MA612L-Partial Differential Equations

Lecture 3 : Applications and Classifications

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Recap

PDE-Notations

Independent Variables: Let us denote the independent variable by
$\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n, n \geq 2$
Dependent Variables: Let us denote the dependent variable or unknown function by $\mathbf{u} = (u_1, u_2, \cdots, u_p) \in \mathbb{R}^p, p \geq 1$
Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{Z}_+^n$ and

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

Then $D^\alpha u$ denotes

$$D^\alpha u = \frac{\partial^\alpha u}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n}$$

PDE

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Definition 1 (PDE-Formal Definition)
Let $\Omega \subset \mathbb{R}^n, m \in \mathbb{N}$ and

$$F : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \mathbb{R}^{n^2 p} \times \cdots \times \mathbb{R}^{n^m p} \rightarrow \mathbb{R}^q$$

A system of partial differential equations of order m is defined by the equation

$$F(\mathbf{x}, \mathbf{u}, D\mathbf{u}, D^2\mathbf{u}, \cdots, D^m\mathbf{u}) = \mathbf{0} \quad \text{or} \quad F(\mathbf{x}, (\partial^\alpha \mathbf{u})_{|\alpha|} \leq m) = \mathbf{0}$$

Here, some m^{th} order derivative of the function $\mathbf{u}$ appears in the system of equations.

PDE

If $u = (u_1)$ is the only dependent variable and F is real-valued, then the above definition can be written as

Definition 2 (PDE-Formal Definition, $p = 1, q = 1$)
Let $\Omega \subset \mathbb{R}^n, m \in \mathbb{N}$ and

$$F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \times \cdots \times \mathbb{R}^{n^m} \to \mathbb{R}$$

The partial differential equation of order m is defined by the equation

$$F(\mathbf{x}, u, Du, D^2u, \cdots, D^m u) = 0$$

Here, some m^{th} order derivative of the function u appears in the equation.

Four Important Linear PDEs

- Transport Equation

$$u_t + au_x = 0$$

- Heat Equation

$$u_t = \alpha^2 u_{xx}$$

- Wave Equation

$$u_{tt} = c^2 u_{xx}$$

- Laplace Equation

$$u_{xx} + u_{yy} = 0$$

In all problems in this presentation, valid domains for time, space, and function space are assumed.

A few more PDEs

A few more $\cdots$

The following are a few important PDEs that arise from physical problems

- Burgers' Equation (Dynamic Gases)

$$u_t + uu_x = 0 \quad \text{or} \quad u_t + uu_x = \mu u_{xx}, t > 0, x \in \mathbb{R}$$

- Eikonal Equation (Pedestrian, Robotic Path)

$$u_x^2 + u_y^2 = 1$$

- Shock Waves (Aerospace, Hypersonic Aircraft, Detonations, Tsunami Prediction)

$$u_x + uu_y = 0$$

A few more · · ·

- Biharmonic (Bending of Thin Elastic Plates)

$$u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$$

- Wave equation with interaction ()

$$u_{tt} = u_{xx} - u^3$$

- Born-Infeld (nonlinear electrodynamics, minimal surfaces, Cosmology, Modified Gravity)

$$(1 - u_t^2)u_{xx} + 2u_x u_t u_{xt} - (1 + u_x^2)u_{tt} = 0$$

A few more $\cdots$

- Monge-Ampere (Differential Geometry, Optimal Transport, Optics, CG)

$$u_{xy}^2 - u_{xx}u_{yy} = f(x, y)$$

- Klein-Gordon (Quantum Physics, QFT, Soliton Theory)

$$u_{tt} - c^2\nabla^2 u + \frac{m^2c^4u}{\hbar^2} = 0$$

- Hamilton-Jacobi (Quantum/Classical Mechanics, Field Theory)

$$-u_t = H(q, u_q, t)$$

A few more · · ·

- Schrodinger's Equation(time independent,Quantum mechanics)

$$iu_t = -\frac{\hbar}{2m}\Delta u + V(x)u(t,x) = 0, t > 0, x \in \mathbb{R}$$

- Euler-Bernoulli Beam Equation

$$u_{tt} + \alpha^4 u_{xxxx} = 0, t > 0, x \in \mathbb{R}$$

- Korteweg-de Vries Equation (Shallow water Waves, Plasma Physics, Optics)

