MA612L-Partial Differential Equations

Lecture 35: Nonlinear PDE - Standard Forms

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Nonlinear PDE- with two variables Complete Integral

Nonlinear PDE- with two variables: Complete Integral

The most general form of a PDE of order one is

$$F(x, y, z, p, q) = 0$$

where

$$z = u(x, y), p = u_x, q = u_y$$

The relation between the dependent and independent variables obtained from the given PDE is called a solution or integral of the PDE, provided the values of the dependent variable and its partial derivatives satisfy the PDE.

Nonlinear PDE- with two variables: Complete Integral

Now consider the relation

$$\phi(x, y, z, a, b) = 0 \tag{1}$$

in the variable x, y and z where z is a dependent variable and a, b are arbitrary constants. Differentiating (1) w.r.t. x and y, we get

$$\phi_x + \phi_z p = 0 \tag{2}$$
$$\phi_y + \phi_z q = 0 \tag{3}$$

Hence, there are two arbitrary constants with three equations; upon eliminating the constant, we will obtain a relation

$$F(x, y, z, p, q) = 0 \tag{4}$$

Nonlinear PDE- with two variables: Complete Integral

The relation (1) is called a solution or integral of the PDE (4). Such a type of solution, which has *as many arbitrary constants as there are independent variables*, is called **complete integral** of (4).

Examples 1

The complete integral of $z = px + qy$ is $z = ax + by$.

If a *particular value* is given *to the arbitrary constants* in the complete integral of a PDE of order one, then the solution obtained is known as the **particular integral** of the given PDE.

Nonlinear PDE- with two variables: Singular Integral

In (1) we have assumed a and b are constants. Instead, if we assume a and b are functions of independent variables, then these don't alter the form of p and q. Hence differentiating, (1) w.r.t. x and y, we obtain

$$\phi_x + \phi_z p + \phi_a a_x + \phi_b b_x = 0 \tag{5}$$

$$\phi_y + \phi_z q + \phi_a a_y + \phi_b b_y = 0 \tag{6}$$

The forms of p and q will be same as in (2) and (3), if we have

$$\phi_a a_x + \phi_b b_x = 0 \tag{7}$$

$$\phi_a a_y + \phi_b b_y = 0 \tag{8}$$

Nonlinear PDE- with two variables: Singular Integral

$$\overbrace{\begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix}}^{R} \begin{bmatrix} \phi_a \\ \phi_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\det R \neq 0$, then we must have $\phi_a = 0, \phi_b = 0.$
From $\phi_a = 0$ and $\phi_b = 0$, we can obtain the value of a and b in terms of the variables. The solutions obtained by solving $\phi_a = 0, \phi_b = 0$ together with $\phi(x, y, z, a, b) = 0$ are called the ***singular integral*** of the given PDE.

Nonlinear PDE- with two variables: General Integral

In the complete integral $\phi(x, y, z, a, b) = 0$ if the arbitrary constants a and b are functionally related, that is, if

$$b = \psi(a) \tag{9}$$

where ψ is an arbitrary function. Hence, (7) $\times\, dx +$ (8) $\times\, dy$ gives

$$\begin{aligned} \phi_a a_x dx + \phi_b b_x dx + \phi_a a_y dy + \phi_b b_y dy &= 0 \\ \phi_a(a_x dx + a_y dy) + \phi_b(b_x dx + b_y dy) &= 0 \\ \phi_a da + \phi_b db &= 0 \end{aligned}$$

$$b = \psi(a) \implies db = \phi_a da$$

$$\implies \phi_a + \phi_b \psi_a = 0 \tag{10}$$

Nonlinear PDE- with two variables: General Integral

From (10), the value of a involving the arbitrary function G may be obtained. Then b is given by equation (9). When these values are used in (1), it takes a new form which is different from the previously obtained integrals. This solution is known as **general integral** of the given PDE.

Examples 2

We have seen that $z = ax + by$ is a complete integral of the PDE $z = px + qy$. Now,

$$z = yf\left(\frac{y}{x}\right)$$

is also an integral of the PDE $z = px + qy$.

Note: Without singular integral and general integral, the complete integral is considered to be an incomplete solution of the given PDE.

Geometrical Interpretation

Complete integral:

- A complete integral, being a relation between x, y and z, is the equation of a surface.
- Since it contains two arbitrary parameters, it belongs to a double infinite system of surface or to a single infinite system of a family of surfaces.

Geometrical Interpretation

General integral: Let $\phi(x, y, z, a, b) = 0$ be a complete integral of $F(x, y, z, p, q) = 0$. A general integral is obtained from $b = \psi(a)$ and $\phi_a + \phi_b \psi_a = 0$.

