strong Maximum Principle

Proof: The above statement is true since the set points in Ω which can be so connected to $\mathbf{x}_0$ by a polygonal path are nonempty, open, and relatively closed in Ω. Select times $t_0 > t_1 > \cdots > t_m = t$. The line segments in $\mathbb{R}^{n+1}$ connecting (x_{i-1}, t_{i-1}) to (x_i, t_i), for $i = 1, 2, \cdots, m$ lie in Ω_T. Hence $u \equiv M$ on each such segment and hence $u(\mathbf{x}, t) = M$.

Strong Maximum Principle

Remarks

- Similar proof can be done for the minimum principle.
- If u attains a maximum or minimum at an interior point, then u is constant at all earlier times.
- The solution will be constant on the time interval $[0, t_0]$ provided the initial and boundary conditions are constant
- The solution may change at times $t > t_0$, provided the boundary conditions alter after t_0.
- The solution will not respond to changes in boundary conditions until these changes happen.

Strong Maximum Principle

The strong maximum principle implies that if Ω is connected and $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ satisfies

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega_T \\ u = 0 & \text{on} \quad \partial\Omega \times [0, T] \\ u = f & \text{on} \quad \Omega \times \{t = 0\} \end{cases}$$

Here $f \geq 0$

Remarks

- if $f > 0$ somewhere in Ω then u is positive everywhere within Ω_T.

Uniqueness on bounded domains

Theorem 2 (Uniqueness on bounded domains)
Let $f \in C(\Gamma_T), h \in C(\Omega_T)$. Then there exists at most one solution $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega}_T)$ of the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = h & \text{in } \Omega_T \\ u = f & \text{on } \Gamma_T \end{cases}$$

Proof: Suppose u_1 and u_2 are two solutions of the above IBVP, then applying the strong maximum principle on $w = u_1 - u_2$ proves the uniqueness.

Heat Equation: Maximum Principle for the Cauchy Problem Self Reading

Maximum Principle for the Cauchy Problem

Prove that

$$\Phi(\mathbf{x}, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}$$

solves the heat equation

$$\Phi_t - \Phi_{xx} = 0$$

Proof:

$$\begin{aligned}\Phi_t &= -\frac{n}{2t(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}} + \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}} \frac{|\mathbf{x}|^2}{4t^2} \\ &= \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}} \left(\frac{|\mathbf{x}|^2}{4t^2} - \frac{n}{2t} \right) = \frac{1}{(4\pi t)^{n/2} 4t^2} e^{-\frac{|\mathbf{x}|^2}{4t}} \left(|\mathbf{x}|^2 - 2nt \right)\end{aligned}$$

Maximum Principle for the Cauchy Problem

Proof (continued):

$$\Phi_x = -\frac{x}{2t(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}$$

$$\implies \Phi_{xx} = -\frac{1}{2t(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}} + \frac{|\mathbf{x}|^2}{4t^2(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}$$

$$= \frac{1}{(4\pi t)^{n/2} 4t^2} e^{-\frac{|\mathbf{x}|^2}{4t}} \left(|\mathbf{x}|^2 - 2nt\right) = \Phi_t$$

Maximum Principle for the Cauchy Problem

Theorem 3 (Maximum Principle for the Cauchy Problem)

Suppose $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solve the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u = f & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

and satisfies the growth estimate

$$u(\mathbf{x},t) \leq Ae^{a|\mathbf{x}|^2} \quad (\mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq T)$$

for constants $A, a > 0$. Then

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} f$$

Maximum Principle for the Cauchy Problem

Proof: Choose your a such that $4aT < 1$. Then there exists $\epsilon > 0$ such that $4aT + 4a\epsilon < 1$. Define a new function $v(\mathbf{x}, t)$ as follows:

$$v(\mathbf{x}, t) := u(\mathbf{x}, t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon-t)}} \quad (\mathbf{x} \in \mathbb{R}^n, t > 0) \tag{3}$$