$$u_t + u_{xxx} + uu_x = 0, t > 0, x \in \mathbb{R}$$

A few more $\cdots$

- Benjamin-Bona-Mahony equation (Shallow Water, Nonlinear Wave)

$$u_t + u_x + u_{txx} + uu_x = 0, t > 0, x \in \mathbb{R}$$

- Vlasov-Poisson equation (Plasma Physics, Astrophysics, kinetic theory)

$$f_t + v.\nabla_x f + E.\nabla_v f = 0, t > 0, x \in \mathbb{R}^n, v \in \mathbb{R}^n$$

$$E = \nabla_x V, \Delta V = \int_{\mathbb{R}^n} f(t, x, v) dv, V = V(x), f \geq 0$$

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Maxwell’s equation

$$\nabla.\mathrm{D} = \rho_v$$

$$\nabla.\mathrm{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{H}}{\partial t} + \mathbf{J}$$

Sloshing Dynamics

The following equations are used in the hydrodynamic analysis of partially filled liquid tanks

$$\Delta u = 0 \tag{1}$$

$$\nabla . n - \nabla \eta . \nabla u = \eta_t \tag{2}$$

$$u_t + g\eta + \frac{1}{2}(\nabla u)^2 = 0 \tag{3}$$

Shallow Water Equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\Big((H+h)u\Big) + \frac{\partial}{\partial y}\Big((H+h)v\Big) = 0, \tag{4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial h}{\partial x} - bu + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{5}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial h}{\partial y} - bv + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{6}$$

Reynolds Transport

Reynolds Transport Theorem

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v.\nabla f$$

Euler equation

$$\frac{D\rho}{Dt} = 0$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{\nabla p}{\rho} + \mathbf{g}$$

$$\nabla.\mathbf{u} = 0$$

Navier-Stokes

Continuity Equation (Conservation of Mass)

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0 \tag{7}$$

Momentum Equation (Conservation of Momentum)

$$\frac{\partial}{\partial t}(\rho\, \mathbf{u}) + \nabla \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) = -\nabla \bar{p} + \mu\, \nabla^2 \mathbf{u} + \tfrac{1}{3}\mu\, \nabla(\nabla \cdot \mathbf{u}) + \rho \mathbf{g} \tag{8}$$

Energy Equation (Conservation of Energy)

$$\frac{\partial}{\partial t}(\rho\, \mathbf{e}) + \nabla.[\rho \mathbf{u}\mathbf{e} + \dot{q}_s - \tau.\mathbf{u}] = p\nabla.\mathbf{u} + \mathbf{f_b}.\mathbf{u} + \dot{q}_u \tag{9}$$

Weather Prediction

$$\text{Momentum:} \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{V}.\nabla\mathbf{V} + 2\mathbf{\Omega} \times \mathbf{V} + \frac{1}{\rho}\nabla p = \mathbf{F} + \mathbf{g} \tag{10}$$

$$\text{Continuity:} \quad \frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0 \tag{11}$$

$$\text{Water substance:} \quad \frac{dq}{gt} = S \tag{12}$$

$$\text{Hydrostatic:} \quad \frac{\partial p}{\partial z} + g\rho = 0 \tag{13}$$

Drones/Helicopter Rotor

$$(\Omega.\nabla)\mathbf{v} = \frac{D\Omega}{Dt} \tag{14}$$

$$\frac{\partial\Omega}{\partial t} + \mathbf{v}.\nabla\boldsymbol{\Omega} = (\Omega.\nabla)\mathbf{v} \tag{15}$$

$$\Omega = \nabla \times \mathbf{v} \tag{16}$$

Minimal Surface

$$(1 + |\nabla u|^2)\Delta u - \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

Goals

Goals

When an industrial problem or real-world, or physical problem is posted to you, you need to model it mathematically. In case your mathematical model falls under the scope of partial differential equations. Then one of the fundamental questions is "Which PDEs should I choose?". No general answer!

