MA612L-Partial Differential Equations

Lecture 34: Nonlinear PDE - General Integral and Charpit's Method

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Nonlinear PDE-Recap

Complete Integrals

$$F(Du, u, \mathbf{x}) = 0 \tag{1}$$

Definition 1 (Complete Integral)
A C^2 function $u = u(\mathbf{x}; \mathbf{a})$ is called a complete integral in $\Omega \times A$ provided

1. $u(\mathbf{x}; \mathbf{a})$ solves the PDE (1) for each $\mathbf{a} \in A$.
2. $\text{rank}(D_{\mathbf{a}} u, D^2_{\mathbf{xa}} u) = n \quad (\mathbf{x} \in \Omega, \mathbf{a} \in A)$

Definition 2 (Particular Integral)
If particular values are given to the parameters $\mathbf{a} \in A$, then the solution is called a particular integral or particular solution of (1).

Envelopes

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Definition 3 (Envelopes)

Let $u = u(\mathbf{x}; \mathbf{a})$ be a C^1 function of $\mathbf{x} \in \Omega, \mathbf{a} \in A$ where $\Omega, A \subset \mathbb{R}^n$ are open sets. Consider the vector equation

$$D_{\mathbf{a}} u(\mathbf{x}; \mathbf{a}) = 0 \quad (\mathbf{x} \in \Omega, \mathbf{a} \in A) \tag{2}$$

Suppose we can solve (2) for the parameter $\mathbf{a}$ as a C^1 function of $\mathbf{x}$.

$$\mathbf{a} = \phi(\mathbf{x}) \tag{3}$$

Then

$$D_{\mathbf{a}} u(\mathbf{x}; \phi(\mathbf{x})) = 0 \quad (\mathbf{x} \in \Omega) \tag{4}$$

We then call

$$v(\mathbf{x}) := u(\mathbf{x}; \phi(\mathbf{x})) = 0 \quad (\mathbf{x} \in \Omega) \tag{5}$$

the envelope of the function $\{u(.; \mathbf{a})\}_{\mathbf{a} \in A}$

Envelopes

Theorem 1 (Construction of New Solutions)
Suppose for each $\mathbf{a} \in A$ as above $u = u(.;\mathbf{a})$ solves (1). Assume further that the envelope v defined by (5) and (4) above exists and is a C^1 function. Then v solves (1) as well.

Proof: From (5), we have

$$\begin{aligned} v_{x_i}(\mathbf{x}) &= u_{x_i}(\mathbf{x};\phi(\mathbf{x})) + \sum_{j=1}^{n} u_{a_j}(\mathbf{x},\phi(\mathbf{x}))\phi^j_{x_i}(\mathbf{x}) \\ &= u_{x_i}(\mathbf{x};\phi(\mathbf{x})) \end{aligned}$$

The second term vanishes due to (4). Hence for each $\mathbf{x} \in \Omega$, we have

$$F(Dv(\mathbf{x}), v(\mathbf{x}), \mathbf{x}) = F(Du(\mathbf{x};\phi(\mathbf{x})), u(\mathbf{x};\phi(\mathbf{x})), \mathbf{x}) = 0$$

Envelopes: Geometric Interpretation

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1. The geometric interpretation for envelope is that for each $\mathbf{x} \in \Omega$, the graph of v is tangent to the graph of $u(.;\mathbf{a})$ for $\mathbf{a} = \phi(\mathbf{x})$. Thus $Dv = D_{\mathbf{x}}u(.;\mathbf{a})$ at $\mathbf{x}$ for $\mathbf{a} = \phi(\mathbf{x})$.
2. The operation of elimination is equivalent to the selection of a representative family from the system of families of surfaces and then finding its envelope.
3. For example, in two dimensions, the above equations represent a curve drawn on the surface of the family whose parameter is a, while the equation obtained by eliminating a between them is the envelope of the family.
4. Hence the envelope touches the surface represented by $F(a, b, z, x, y) = 0$ and $b = \phi(a)$ along the curve represented by $F_a(a, b, z, x, y) = 0$. This curve is called the characteristic of the envelope.