We can observe that

$$v = u - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon-t)}}$$

$$\implies v_t = u_t - \frac{\mu}{4(T+\epsilon-t)^{n/2+2}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon-t)}} \left(|\mathbf{x}|^2 + 2n(T+\epsilon-t)\right)$$

$$\Delta v = \Delta v - \frac{\mu}{4(T+\epsilon-t)^{n/2+2}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon-t)}} \left(|\mathbf{x}|^2 + 2n(T+\epsilon-t)\right)$$

$$\implies v_t - \Delta v = 0$$

Maximum Principle for the Cauchy Problem

Proof: Fix $r > 0$ and set $U := B^0(\mathbf{y}, r)$, $\Omega_T = B^0(\mathbf{y}, r) \times (0, T]$. Then, as per the maximum principle, we have

$$\max_{\overline{\Omega}_T} u = \max_{\Gamma_T} u$$

Now, if $\mathbf{x} \in \mathbb{R}^n$

$$\begin{aligned} v(\mathbf{x}, 0) &= u(\mathbf{x}, 0) - \frac{\mu}{(T+\epsilon)^{n/2}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon)}} \\ &\leq u(\mathbf{x}, 0) = f(\mathbf{x}) \end{aligned}$$

Now, if $|\mathbf{x} - \mathbf{y}| = r, 0 \leq t \leq T$, then

$$v(\mathbf{x}, t) = u(\mathbf{x}, t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}}$$

Maximum Principle for the Cauchy Problem

Proof: Now, if $|\mathbf{x}-\mathbf{y}| = r, 0 \le t \le T$, then

$$\begin{aligned} v(\mathbf{x},t) &= u(\mathbf{x},t) - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \\ &\le Ae^{a|\mathbf{x}|^2} - \frac{\mu}{(T+\epsilon-t)^{n/2}} e^{\frac{r^2}{4(T+\epsilon-t)}} \\ &\le Ae^{a(|\mathbf{y}|+r)^2} - \frac{\mu}{(T+\epsilon)^{n/2}} e^{\frac{r^2}{4(T+\epsilon)}} \end{aligned}$$

Since $4a(T+\epsilon) < 1$, we have $a < \frac{1}{4(T+\epsilon)}$. Therefore, there exists $\gamma > 0$ such that $a + \gamma = \frac{1}{4(T+\epsilon)}$. Hence, for sufficiently large r. we have

$$v(\mathbf{x},t) \le Ae^{a(|\mathbf{y}|+r)^2} - \mu(4(a+\gamma)^{n/2})e^{(a+\gamma)r^2} \le \sup_{\mathbb{R}^n} g$$

Maximum Principle for the Cauchy Problem

Proof: Therefore, we have

$$v(y,t) \leq \sup_{\mathbb{R}^n} g$$

for all $\mathbf{y} \in \mathbb{R}^n, 0 \leq t \leq T$ (provided you find a). If $\mu \to 0$, we obtain a theorem. Suppose, there exists no a such that $4aT < 1$, then repeatedly apply the result above on time time intervals $[0, T_1], [T_1, 2T_1] \cdots$ for $T_1 = \frac{1}{8a}$. Hence the theorem.

Uniqueness for the Cauchy Problem

Theorem 4 (Uniqueness for the Cauchy Problem)

Suppose $f \in C(\mathbb{R}^n), h \in C(\mathbb{R}^n \times [0,T])$. Then there exists at most one solution $u \in C_1^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ that solves the initial and boundary value problem

$$\begin{cases} u_t - \Delta u = h & \text{in } \mathbb{R}^n \times (0,T) \\ u = f & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

satisfying the growth estimate

$$u(\mathbf{x},t) \le Ae^{a|\mathbf{x}|^2} \quad (\mathbf{x} \in \mathbb{R}^n, 0 \le t \le T)$$

for constants $A, a > 0$. Then

Proof: Apply above theorem on $w = u_1 - u_2$ and claim $w = 0$.

Heat Equation: Regularity

Closed Circular Cylinder

Already, we have seen the parabolic cylinder.