- Discussion of some important physical systems (dated back to the 18th and 19th centuries)
- Existence and Uniqueness

Goals

Let us try to answer the following questions in this lecture:

- Does the PDE have any solutions?
- What are all the necessary conditions to solve a PDE?
- Are the solutions unique for a given data?
- What are the basic qualitative properties of the solution?
- What is the nature of singularities, if any?
- How does a small perturbation in the input data affect the solution?
- What types of quantitative estimates can be obtained for a given solution?
- How can we define the norm of a solution and find its error estimates with respect to the norm?

Clay Math Problem: Million Dollar Question

Universal Solution??

- It is not necessary that all PDEs have a solution. For example, $u_x^2 + 1 = 0$ has no real solution.
- We can't expect a general existence theorem for a general PDE
- Remember: We do not expect a solution for all PDES as we do in linear algebra or matrices in school days.
- Recall: All polynomials or systems of linear equations, or implicit functions, do not always have solutions.
- From Linear Algebra/Matrix Theory: Under certain conditions, a linear system of equations has solution(s). We characterize them depending on the rank of the matrix and the corresponding augmented matrix
- How about non-linear equations?

Universal Solution??

In the ODE course:

- Peano's existence theorem
- Picard's existence and uniqueness theorem
- These theorems address the existence of solutions of IVP for first-order ODEs
- Extends this to study any ODE in normal form

In PDE:

- Can you expect a similar one for PDE? Not possible
- Different types of problems have different conditions
- Let us see one of the million-dollar questions

Clay Mathematics

- Clay Mathematics was founded by T.Clay and his wife in 1998, stating 7 unsolved problems in 2000 called the Millennium Prize Problems
- The Correct solutions discoverer of any of these 7 problems will get 1 Million US dollars
- Poincaré Conjecture is the only problem solved so far (by Perelman, 2003, but he declined the money)
- Navier-Stokes existence and smoothness is one of the Millennium problems in the remaining 6 unsolved problems.

Source: Rest of the contents for this presentation is taken from the following link: http://claymath.org/sites/default/files/navierstokes.pdf

Navier-Stokes

Consider the following three-dimensional Navier-Stokes equation

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= -p_x + \nu(u_{xx} + u_{yy} + u_{zz}) + f_1(x,y,z,t) \\ v_t + uv_x + vv_y + wv_z &= -p_y + \nu(v_{xx} + v_{yy} + v_{zz}) + f_2(x,y,z,t) \\ w_t + uw_x + vw_y + ww_z &= -p_z + \nu(w_{xx} + w_{yy} + w_{zz}) + f_3(x,y,z,t) \end{aligned} \tag{17}$$

$$u_x + v_y + w_z = 0 \tag{18}$$

It is a second-order PDE with four variables $u(x,y,t,z)$, $v(x,y,z,t)$,$w(x,y,z,t)$, $p(x,y,z,t)$, ν kinematic viscosity of the fluid. f_is are external force. We restrict to incompressible (ρ is constant) fluids, which is represented by the last equation. $\nu = 0$ implies Euler equations.

Navier-Stokes

- The Navier-Stokes equations are fundamental in fluid mechanics.
- Difficult to solve either analytically or numerically
- When the initial condition

$$u(x,y,z,0) = u_0(x,y,z) \tag{19}$$

  is Finding the existence or nonexistence of solutions for all future times is a major unresolved problem in mathematics

Let $\mathbf{x} = (x,y,z) \in \mathbb{R}^3$, $\mathbf{u}(\mathbf{x},t) = (u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)) \in \mathbb{R}^3$, $\mathbf{p}(\mathbf{x},t) \in \mathbb{R}$ and $\mathbf{f}(\mathbf{x},t) = (f_1(x,y,z,t), f_2(x,y,z,t), f_3(x,y,z,t)) \in \mathbb{R}^3$.

