MA612L-Partial Differential Equations

Lecture 36: Nonlinear PDE - Characteristic ODE

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Characteristic ODE

Characteristic ODE

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Consider the nonlinear first-order PDE

$$F(Du, u, x) = 0 \text{ in } \Omega \qquad (1)$$

subject to the boundary condition

$$u = g \text{ on } \Gamma \qquad (2)$$

As we have seen in Jacobi's method, we generalize this concept as follows: Suppose u solves (1) and (2). Fix $\mathbf{x} \in \Omega$. We would like to calculate $u(\mathbf{x})$ by finding some curve lying within Ω connecting $\mathbf{x}$ with a point $\mathbf{x}^0 \in \Gamma$.

Characteristic ODE

Suppose, we can describe $\mathbf{x}$ parametrically as follows:

$$\mathbf{x}(s) = (x_1(s), x_2(s), \cdots, x_n(s)), s \in \mathbb{R} \tag{3}$$

Assume that $u \in C^2$ is a solution of (1). Define

$$z(s) := u(\mathbf{x}(s)) \tag{4}$$

and

$$\mathbf{p}(s) := Du(\mathbf{x}(s)) = (p_1(s), p_2(s), \cdots, p_n(s)) = (u_{x_1}(s), u_{x_2}(s), \cdots, u_{x_n}(s)) \tag{5}$$

Here z gives the values of u along the curve and $\mathbf{p}$ gives the values of the gradient Du. Now, we must choose x in such a way that we can compute z and $\mathbf{p}$.

Characteristic ODE

If we differentiate (5) w.r.t. s, we get

$$\frac{dp_i}{ds} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(\mathbf{x}(s))\frac{dx_j}{ds} \tag{6}$$

Similarly differentiating (1) w.r.t. x_i, we obtain

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, \mathbf{x})\frac{\partial^2 u}{\partial x_j \partial x_i} + \frac{\partial F}{\partial z}(Du, u, \mathbf{x})\frac{\partial u}{\partial x_i} + \frac{\partial F}{\partial x_i}(Du, u, \mathbf{x}) = 0 \tag{7}$$

Now, choose your $\mathbf{x}(s)$ such that

$$\frac{dx_j}{ds}(s) = \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \tag{8}$$

Characteristic ODE

Hence, we obtain

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \frac{\partial^2 u}{\partial x_j \partial x_i}(\mathbf{x}(s)) + \frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p_i(s) + \frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) = 0$$

Using the above expression and (8) in (6), we obtain that

$$\frac{dp_i}{ds} = -\frac{\partial F}{\partial z}(\mathbf{p}(s), z(s), \mathbf{x}(s)) p_i(s) - \frac{\partial F}{\partial x_i}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \tag{9}$$

Characteristic ODE

Again differentiating (4) w.r.t. s, we obtain

$$\frac{dz}{ds} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}(\mathbf{x}(s))\frac{dx_j}{ds} = \sum_{j=1}^{n} p_j(s)\frac{\partial F}{\partial p_j}(\mathbf{p}(s), z(s), \mathbf{x}(s)) \tag{10}$$

Hence we obtain

$$\begin{cases} (a) \quad \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ (b) \quad \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)).\mathbf{p}(s) \\ (c) \quad \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \tag{11}$$

Characteristic ODE

1. These $2n+1$ first-order ODEs are called characteristic equations of the nonlinear PDE (1).
2. The functions (5) and (4) are called characteristics.
3. We refer $\mathbf{x}(s)$ as the projected characteristic.
4. The projection of the full characteristics $(\mathbf{p}(.), z(.), \mathbf{x}(.)) \subset \mathbb{R}^{2n+1}$ onto the physical region $\Omega \subset \mathbb{R}^n$

Theorem 1

Let $u \in C^2(\Omega)$ solve the nonlinear first order PDE (1) in Ω. Assume $\mathbf{x}(.)$ solves the ODE (11)(c). Then $\mathbf{p}(.)$ solve the ODE (11)(a) and $z(.)$ solve the ODE (11)(b) for those s such that $x(s) \in \Omega$.

Characteristic ODE: F linear

Let us consider the linear PDE and homogeneous, then

$$F(Du, u, \mathbf{x}) = \mathbf{b}(\mathbf{x}).Du(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = 0 \tag{12}$$

Then

$$F(\mathbf{p}, z, \mathbf{x}) = \mathbf{b}(\mathbf{x}).\mathbf{p} + c(\mathbf{x})z$$

Hence

$$D_{\mathbf{p}}F = \mathbf{b}(\mathbf{x})$$

and

$$\begin{cases} (b) \quad \dot{z}(s) = \mathbf{b}(\mathbf{x}(s)).\mathbf{p}(s) = -c(\mathbf{x}(s))z(s) \\ (c) \quad \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)) \end{cases} \tag{13}$$

Here, $\mathbf{p}(.)$ is not needed.