Definition 1 (Closed Circular Cylinder)
The closed circular cylinder of radius r and height r^2 is given by

$$C(\mathbf{x}, t; r) = \{(\mathbf{y}, s) : |\mathbf{x} - \mathbf{y}| \leq r, t - r^2 \leq s \leq t\} \tag{4}$$

where $(\mathbf{x}, t)$ denote the top center point of the $C(\mathbf{x}, t; r)$

Smoothness

Theorem 5 (Smoothness)

Suppose $u \in C_1^2(\Omega_T)$ solves the heat equation in Ω_T, then

$$u \in C^\infty(\Omega_T)$$

Proof: Now, fix an $(\mathbf{x}_0, t_0) \in \Omega_T$ and choose $r > 0$ so small that $C := C(\mathbf{x}_0, t_0; r) \subset \Omega_T$. Define also the smaller cylinders $C' := C(\mathbf{x}_0, t_0; \frac{3}{4}r)$, $C'' := C(\mathbf{x}_0, t_0; \frac{1}{2}r)$. Select a smooth cutoff function $\zeta = \zeta(\mathbf{x}, t)$ such that

$$\begin{cases} 0 \leq \zeta \leq 1, \zeta \equiv 1 & \text{on} C' \\ \zeta \equiv 0 & \text{near the parabolic boundary of} C \end{cases}$$

Extend $\zeta \equiv 0$ in $(\mathbb{R}^n \times [0, t_0]) \setminus C$

Smoothness

Proof: Now define

$$K(\mathbf{x},\mathbf{y},t,s) := \Phi(\mathbf{x}-\mathbf{y},t-s)(\zeta_s(\mathbf{y},s)+\Delta\zeta(\mathbf{y},s)) \\ + 2D_{\mathbf{y}}\Phi(\mathbf{x}-\mathbf{y},t-s).D\zeta(\mathbf{y},s) \tag{5}$$

Since $\zeta \equiv 1$ on C', we have

$$K(x,y,t,s) = 0$$

for all points $(\mathbf{y},s) \in C'$. Since Φ is smooth and we have selected our ζ as smooth, it is obvious that K is smooth on $C \setminus C'$.

Question
How do we obtain the function K?

Smoothness

Now, if we set

$$u(\mathbf{x},t) = \iint\limits_{C} K(\mathbf{x},\mathbf{y},t,s)u(\mathbf{y},s)d\mathbf{y}ds \tag{6}$$

Then, obtain that u is C^∞ within C''.
Let us answer the question by the following assumption. Suppose $u \in C^\infty$, then define

$$v(\mathbf{x},t) := \zeta(\mathbf{x},t)u(\mathbf{x},t) \quad (\mathbf{x} \in \mathbb{R}^n, 0 \leq t \leq t_0)$$

Then

$$v_t = \zeta u_t + \zeta_t u \tag{7}$$

$$\Delta v = \zeta \Delta u + 2D\zeta . Du + u\Delta\zeta \tag{8}$$

Smoothness

Proof:
Note that

$$v = 0 \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\}$$

and

$$v_t - \Delta v = \zeta_t u - 2D\zeta . Du - u\Delta\zeta =: \overline{h} \quad \text{in} \quad \mathbb{R}^n \times (0, t_0)$$

Now, set

$$\overline{v}(\mathbf{x}, t) := \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t - s)\overline{h}(\mathbf{y}, s) d\mathbf{y} ds$$

Smoothness

Proof: Then according to inhomogeneous IVP problem, $\overline{v}$ solves the following problem

$$\begin{cases} \overline{v}_t - \Delta \overline{v} = \overline{h} & \text{in } \mathbb{R}^n \times (0, t_0) \\ \overline{v} = 0 & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Since $|v|, |\overline{v}| \leq A$ for some constant A, by uniqueness theorem, we get $v \equiv \overline{v}$
Hence

$$v(\mathbf{x}, t) := \int_0^t \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}, t - s) \overline{h}(\mathbf{y}, s) d\mathbf{y} ds$$