Navier-Stokes

For physically reasonable, we would like to make sure $\mathbf{u}(\mathbf{x}, t)$ does not grow large as $|\mathbf{x}| \to \infty$. Let us keep some restrictions on $\mathbf{f}$ and $\mathbf{u_0}$.

$$|\partial_{\mathbf{x}}^{\alpha} \mathbf{u_0}(\mathbf{x})| \leq C_{\alpha K}(1+|x|)^{-K}, \quad \text{on} \quad \mathbb{R}^3, \quad \text{for any} \quad \alpha, K \tag{20}$$

$$|\partial_{\mathbf{x}}^{\alpha} \partial_t \mathbf{f}(\mathbf{x}, t)| \leq C_{\alpha m K}(1+|x|+t)^{-K}, \quad \text{on} \quad \mathbb{R}^3, t \geq 0 \quad \text{for any} \quad \alpha, m, K \tag{21}$$

Solution of (17), (18) and (19) is accepted as physically reasonable only if it satisfies

$$\mathbf{p}, \mathbf{u} \in C^{\infty}(\mathbb{R}^3, [0, \infty)) \tag{22}$$

$$\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x} < C, \quad \text{for all} \quad t \geq 0 \tag{23}$$

Navier-Stokes

If problems at infinity are ruled out, we can look for spatially periodic solutions. Therefore, restriction on $\mathbf{f}$ and $\mathbf{u_0}$ becomes

$$\mathbf{u_0}(\mathbf{x}+\mathbf{e_j}) = \mathbf{u_0}(x), \quad \mathbf{f}(\mathbf{x}+\mathbf{e_j},t) = \mathbf{f}(x,t), 1 \le j \le 3 \tag{24}$$

$$|\partial_{\mathbf{x}}^{\alpha}\partial_t \mathbf{f}(\mathbf{x},t)| \le C_{\alpha m K}(1+t)^{-K}, \quad \text{on} \quad \mathbb{R}^3, t \ge 0 \ \text{ for any } \ \alpha, m, K \tag{25}$$

Solution of (17) is accepted as physically reasonable only if it satisfies

$$\mathbf{u}(\mathbf{x}+\mathbf{e_j},t) = \mathbf{u}(x,t) \ \text{ on } \ \mathbb{R}^3 \times [0,\infty) \ \text{ for } \ 1 \le j \le 3 \tag{26}$$

$$\mathbf{p}, \mathbf{u} \in C^{\infty}(\mathbb{R}^3, [0,\infty)) \tag{27}$$

Millennium Problem

Unsolved Problems 2.1 (Existence and Uniqueness of NS Solution-A)
Let $\nu > 0$. Let $\mathbf{u_0}(\mathbf{x})$ be any smooth function, that is, $u_0 \in C^\infty(\mathbb{R}^3)$, divergence free vector field which satisfies the condition (20). Let $\mathbf{f} = 0$. Then there exist smooth functions $\mathbf{p}(\mathbf{x}, t), u(\mathbf{x}, t)$,$v(\mathbf{x}, t), w(\mathbf{x}, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy the above (17), (18), (19), (22) and (23).

Millennium Problem

Unsolved Problems 2.2 (Existence and Uniqueness of NS Solution-B)

Let $\nu > 0$. Let $\mathbf{u_0}(\mathbf{x})$ be any smooth, that is, $u_0 \in C^\infty(\mathbb{R}^3)$, divergence free vector field which satisfies the condition (24). Let $\mathbf{f} = 0$. Then there exist smooth functions $\mathbf{p}(\mathbf{x}, t), u(\mathbf{x}, t)$,$v(\mathbf{x}, t), w(\mathbf{x}, t)$ on $\mathbb{R}^3 \times [0, \infty)$ that satisfy the above (17), (18), (19), (26) and (27).

Millennium Problem

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Unsolved Problems 2.3 (Breakdown of NS Solution-C)

Let $\nu > 0$. Then there exist a smooth, divergence free vector field $\mathbf{u_0}(\mathbf{x})$, that is $u_0 \in C^\infty(\mathbb{R}^3)$, and a smooth $\mathbf{f}(\mathbf{x}, \mathbf{t})$ on $\mathbb{R}^3 \times [0, \infty)$ satisfying (20) and (21) for which there exist no solutions $(\mathbf{p}, \mathbf{u})$ of (17), (18), (19), (22) and (23) on $\mathbb{R}^3 \times [0, \infty)$

Millennium Problem

Unsolved Problems 2.4 (Breakdown of NS Solution-D)