Envelopes: Example

Examples 2

Consider the PDE

$$u^2(1 + |Du|^2) = 1$$

A complete integral is

$$u(\mathbf{x}, \mathbf{a}) = \pm\sqrt{1 - |\mathbf{x} - \mathbf{a}|^2} \quad (|\mathbf{x} - \mathbf{a}| < 1)$$

Now,

$$D_{\mathbf{a}}u = \pm\frac{\mathbf{x} - \mathbf{a}}{\sqrt{1 - |\mathbf{x} - \mathbf{a}|^2}} = 0$$

provided $\mathbf{a} = \phi(\mathbf{x})\mathbf{x}$. Thus, $= u(x; a = x) = \pm 1 \implies v \equiv \pm 1$ are singular integrals of the given PDE.

Envelopes: Example

For,

$$D_x u = \pm \frac{1}{2\sqrt{1 - |\mathbf{x} - \mathbf{a}|^2}}(-2)(\mathbf{x} - \mathbf{a}) = \pm \frac{\mathbf{x} - \mathbf{a}}{\sqrt{1 - |\mathbf{x} - \mathbf{a}|^2}}$$

$$1 + |Du|^2 = 1 + \frac{|\mathbf{x} - \mathbf{a}|}{1 - |\mathbf{x} - \mathbf{a}|^2} = 1$$

Envelopes: General Integral

To generate still more solutions of (1) from a complete integral, let us vary the above construction. Choose any $A \subset \mathbb{R}^{n-1}$ and any C^1 function $h : A' \to \mathbb{R}$ such that the graph lies within A. Let us write

$$\mathbf{a} = (a_1, a_2, \cdots, a_n) = (\mathbf{a}', a_n), \quad \mathbf{a}' = (a_1, a_2, \cdots, a_{n-1}) \in \mathbb{R}^{n-1}$$

Definition 4 (General Integral)
The general integral (depending on h) is the envelope $v' = v'(\mathbf{x})$ of the functions

$$u'(\mathbf{x}; \mathbf{a}) = u(\mathbf{x}; \mathbf{a}', h(\mathbf{a}')) \quad (\mathbf{x} \in \Omega, \mathbf{a}' \in A')$$

provided this envelope exists in C^1.

Envelopes

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Examples 3

An alternative form for a complete integral of the eikonal equation is given as follows for $n = 2$

$$|Du| = 1$$

$$u(\mathbf{x}; \mathbf{a}) = x_1 \cos a_1 + x_2 \sin a_1 + a_2 \quad \mathbf{x}, \mathbf{a} \in \mathbb{R}^2 \tag{6}$$

If we set $h \equiv 0$ such that

$$u'(\mathbf{x}; a_1) = x_1 \cos a_1 + x_2 \sin a_1$$

represents the subfamily of planar solutions of $|Du| = 1$, whose graphs pass through the point $(0, 0, 0) \in \mathbb{R}^3$, then the envelope is computed by solving

$$D_{a_1} u'(\mathbf{x}; a_1) = -x_1 \sin a_1 + x_2 \cos a_1 = 0$$

Envelopes

$$\implies a_1 = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

Hence,

$$u'(\mathbf{x};a_1) = x_1 \cos\left(\tan^{-1}\left(\frac{x_2}{x_1}\right)\right) + x_2 \sin\left(\tan^{-1}\left(\frac{x_2}{x_1}\right)\right) = \pm|\mathbf{x}|$$

solves $|Du| = 1$ for $\mathbf{x} \neq 0$

General Integral: Remarks

Remarks

1. To compute envelope, we restrict only to parameters $\mathbf{a}$ of the form $a = (\mathbf{a}', h(\mathbf{a}'))$
2. From a complete integral with n arbitrary constants, to construct a general integral, we need an arbitrary function h of $n-1$ variables.

Question

Suppose we find a solution of (1) depending on an arbitrary function h, does it mean we have found all solutions of (1)?

General Integral: Remarks

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Answer: No Suppose our PDE is of the following form

$$F(Du, u, \mathbf{x}) = F_1(Du, u, \mathbf{x})F_2(Du, u, \mathbf{x}) = 0$$

Suppose $u_1(\mathbf{x}; \mathbf{a})$ is a complete integral of $F_1(Du, u, \mathbf{x}) = 0$ and we found a general integral corresponding to any function h, we have still missed to find all solutions of the PDE $F_2(Du, u, \mathbf{x}) = 0$.

Question

How do we find these solutions?

Remember Characteristic ODE. Let us have a look at a few more methods to find the complete integral of a nonlinear PDE, namely Charpit's method and Jacobi's method.