- Operation of elimination is equivalent to the selection of a representative family from the system of families of surfaces and then finding its envelope.
- They represent a curve drawn on the surface of the family whose parameter is a
- The equation obtained by eliminating a between them is the envelope of the family.
- This curve is called the characteristic of the envelope, and the general integral thus represents the envelope of a family of surfaces considered as composed of its characteristics.

Geometrical Interpretation

Singular integral: For singular integral, we need to eliminate a and b from

$$\phi(x, y, z, a, b) = 0, \phi_a = 0, \phi_b = 0$$

- Operation of elimination is equivalent to finding the envelope of all the surfaces included in the complete integral.
- Three equations used in finding singular integrals give the point of contact of the particular surface represented by $\phi(x, y, z, a, b)$ with the general envelope.
- The singular integral thus represents the general envelope of all the surfaces included in he complete integral.

Standard Form I

Standard Form I

Nonlinear PDE of type

$$F(p, q) = 0 \tag{11}$$

Let us take the trial solution

$$z = ax + by + c \tag{12}$$

Then

$$z_x = p = a, \quad z_y = q = b \tag{13}$$

Hence $F(a, b) = 0$. Hence (12) is a solution of (11) if $F(a, b) = 0$. Solving for $F(a, b) = 0$ for b, we obtain $b = f(a)$. Therefore, the complete integral is

$$z = ax + yf(a) + c \tag{14}$$

Standard Form I

To find the singular integral, we have to eliminate a and c from (14) and the equations $z_a = 0, z_c = 0$. However, $z_c = 0 \implies 1 = 0$ which is not valid.
Therefore, there is no singular integral.
To find general integral, let us take $c = \psi(a)$ in (14) and get

$$z = ax + yf(a) + \psi(a) \tag{15}$$

Differentiating (15) w.r.t. a, we get

$$0 = x + yf'(a) + \phi'(a) \tag{16}$$

Eliminating a from (15) and (16), we get the general integral.

Standard Form I

Examples 3

Find the complete integral of the PDE $pq = 2$.

We have

$$F(p, q) = pq - 2 = 0$$

The complete integral is given by $z = ax + by + c$, where

$$F(a, b) = ab - 2 = 0 \implies b = \frac{2}{a}$$

Hence, the complete integral is given by

$$z = ax + \frac{2y}{a} + c$$

Standard Form I

Examples 4

Find the complete integral of the PDE $\sqrt{p} + \sqrt{q} = 1$.

We have

$$F(p, q) = pq - 2 = 0$$

The complete integral is given by $z = ax + by + c$, where

$$F(a, b) = \sqrt{a} + \sqrt{b} - 1 = 0 \implies b = (1 - \sqrt{a})^2$$

Hence, the complete integral is given by

$$z = ax + (1 - \sqrt{a})^2 y + c$$

Standard Form I

Exercise 1: Standard Form I

Find the complete integral of the following PDEs

- $p + q = pq$
- $p = e^q$
- $p^2 - q^2 = 4$
- $p + \sin q = 0$
- $q - \sin p = 0$

Standard Form II: Clairaut's Equation

Standard Form II

Nonlinear PDE of type

$$z = px + qy + f(p, q) \tag{17}$$

is known as Clairaut's type of PDE. The complete integral is

$$z = ax + by + f(a, b) \tag{18}$$

The singular integrals can be obtained as discussed earlier.

Standard Form II

Examples 5

Find the complete and singular integral of the PDE $z = px + qy + pq$.
The complete integral is given by

$$z = ax + by + ab$$

To find the singular integral, differentiate the complete integral w.r.to. a and b. Then

$$x + b = 0 \implies b = -x$$
$$y + a = 0 \implies a = -y$$
$$\implies z = -yx - xy + xy \implies z + zy = 0$$

Standard Form II

Exercise 2: Standard Form II

- Find the complete integral of the following PDEs
  - $pqz = p^2(xq + p^2) + q^2(yp + q^2)$
  - $(p - q)(z - px - qy) = 1$
- Prove that the singular integral of $z = px + qy + \sqrt{p^2 + q^2 + 1}$ is the unit sphere with centre at the origin.
- Prove that the complete integral of $z = px + qy - 2p - 3q$ represents all possible planes through the point $(2, 3, 0)$.

Standard Form III

Standard Form III

Let us consider the Nonlinear PDE of type

$$F(z, p, q) = 0 \tag{19}$$

Let $z = \phi(x + ay)$ be a trial solution. Let $u = x + ay$, then $p = z_x = z_u u_x = z_u$ and $q = z_y = z_u u_y = az_u$ Hence, we get

$$F(z, z_u, az_u) = 0$$

This is an ODE, and its solution is the complete integral which is given by $z = f(u + b)$

$$z = f(x + ay + b) \tag{20}$$

The singular integrals can be obtained as discussed earlier.