Let $\nu > 0$. Then there exist a smooth, divergence free vector field $\mathbf{u_0}(\mathbf{x})$, that is $u_0 \in C^\infty(\mathbb{R}^3)$, and a smooth $\mathbf{f}(\mathbf{x}, \mathbf{t})$ on $\mathbb{R}^3 \times [0, \infty)$ satisfying (24) and (25), for which there exist no solutions $(\mathbf{p}, \mathbf{u})$ of (17), (18), (19), (26) and (27) on $\mathbb{R}^3 \times [0, \infty)$

Classification of PDEs

Why Classification

- Based on the number of properties, we can group families of similar equations
- In fact, a few researchers see no advantage in the classification process
- Some classifications are given a few branding, like Navier-Stokes, Heat Equation, etc
- Some classifications help to identify or guess, or predict the properties of solutions of PDEs in that class.
- Some classification helps to identify the allowable initial and boundary conditions
- A few classification helps to select an effective numerical method
- Classifications are done using characteristics, order, linearity, and so on.

PDE

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Definition 1 (PDE-Formal Definition)

Let $\Omega \subset \mathbb{R}^n, m \in \mathbb{N}$ and

$$F : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \mathbb{R}^{n^2 p} \times \cdots \times \mathbb{R}^{n^m p} \to \mathbb{R}^q$$

A system of partial differential equations of order m is defined by the equation

$$F(\mathbf{x}, \mathbf{u}, D\mathbf{u}, D^2\mathbf{u}, \cdots, D^m\mathbf{u}) = \mathbf{0} \tag{28}$$

Here, some m^{th} order derivative of the function $\mathbf{u}$ appears in the system of equations.

Classification-I - System

Based on the number of equations, we can classify PDEs.

Definition 2

If a PDE (28) consists of more than one equation, it is called a system of PDEs. Otherwise, it is called a single PDE or a scalar PDE, or simply PDE.

Exercise 1:

Classify all PDEs given in our last class into a system of PDEs and a single PDE

Classification-II - Order

Based on the highest order derivative, we can classify PDEs.

Definition 3

If the highest order derivative appearing in the PDE is m, then such PDEs are classified as m^{th} order PDEs.

Exercise 2:

Find the order of PDEs of all PDEs discussed in our last class.

Classification-III - Linear/Nonlinear

Through algebra, we can also classify PDEs. In algebra, we categorize algebraic equations as linear and nonlinear equations. To define linearity, let us rewrite the equation (28) as

$$\mathcal{L}u = f \tag{29}$$

where $\mathcal{L}$ is an operator which assigns u a new function $\mathcal{L}u$. Here f is a function of $\mathbf{x}$ only.

Definition 4

The operator $\mathcal{L}$ is called linear if

$$\mathcal{L}(\alpha u + \beta v) = \alpha \mathcal{L}u + \beta \mathcal{L}v \tag{30}$$

for any function u and v and constants α and β.

Classification-III - Linear

Definition 5

If the operator $\mathcal{L}$ in (29) is linear, then the PDE is called a linear PDE. Equivalently, an m^{th}-order PDE is linear if it can be written as

$$\sum_{|\alpha| \leq m} a_\alpha(\mathbf{x}) D^\alpha u = f(\mathbf{x}) \tag{31}$$

Here a_α's are functions of $\mathbf{x}$ only.

Example 3

1. $u_t + u_x = 0$
2. $u_{xx} + u_{yy} = 0$
3. $u_t + x^2 u_x = 0$

Classification-III - Nonlinear

Definition 6

If the operator $\mathcal{L}$ in (29) is not linear (or equivalently, it can't be written in the form of (31)), then the PDE is called a nonlinear PDE.

Example 4

1. $u_t + uu_x = 0$
2. $u_x^2 + u_y^2 = 0$

Exercise 3:

Identify the list of linear and nonlinear PDEs from all PDEs discussed in our last class.

Classification-IV - Quasilinear

We have categorized PDEs as linear and nonlinear already. The PDEs can be further categorized based on the linearity of different derivatives. For example,

- Quasilinear
- Non-Quasilinear or Fully nonlinear

Definition 7

The equation (28) of order m is called quasilinear if it is linear in the derivatives of order m with coefficients that depend on the independent variables and derivatives of the unknown function of order strictly less than m.