Charpit's Method

If the PDE is of the following form in two variables,

$$F(x, y, z, p, q) = 0 \tag{7}$$

then the corresponding characteristic or Charpit's auxiliary equation is given by:

$$\frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{dz}{-F_p p - F_q q} = \frac{dp}{F_x + F_z p} = \frac{dq}{F_y + F_z q} = \frac{dG}{0} \tag{8}$$

How did we get (8)?

Charpit's Method

Our goal is to find a surface $z = z(x, y)$ satisfying $F(x, y, z, p, q) = 0$

$$dF = F_x dx + F_y dy + F_z dz + F_p dp + F_q dq = 0$$

Since $dz = pdx + qdy$, we have

$$dF = (F_x + pF_z)dx + (F_y + qF_z)dy + F_z dz + F_p dp + F_q dq = 0$$

We want to find a system of ODEs that describes the characteristic curves along which F is constant. There are infinitely many possible sets that can satisfy. Charpit's choice was
$x' = F_p, y' = F_q, p' = -(F_x + pF_z), q' = -(F_y + qF_z)$

Charpit's Method: Summary

Steps involved in Charpit's method are as follows

Charpit's method: Steps

1. Convert given PDE to the form
   $F(x, y, z, p, q) = 0, \quad z = u, p = z_x = u_z, q = z_y = u_y$
2. Write Charpit's auxiliary equations (8)
3. Compute F_x, F_y, F_z, F_p, F_q and substitute in (8)
4. Solve the Characteristic ODEs to solve for p and q
5. Substitute p and q in the following equation to obtain the complete integral.

$$dz = pdx + qdy \tag{9}$$

Charpit's Method

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Examples 4

Find a complete integral of the PDE $3p^2 = q$ using Charpit's method.

$$F(x, y, z, p, q) = 3p^2 - q = 0$$

The auxiliary equations are

$$\frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{dz}{-F_p p - F_q q} = \frac{dp}{F_x + F_z p} = \frac{dq}{F_y + F_z q} = \frac{dG}{0}$$

$$\frac{dx}{-6p} = \frac{dy}{1} = \frac{dz}{-6p^2 + q} = \frac{dp}{0} = \frac{dq}{0} = \frac{dG}{0}$$

$$dp = 0 \implies p = a \implies q = 3a^2$$

$$dz = adx + 3a^2 dy \implies z = ax + 3a^2 y + b$$

Charpit's Method

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Examples 5

Find a complete integral of the PDE $p^2 - y^2 q = y^2 - x^2$ using Charpit's method.

$$F(x, y, z, p, q) = p^2 - y^2 q - y^2 + x^2 = 0$$

(Auxiliary): $$\frac{dx}{-F_p} = \frac{dy}{-F_q} = \frac{dz}{-F_p p - F_q q} = \frac{dp}{F_x + F_z p} = \frac{dq}{F_y + F_z q} = \frac{dG}{0}$$

$$\frac{dx}{-2p} = \frac{dy}{y^2} = \frac{dz}{-2p^2 + y^2 q} = \frac{dp}{2x} = \frac{dq}{-2yq - 2y} = \frac{dG}{0}$$

$$pdp + xdx = 0 \implies p^2 + x^2 = a^2 \implies q = \frac{a^2}{y^2} - 1$$

$$dz = \frac{1}{\sqrt{a^2 - x^2}} dx + \left(\frac{a^2}{y^2} - 1\right) dy \implies z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{a^2}{y^2} - y$$

Nonlinear PDE-Jacobi's Method

Jacobi's Method

- This method is used for solving nonlinear PDEs of order one involving three or more independent variables.
- The basic idea is almost the same as Charpit's method
- Let us have a look at the development of this procedure for three independent variables, which can be generalized for n variables also

Consider the following first-order nonlinear PDEs of the form

$$F(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \tag{10}$$

Here $p_i = u_{x_i}, i = 1, 2, 3$. Note that here u does not appear.

Jacobi's Method

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The fundamental idea of Jacobi's method is to introduce two first-order PDES involving two arbitrary constants a and b in the following form

$$G_1(x_1, x_2, x_3, p_1, p_2, p_3, a) = 0 \tag{11}$$

$$G_2(x_1, x_2, x_3, p_1, p_2, p_3, b) = 0 \tag{12}$$

Further, G_1 and G_2 must satisfy the following

$$\frac{\partial(F, G_1, G_2)}{\partial(p_1, p_2, p_3)} \neq 0 \tag{13}$$

Equations (10), (11) and (12) can be solved for p_1, p_2 and p_3. Substitute p_1, p_2 and p_3 in the following equation to obtain the complete integral.