Smoothness

Proof: If $(\mathbf{x}, t) \in C''$, as $\zeta \equiv 0$ outside the cylinder C, we get

$$u(\mathbf{x},t) = \iint\limits_{C} \Phi(\mathbf{x}-\mathbf{y}, t-s)(\zeta_s u - 2D\zeta.Du - u\Delta\zeta)(\mathbf{y},s)d\mathbf{y}ds$$

$$= \iint\limits_{C} \Phi(\mathbf{x}-\mathbf{y}, t-s)[(\zeta_s(\mathbf{y},s) - \Delta\zeta(\mathbf{y},s))u(\mathbf{y},s)d\mathbf{y}ds$$

$$- \iint\limits_{C}^{t} \Phi(\mathbf{x}-\mathbf{y}, t-s)2D\zeta(\mathbf{y},s).Du(\mathbf{y},s)d\mathbf{y}ds$$

Smoothness

Proof: By applying integration by parts, we obtain that

$$u(\mathbf{x},t) = \iint\limits_{C} \Phi(\mathbf{x}-\mathbf{y},t-s)[(\zeta_s(\mathbf{y},s) - \Delta\zeta(\mathbf{y},s))u(\mathbf{y},s)d\mathbf{y}ds$$

$$+ \iint\limits_{C}^{t} 2D_y\Phi(\mathbf{x}-\mathbf{y},t-s).D\zeta(\mathbf{y},s)u(\mathbf{y},s)d\mathbf{y}ds$$

$$+ \iint\limits_{C}^{t} 2\Phi(\mathbf{x}-\mathbf{y},t-s)\Delta\zeta(\mathbf{y},s)u(\mathbf{y},s)d\mathbf{y}ds$$

Smoothness

Proof: By applying integration by parts, we obtain that

$$u(\mathbf{x},t) = \iint\limits_{C} [\Phi(\mathbf{x}-\mathbf{y},t-s)(\zeta_s(\mathbf{y},s)+\Delta\zeta(\mathbf{y},s))$$

$$+ 2D_{\mathbf{y}}\Phi(\mathbf{x}-\mathbf{y},t-s).D\zeta(\mathbf{y},s)])u(\mathbf{y},s)d\mathbf{y}ds$$

Now, if $u \in C_1^2(\Omega_T)$, then we have obtained the above equation. By using $u^\epsilon = \eta_\epsilon * u$ and $\epsilon \to 0$, we obtain the result.

Heat Equation: Local Estimates Self Reading

Local Estimates

Theorem 6 (Local Estimates)
For each pair of integers $k, l = 0, 1, \cdots$, there exists a constant C_{kl} such that

$$\max_{C(\mathbf{x},t;r/2)} |D_{\mathbf{x}}^k D_t^l u| \leq \frac{C_{kl}}{r^{k+2l+n+2}} \|u\|_{L^1(C(\mathbf{x},t;r))}$$

for all cylinders $C(\mathbf{x}, t; r/2) \subset C(\mathbf{x}, t; r) \subset \Omega_T$ and all solutions of the heat equation in Ω_T.

Proof: Now, fix an $(\mathbf{x}_0, t_0) \in \Omega_T$. By shifting the coordinates, we get the point as $(\mathbf{0}, 0)$. Let $C(1) := C(\mathbf{0}, 0; 1)$ and $C\left(\frac{1}{2}\right) := C\left(\mathbf{0}, 0; \frac{1}{2}\right)$. Suppose $C(1) \subset \Omega_T$, then as in the previous theorem, for some smooth function K, we have

$$u(\mathbf{x}, t) = \iint_{C(1)} K(\mathbf{x}, \mathbf{y}, t, s) u(\mathbf{y}, s) d\mathbf{y} ds \quad \left((\mathbf{x}, t) \in C\left(\frac{1}{2}\right)\right) \tag{9}$$

Local Estimates

Proof (continued): Therefore,

$$
\begin{aligned}
|D^k_{\mathbf{x}} D^l_t u(\mathbf{x}, t)| &\leq \iint\limits_{C(1)} |D^k_{\mathbf{x}} D^l_t |K(\mathbf{x}, \mathbf{y}, t, s)||u(\mathbf{y}, s)| d\mathbf{y} ds \quad ((\mathbf{x}, t) \in C\left(\frac{1}{2}\right)) \\
&\leq C_{kl} \|u\|_{L^1(C(1))}
\end{aligned}
$$