Classification-IV - Quasilinear

Definition 8

Equivalently, an m^{th} order PDE is quasilinear if it can be written in the form

$$\sum_{|\alpha|=m} a_\alpha(\mathbf{x}, u, Du, \cdots D^{m-1}u) D^\alpha u + a_0(\mathbf{x}, u, Du, \cdots D^{m-1}u) = 0 \qquad (32)$$

Here a_α's are functions of $\mathbf{x}$ and derivatives of the unknown function of order less than m.

Definition 9

An m^{th} order PDE is called fully nonlinear if it is not linear in the derivatives of order m. Equivalently, a PDE that is not quasilinear is called a fully nonlinear PDE.

Classification-IV - Quasilinear

Definition 10

The equation (28) of order m is called **quasilinear** if it is linear in the derivatives of order m with coefficients that depend on the independent variables and derivatives of the unknown function of order strictly less than m. Equivalently,

$$\sum_{|\alpha|=m} a_\alpha(\mathbf{x}, u, Du, \cdots D^{m-1}u)D^\alpha u + a_0(\mathbf{x}, u, Du, \cdots D^{m-1}u) = 0 \qquad (33)$$

An m^{th} order PDE is called **fully nonlinear** if it is not linear in the derivatives of order m. Equivalently, a PDE that is not quasilinear is called a fully nonlinear PDE.

Quasilinear

Example,

$$u_x + uu_y = 0$$

For,

$$(c_1 u_{1x} + c_2 u_{2x}) + u(c_1 u_{1y} + c_2 u_{2y}) = c_1 u_{1x} + c_2 u_{2x} + c_1 u u_{1y} + c_2 u u_{2y}$$

Example 5

1. $u_x + uu_y = 0$ is quasilinear
2. $u_t + a(u)u_x = 0$ is quasilinear
3. $u_x^2 + u_y^2 = 0$ is not quasilinear. It is fully nonlinear
4. $div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$ is fully nonlinear
5. $u_t + u_x^2 - u = \cos(xt)$ is fully nonlinear

Classification-V - Semilinear

Quasilinear PDEs are further categorized into

- Semilinear
- Non-semilinear

Definition 11

A quasilinear PDE of order m is called a semilinear PDE if the coefficients of derivatives of order m are functions of the independent variables alone.

Classification-V - Semilinear

Definition 12

A quasilinear PDE of order m is called a **semilinear PDE** if the coefficients of derivatives of order m are functions of the independent variables alone. Equivalently

$$\sum_{|\alpha|=m} a_\alpha(\mathbf{x}) D^\alpha u + a_0(\mathbf{x}, u, Du, \cdots D^{m-1}u) = 0 \tag{34}$$

Here a_α's are functions of $\mathbf{x}$ alone.

Classification-V - Semilinear

Definition 13

Equivalently, an m^{th} order PDE is semilinear if it can be written in the form

$$\sum_{|\alpha|=m} a_\alpha(\mathbf{x}) D^\alpha u + a_0(\mathbf{x}, u, Du, \cdots D^{m-1}u) = 0 \quad (35)$$

Here a_α's are functions of $\mathbf{x}$ alone.

Example 6

1. $u_t + u_x + u^2 = 0$ is semilinear
2. $u_t + u_{xxx} + uu_x = 0$ is semilinear
3. $xu_x + yu_y = u$ is semilinear
4. $u_t + uu_x = 0$ is not semilinear

Classification-VI - Almost linear

Definition 14

An m^{th} order semilinear PDE is called almost linear if it can be written in the form

$$\sum_{|\alpha|\leq m} a_\alpha(\mathbf{x})D^\alpha u + f(\mathbf{x}, u) = 0 \tag{36}$$

Here a_α's are function of $\mathbf{x}$ alone or if it is of the form

$$\mathcal{L}u = f(x, u) \tag{37}$$

where $f(x, u)$ is a nonlinear function with respect to u and $\mathcal{L}$ is a linear operator.