Characteristic ODE: F linear

Example 2

Solve the following PDE

$$\begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } \Omega \\ u = g & \text{on } \Gamma \end{cases} \tag{14}$$

where $\Omega = \{x_1 > 0, x_2 > 0\}, \Gamma = \{x_1 > 0, x_2 = 0\} \subset \partial\Omega$. This PDE is of the form linear PDE, with $\mathbf{b} = (-x_2, x_1)$ and $c = -1$. Thus, the equation will become

$$\begin{cases} (b) & \dot{z}(s) = z \\ (c) & \dot{x_1} = -x_2, \dot{x_2} = x_1 \end{cases} \tag{15}$$

Characteristic ODE: F linear

Upon solving, we obtain

$$\begin{cases} (b) & z(s) = z_0 e^s = g(x_0) e^s \\ (c) & x_1(s) = x_0 \cos s, x_2(s) = x_0 \sin s \end{cases} \tag{16}$$

where $x_0 \geq 0, 0 \leq s \leq \frac{\pi}{2}$. Fix a point $(x_1, x_2) \in \Omega$. Let $s > 0, x_0 > 0$ such that $(x_1, x_2) = (x_0 \cos s, x_0 \sin s)$. That is, $x_0 = \sqrt{x_1^2 + x_2^2}, s = \tan^{-1}\left(\frac{x_2}{x_1}\right)$. Hence

$$u(x_1, x_2) = g\left(\sqrt{x_1^2 + x_2^2}\right) e^{\tan^{-1}\left(\frac{x_2}{x_1}\right)}$$

Characteristic ODE: F Quasilinear

Let us consider the quasilinear PDE and homogeneous, then

$$F(Du, u, \mathbf{x}) = \mathbf{b}(\mathbf{x}, u(\mathbf{x})).Du(\mathbf{x}) + c(\mathbf{x}, u(\mathbf{x})) = 0 \tag{17}$$

Then

$$F(\mathbf{p}, z, \mathbf{x}) = \mathbf{b}(\mathbf{x}, z).\mathbf{p} + c(\mathbf{x}, z)$$

Hence

$$D_{\mathbf{p}}F = \mathbf{b}(\mathbf{x}, z)$$

and

$$\begin{cases} (b) \quad \dot{z}(s) = \mathbf{b}(\mathbf{x}(s), z(s)).\mathbf{p}(s) = -c(\mathbf{x}(s), z(s)) \\ (c) \quad \dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s), z(s)) \end{cases} \tag{18}$$

Here, $\mathbf{p}(.)$ is not needed.

Characteristic ODE: F Quasilinear

Example 3

Solve the following PDE

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 & \text{in } \Omega \\ u = g & \text{on } \Gamma \end{cases} \tag{19}$$

where $\Omega = \{x_2 > 0\}, \Gamma = \{x_2 = 0\} = \partial\Omega$. This PDE is of the form quasilinear PDE, with $\mathbf{b} = (1,1)$ and $c = -z^2$. Thus the equation will become

$$\begin{cases} (b) & \dot{z}(s) = z^2 \\ (c) & \dot{x_1} = 1, \dot{x_2} = 1 \end{cases} \tag{20}$$

Characteristic ODE: F Quasilinear

Upon solving, we obtain

$$\begin{cases} (b) & z(s) = \frac{z_0}{1-sz_0} = \frac{g(x_0)}{1-sg(x_0)} \\ (c) & x_1(s) = x_0 + s, x_2(s) = s \end{cases} \tag{21}$$

where $x_0 \in \mathbb{R}, s \geq 0$. Fix a point $(x_1, x_2) \in \Omega$. Let $s > 0, x_0 > 0$ such that $(x_1, x_2) = (x_0 + s, s)$. That is, $x_0 = x_1 - x_2, s = x_2$. Hence

$$u(x_1, x_2) = \frac{g(x_0)}{1 - sg(x_0)} = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}$$

Characteristic ODE: F Fully nonlinear

Example 4

Solve the following PDE

$$\begin{cases} u_{x_1}u_{x_2} = u & \text{in } \Omega \\ u = x_2^2 & \text{on } \Gamma \end{cases} \tag{22}$$

where $\Omega = \{x_1 > 0\}, \Gamma = \{x_1 = 0\} = \partial\Omega$. Here

$$F(\mathbf{p}, z, \mathbf{x}) = p_1 p_2 - z$$

.

$$\begin{cases} (a) & \dot{p}_1 = p_1, \dot{p}_2 = p_2 \\ (b) & \dot{z}(s) = 2p_1p_2 \\ (c) & \dot{x}_1 = p_2, \dot{x}_2 = p_1 \end{cases} \tag{23}$$

Characteristic ODE: F Fully nonlinear

Upon solving, we obtain

$$\begin{cases} (a) & p_1(s) = \frac{x_0}{2}e^s, p_2(s) = 2x_0e^s \\ (b) & z(s) = x_0^2e^{2s} \\ (c) & x_1(s) = 2x_0(e^s - 1), x_2(s) = \frac{x_0}{2}(e^s + 1) \end{cases} \tag{24}$$

Fix a point $(x_1, x_2) \in \Omega$. Let $s > 0, x_0 > 0$ such that $(x_1, x_2) = (2x_0(e^s - 1), \frac{x_0}{2}(e^s + 1))$. That is, $x_0 = \frac{1}{4}(4x_2 - x_1), e^s = \frac{x_1+4x_2}{4x_2-x_1}$.
Hence

$$u(x_1, x_2) = x_0^2e^{2s} = \frac{(x_1 + 4x_2)^2}{16}$$

Thanks

Doubts and Suggestions

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