Standard Form III

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Examples 6

Find the complete integral of the PDE $z = p^2 - q^2$.

As we discussed above, $p = z_u, q = az_u$ and hence

$$z = z_u^2 - a^2 z_u^2$$

Upon solving, we obtain

$$\sqrt{1-a^2} 2\sqrt{z} = u + b$$

Therefore, the complete integral is given by

$$4(1-a^2)z = (x + ay + b)^2$$

Standard Form III

Examples 7

Solve the following PDE completely $z = p^2 + q^2$.
Here, we need to find a complete, singular, and general integral.
As we discussed above, $p = z_u, q = az_u$ and hence

$$z = z_u^2 + a^2 z_u^2$$

Upon solving, we obtain

$$\sqrt{1 + a^2} 2\sqrt{z} = u + b$$

Therefore, the complete integral is given by

$$4(1 + a^2)z = (x + ay + b)^2$$

Standard Form III

Differentiating the complete integral w.r.t. a and b, we get

$$8az = 2y(x + ay + b) \quad \text{and} \quad 0 = 2(x + ay + b)$$

$$\implies 8az = 0 \implies z = 0$$

The singular integral is $z = 0$.
Now, let $b = \phi(a)$ then the complete integral becomes

$$4(1 + a^2)z = (x + ay + \phi(a))^2$$

Differentiating this w.r.t. a we get

$$8az = 2(x + ay + \phi(a))(y + \phi'(a))$$

By eliminating a between the above two equations, we obtain a general integral.

Standard Form III

Exercise 3: Standard Form III

- Find the complete integral of the following PDEs
  - $p^2z^2 + q^2 = p^2q$
  - $p^3 + q^3 = 3pqz$
- Solve the PDE completely $p^2 + pq = 4z$
- Solve the PDE completely $p(1 - q^2) = q(1 - z)$

Standard Form IV

Standard Form IV

Let us consider the Nonlinear PDE of type

$$F_1(x,p) = F_2(y,q) \tag{21}$$

This is a separable PDE. Hence

$$F_1(x,p) = F_2(y,q) = a \tag{22}$$

Upon solving for p and q we get

$$p = f_1(x,a), \quad q = f_2(y,a) \tag{23}$$

$$dz = pdx + qdy$$

gives complete integral, which is given

$$z = \int f_1(x,a)dx + \int f_2(y,a)dy + b \tag{24}$$

There is no singular integral as $z_b = 0 \implies 1 = 0$

Standard Form IV

Examples 8

Find the complete integral of the PDE $p + q = \sin x + \sin y$.

This can be written as $p - \sin x = q - \sin y$ As we discussed above, $p = a + \sin x, q = \sin y - a$ Therefore, the complete integral is given by

$$z = a(x - y) - \cos x - \cos y + b$$

Few More Types

Few More types

MA612L PARTIAL DIFFERENTIAL EQUATIONS

For the type $F(x^m p, y^n q) = 0, m \neq 1, n \neq 1$**:**

- Take $X = x^{1-m}, Y = y^{1-n}$, then we can translate this to

$$F[(1-m)P, (1-n)Q] = 0$$

where $P = z_X, Q = z_Y$ is of the Standard form I.

For the type $F(x^m p, y^n q) = 0, m = 1, n = 1$**:**

- Take $X = \log x, Y = \log y$, then we can translate this to

$$F[P, Q] = 0$$

where $P = xp, Q = yq$ is of the Standard form I.

Few More types

For the type $F(x^m p, y^n q, z) = 0$:

- By above X and Y, we can transform this to the form $F(z, P, Q) = 0$ which is Standard form III.

For the type $F(z^k p, z^k q) = 0$:

- Take

$$Z = \begin{cases} z^{k+1} & k \neq -1 \\ \log z & k = -1 \end{cases}$$

then we can translate this to $F[P, Q] = 0$ is of the Standard form I.

For the type $F_1(x, z^k p) = F_2(y, z^k q)$:

- The above Z transform this to the Standard form IV.

Few More types

For the type $F(x^m z^k p, y^n z^k q) = 0$**:** Use the following substitution to transform this to the standard form $F(P, Q) = 0$

$$X = \begin{cases} x^{1-m} & m \neq 1 \\ \log x & m = 1 \end{cases}$$

$$Y = \begin{cases} x^{1-n} & n \neq 1 \\ \log n & n = 1 \end{cases}$$

$$Z = \begin{cases} z^{k+1} & k \neq -1 \\ \log z & k = -1 \end{cases}$$

Thanks

Doubts and Suggestions

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