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \tag{14}$$

Jacobi's Method

Differentiation (10) and (11) w.r.t. x_1, we get

$$F_{x_1} + F_{p_1} p_{1_{x_1}} + F_{p_2} p_{2_{x_1}} + F_{p_3} p_{3_{x_1}} = 0 \tag{15}$$

$$G_{1_{x_1}} + G_{1_{p_1}} p_{1_{x_1}} + G_{1_{p_2}} p_{2_{x_1}} + G_{1_{p_3}} p_{3_{x_1}} = 0 \tag{16}$$

Eliminating $p_{1_{x_1}}$ between (15) and (16), we obtain that

$$(F_{x_1} G_{1_{p_1}} - F_{p_1} G_{1_{x_1}}) + (F_{p_2} G_{1_{p_1}} - F_{p_1} G_{1_{p_2}}) p_{2_{x_1}} + (F_{p_3} G_{1_{p_1}} - F_{p_1} G_{1_{p_3}}) p_{3_{x_1}} = 0 \tag{17}$$

$$\frac{\partial(F, G_1)}{\partial(x_1, p_1)} + \frac{\partial(F, G_1)}{\partial(p_2, p_1)} p_{2_{x_1}} + \frac{\partial(F, G_1)}{\partial(p_3, p_1)} p_{3_{x_1}} = 0 \tag{18}$$

Jacobi's Method

Similarly

$$\frac{\partial(F, G_1)}{\partial(x_2, p_2)} + \frac{\partial(F, G_1)}{\partial(p_1, p_2)} p_{1_{x_2}} + \frac{\partial(F, G_1)}{\partial(p_3, p_2)} p_{3_{x_2}} = 0 \tag{19}$$

and

$$\frac{\partial(F, G_1)}{\partial(x_3, p_3)} + \frac{\partial(F, G_1)}{\partial(p_1, p_3)} p_{1_{x_3}} + \frac{\partial(F, G_1)}{\partial(p_2, p_3)} p_{2_{x_3}} = 0 \tag{20}$$

Jacobi's Method

Suppose, if we impose the condition,

$$p_{i_{x_j}} = p_{j_{x_i}},$$

by adding (18),(19) and (20), we obtain that

$$\frac{\partial(F,G_1)}{\partial(x_1,p_1)} + \frac{\partial(F,G_1)}{\partial(x_2,p_2)} + \frac{\partial(F,G_1)}{\partial(x_3,p_3)} = 0 \tag{21}$$

because

$$\frac{\partial(F,G_1)}{\partial(p_i,p_j)} = -\frac{\partial(F,G_1)}{\partial(p_j,p_i)} \tag{22}$$

Jacobi's Method

Similarly we get

$$\frac{\partial(F, G_2)}{\partial(x_1, p_1)} + \frac{\partial(F, G_2)}{\partial(x_2, p_2)} + \frac{\partial(F, G_2)}{\partial(x_3, p_3)} = 0 \qquad (23)$$

and

$$\frac{\partial(G_1, G_2)}{\partial(x_1, p_1)} + \frac{\partial(G_1, G_2)}{\partial(x_2, p_2)} + \frac{\partial(G_1, G_2)}{\partial(x_3, p_3)} = 0 \qquad (24)$$

(23) and (24) are semi-linear PDEs of the form

$$F_{x_1}\frac{\partial G_1}{\partial p_1} - F_{p_1}\frac{\partial G_1}{\partial x_1} + F_{x_2}\frac{\partial G_1}{\partial p_2} - F_{p_2}\frac{\partial G_1}{\partial x_2} + F_{x_3}\frac{\partial G_1}{\partial p_3} - F_{p_3}\frac{\partial G_1}{\partial x_3} = 0 \qquad (25)$$

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Jacobi's Method

Therefore, the Jacobi Auxiliary equation is given by

$$\frac{dx_1}{-F_{p_1}} = \frac{dx_2}{-F_{p_2}} = \frac{dx_3}{-F_{p_3}} = \frac{dp_1}{F_{x_1}} = \frac{dp_2}{F_{x_2}} = \frac{dp_3}{F_{x_3}} \tag{26}$$

The rest of the procedures are similar to Charpit's method. Substitute p_1, p_2 and p_3 in the following equation to obtain the complete integral.