Example 7

1. $u_t + u_x + u^2 = 0$ is almost linear
2. $xu_x + yu_y = u$ is almost linear

Classification-VII - In/Homogeneous

Suppose (28) can be written in the following form

$$\mathcal{D}(u) = f(\mathbf{x}) \tag{38}$$

Definition 15

If $f \equiv 0$ in (38), then the PDE is called a homogeneous PDE. If $f \neq 0$, then the PDE is an inhomogeneous PDE[1].

Example 8

1. $u_t + uu_x = 0$ is homogeneous
2. $2u_y - 5u^3 = x$ is inhomogeneous
3. $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f(r,\theta)$ is inhomogeneous if $f \neq 0$

[1] In some textbooks it is also called nonhomogeneous PDE. Also, many textbooks usually classify only linear PDE as homogeneous and nonhomogeneous

Examples

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Example 9

PDE	O	Lin	AL	Sem	Qua	HG	FNL
$u_t + u_x + u^2 = 0$	1	✗	✓	✓	✓	✓	✗
$u_{xx} + u_{yy} = 0$	2	✓	✓	✓	✓	✓	✗
$u_x^2 + u_y^2 = x^2 + y^2$	1	✗	✗	✗	✗	✗	✓
$u_x + 5u = x^2 y$	1	✓	✓	✓	✓	✗	✗

O - Order, Lin - Linear, AL - Almost linear, Sem - Semilinear, Qua - Quasilinear, HG - Homogeneous, FNL - Fully nonlinear.

First-order PDEs in Two Variables

The general first-order PDEs in two variables can be written in the form

$$F(x, y, u, u_x, u_y) = 0 \tag{39}$$

The first-order **linear** PDE is of the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u + f(x, y) \tag{40}$$

The first-order **semilinear** PDE is of the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u) \tag{41}$$

The first-order **quasilinear** PDE is of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{42}$$

Second-order PDEs in Two Variables

The general second-order PDEs in two variables can be written in the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

The second-order **linear** PDE is of the form

$$a_1(x,y)u_{xx} + a_2(x,y)u_{xy} + a_3(x,y)u_{yy} + a_4(x,y)u_x + a_5u_y + a_6(x,y)u = f(x,y)$$

The second-order **semilinear** PDE is of the form

$$a_1(x,y)u_{xx} + a_2(x,y)u_{xy} + a_3(x,y)u_{yy} = f(x, y, u, u_x, u_y)$$

The second-order **quasilinear** PDE is of the form

$$a_1(x,y,u,u_x,u_y)u_{xx} + a_2(x,y,u,u_x,u_y)u_{xy} + a_3(x,y,u,u_x,u_y)u_{yy} = f(x,y,u,u_x,u_y)$$

Classification

Remarks 1

A few authors classify

- only linear PDE as homogeneous and nonhomogeneous
- nonlinear PDE as semilinear and non-semilinear
- non-semilinear PDE as quasilinear and non-quasilinear/fully nonlinear

Later, we will see some more classifications like parabolic, elliptic, and hyperbolic when we discuss second-order PDE, which can be extended further for higher-order PDEs.

Remarks 2

One can prove that

$$\text{Linaer PDE} \subsetneq \text{Semilinear PDE} \subsetneq \text{Quasilinear PDE} \subsetneq \text{PDE}$$

(Prove that the inclusion is strict!)

Exercise

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Exercise 4: Hard

Create a table like Example 9 and fill in the tick marks accordingly.

1. $u_{xxx} - 4u_{xxyy} + u_{yyzz} = 0$
2. $u_x^2 u_{tt} - 0.5u = 1 - u^2$
3. $u_{tt} u_{xxx} - u_x u_{ttt} = x^2 + t^2$
4. $e^{u_{xtt}} - u_{xt} u_{xxx} + u^2 = 0$
5. $2\cos(xt) u_t - xe^t u_x - 9u = e^t \sin x$
6. $uu_t + u^2 u_x + u = e^x$
7. $\sqrt{1 + x^2 y^2} u_{xyy} - \cos(xy^3) u_{xxy} + e^{-y^3} u_x - (5x^2 - 2xy + 3y^2)u = 0$