for some constant C_{kl}. Now, instead of $C(1)$, let us consider $C(r) := C(\mathbf{0}, 0; r) \subset \Omega_T$ and $C\left(\frac{r}{2}\right) := C\left(\mathbf{0}, 0; \frac{r}{2}\right)$. Define

$$
v(\mathbf{x}, t) := u(r\mathbf{x}, r^2 t)
$$

Then

$$
v_t - \Delta v = 0 \quad \text{in} \quad C(1)
$$

Local Estimates

Proof (continued): Therefore,

$$|D_{\mathbf{x}}^k D_t^l v(\mathbf{x},t)| \leq C_{kl} \|v\|_{L^1(C(1))} \quad ((\mathbf{x},t) \in C\left(\frac{1}{2}\right))$$

But $D_{\mathbf{x}}^k D_t^l v(\mathbf{x},t) = r^{2l+k} D_{\mathbf{x}}^k D_t^l u(r\mathbf{x}, r^2 t)$ and

$$\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}} \|u\|_{L^1(C(r))}$$

Therefore,

$$D_{\mathbf{x}}^k D_t^l u(\mathbf{x},t) = \frac{C_{kl}}{r^{2l+k+n+2}} \|u\|_{L^1(C(r))}$$

Local Estimates

Remarks

If u solves the heat equation in Ω_T, then for each fixed time $0 < t \leq T$, the mapping $\mathbf{x} \to u(\mathbf{x}, t)$ is analytic. However, $t \to u(\mathbf{x}, t)$ is not in general analytic.

Heat Equation: Energy Methods

Energy Methods

As we did in the Laplace equation, let us prove the uniqueness of the following IBVP using energy methods

$$\begin{cases} u_t - \Delta u = h & \text{in } \Omega_T \\ u = f & \text{on } \Gamma_T \end{cases} \tag{10}$$

Theorem 7 (Energy Methods)
Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded and that $\partial\Omega$ is C^1. Suppose the terminal time $T > 0$ is given. Then there exists at most one solution $u \in C_1^2(\overline{\Omega}_T)$ for (10).

Proof: Suppose u_1 and u_2 are two solutions of (10). Define $w = u_1 - u_2$

Energy Methods

Proof: Then w solve the following problem

$$\begin{cases} w_t - \Delta w = 0 & \text{in } \Omega_T \\ w = 0 & \text{on } \Gamma_T \end{cases} \tag{11}$$

Now, set

$$e(t) := \int_{\Omega} w^2(\mathbf{x}, t) d\mathbf{x}, \quad (0 \leq t \leq T)$$

Then

$$\dot{e}(t) = 2 \int_{\Omega} w w_t d\mathbf{x} = 2 \int_{\Omega} w \Delta w d\mathbf{x} = -2 \int_{\Omega} |Dw^2| d\mathbf{x} \leq 0$$

Hence $e(t) \leq e(0) = 0$. Therefore, $w \equiv 0$ and hence the proof.

Backward Uniqueness

Suppose u and $\tilde{u}$ are two smooth solutions of the heat equation in Ω_T with same boundary conditions on $\partial\Omega$:

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T \\ u = f & \text{on } \Gamma_T \end{cases} \tag{12}$$

$$\begin{cases} \tilde{u}_t - \Delta \tilde{u} = 0 & \text{in } \Omega_T \\ \tilde{u} = f & \text{on } \Gamma_T \end{cases} \tag{13}$$

for some function f. The proof of the following theorem is left as an exercise. [Refer to Evans Book]

Backward Uniqueness

Theorem 8 (Backward Uniqueness)
Suppose $u, \tilde{u} \in C^2(\overline{\Omega}_T)$ solves (12) and (13). If

$$u(\mathbf{x}, T) = \tilde{u}(\mathbf{x}, T) \quad (\mathbf{x} \in \Omega)$$

then

$$u \equiv \tilde{u} \quad \text{within} \quad \Omega_T$$

Interpretation

- If two temperature distributions on Ω agree at some time $T > 0$ and have had the same boundary values for time $0 \leq t < T$, then these temperatures must have been identically equal within Ω at all earlier times.

Thanks

Doubts and Suggestions

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