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \tag{27}$$

Jacobi's Method

In general, we can extend this idea to the following PDEs.

$$F(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n) = 0 \tag{28}$$

Therefore, the Jacobi Auxiliary equation is given by

$$\frac{dx_1}{-F_{p_1}} = \frac{dx_2}{-F_{p_2}} = \cdots = \frac{dx_n}{-F_{p_n}} = \frac{dp_1}{F_{x_1}} = \frac{dp_2}{F_{x_2}} = \cdots = \frac{dp_n}{F_{x_n}} \tag{29}$$

The rest of the procedures are similar to Charpit's method. Substitute $p_1, p_2, \cdots, p_n$ in the following equation to obtain the complete integral.

$$dz = p_1 dx_1 + p_2 dx_2 + \cdots + p_n dx_n \tag{30}$$

Jacobi’s Method

Example 6

Find the complete integral of the following PDE: $p_1^2 x_1 + p_2^2 x_2 = p_3^2 x_3$ by Jacobi’s method.

$$F(x_1, x_2, x_3, p_1, p_2, , p_3) = p_1^2 x_1 + p_2^2 x_2 - p_3^2 x_3 = 0 \tag{31}$$

The Auxiliary equation is given by

$$\frac{dx_1}{-F_{p_1}} = \frac{dx_2}{-F_{p_2}} = \frac{dx_3}{-F_{p_3}} = \frac{dp_1}{F_{x_1}} = \frac{dp_2}{F_{x_2}} = \frac{dp_3}{F_{x_3}}$$

$$\frac{dx_1}{-2x_1p_1} = \frac{dx_2}{-2x_2p_2} = \frac{dx_3}{2x_3p_3} = \frac{dp_1}{p_1^2} = \frac{dp_2}{p_2^2} = \frac{dp_3}{-p_3^2}$$

Jacobi's Method

$$\frac{dp_1}{p_1^2} = \frac{dx_1}{-2x_1p_1} \implies \frac{dp_1}{p_1} = \frac{dx_1}{-2x_1}$$

$$\implies \ln p_1 = -2\ln x_1 + \ln a \implies p_1 = \frac{a_1}{\sqrt{x_1}}$$

Similarly

$$p_2 = \frac{a_2}{\sqrt{x_2}}, p_3 = \frac{a_3}{\sqrt{x_3}}$$

Using these values in the given equation, we obtain

$$a_1^2 + a_2^2 = a_3^2$$

$$dz = \frac{a_1}{\sqrt{x_1}}dx_1 + \frac{a_2}{\sqrt{x_2}}dx_2 + \frac{\sqrt{a_1^2 + a_2^2}}{\sqrt{x_3}}dx_3$$

$$\implies z = 2a_1\sqrt{x_1} + 2a_2\sqrt{x_2} + 2\sqrt{a_1^2 + a_2^2}\sqrt{x_3} + b$$

Jacobi's Method

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Example 7

Find the complete integral of the following PDE: $p_1^3 + p_2^3 + p_3 = 1$ by Jacobi's method.

$$F(x_1, x_2, x_3, p_1, p_2, , p_3) = p_1^3 + p_2^3 + p_3 - 1 = 0 \tag{32}$$

The Auxiliary equation is given by

$$\frac{dx_1}{-F_{p_1}} = \frac{dx_2}{-F_{p_2}} = \frac{dx_3}{-F_{p_3}} = \frac{dp_1}{F_{x_1}} = \frac{dp_2}{F_{x_2}} = \frac{dp_3}{F_{x_3}}$$

$$\frac{dx_1}{-3p_1^2} = \frac{dx_2}{-3p_2^2} = \frac{dx_3}{1} = \frac{dp_1}{0} = \frac{dp_2}{0} = \frac{dp_3}{0}$$

$$p_1 = a_1, p_2 = a_2, p_3 = a_3 \implies a_3 = 1 - a_1^3 - a_2^3$$

$$z = a_1x_1 + a_2x_2 + (1 - a_1^2 - a_2^3)x_3 + b$$

Jacobi's Method

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Exercise 1: Jacobi's Method

Find the complete integral of the following PDEs

- $2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0$
- $x_3^2p_1^2p_2^2p_3^2 + p_1^2p_2^2 - p_3^2 = 0$
- $p_1x_1 + p_2x_2 = p_3^2$
- $p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0$
- $(x_2 + x_3)(p_2 + p_3)^2 - x_4p_1p_4 = 0$

Thanks

Doubts and Suggestions

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