MA612L-Partial Differential Equations

Lecture 37: Nonlinear PDE - Local Solution

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

November 7, 2025

Recap

Characteristic ODE

$$F(Du, u, x) = 0 \text{ in } \Omega \tag{1}$$

$$\textbf{BC:} \quad u = g \text{ on } \Gamma \tag{2}$$

Suppose u solves (1) and (2). Fix $\mathbf{x} \in \Omega$. We would like to calculate $u(\mathbf{x})$ by finding some curve lying within Ω connecting $\mathbf{x}$ with a point $\mathbf{x}^0 \in \Gamma$. The Characteristic ODE is given by

$$\begin{cases} (a) & \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ (b) & \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)).\mathbf{p}(s) \\ (c) & \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \tag{3}$$

With Boundary Conditions

Straightening the boundary

Main Goal: Let us use characteristic ODE (3) to **solve BVP at least in a small region near an appropriate portion of** Γ of $\partial\Omega$.
Fix an $\mathbf{x}_0 \in \partial\Omega$.
Find smooth mapping $\Phi, \Psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi = \Psi^{-1}$ and Φ straightens out $\partial\Omega$ near $\mathbf{x}_0$.
Given any function $u : \Omega \to \mathbb{R}$, let us write $V := \Phi(\Omega)$ and set

$$v(y) := u(\Phi(\mathbf{y})), \mathbf{y} \in V$$

Then

$$u(x) = v(\Phi(\mathbf{x})), \mathbf{x} \in \Omega$$

If $u \in C^1$ solves our BVP, what does v solve?

Straightening the boundary

We can see that

$$\frac{\partial u}{\partial x_i}(\mathbf{x}) = \sum_{k=1}^{n} \frac{\partial v}{\partial y_k}(\Phi(\mathbf{x}))\frac{\partial \Phi_k}{\partial x_i}(\mathbf{x}) \implies Du(\mathbf{x}) = Dv(\mathbf{y})D\Phi(\mathbf{x}) \tag{4}$$

Hence

$$\begin{aligned} 0 &= F(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) \\ &= F(Dv(\mathbf{y})D\Phi(\Psi(\mathbf{y})), v(\mathbf{y}), \Psi(\mathbf{y})) \end{aligned}$$

It is an expression of the form

$$G(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y}) = 0 \quad \text{in } V$$

Straightening the boundary

Further $v = h$ on $\Delta = \Phi(\Gamma)$ and

$$h(\mathbf{y}) = g(\Psi(\mathbf{y}))$$

That is, the problem transforms to

$$\begin{cases} G(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y}) = 0 & \text{in } V \\ v = h & \text{on } \Delta \end{cases} \tag{5}$$

Compatibility Conditions on Boundary Data

Assumptions: For a given point $\mathbf{x}_0 \in \Gamma$, we may further assume that Γ is flat near $\mathbf{x}_0$, lying in the plane $\{x_n = 0\}$.
Goal: To initialize the characteristic ODE to construct a solution at least near $\mathbf{x}_0$, and for this, we must discover appropriate initial conditions

$$\mathbf{p}(0) = \mathbf{p}_0$$
$$z(0) = z_0$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

If the curve $\mathbf{x}(.)$ passes through $\mathbf{x}_0$, we should insist that

$$z_0 = g(x_0) \qquad (6)$$

Compatibility Conditions on Boundary Data

What do we require for $\mathbf{p}(0) = p_0$? Since

$$u(x_1, x_2, \cdots, x_{n-1}, 0) = g(x_1, x_2, \cdots, x_{n-1}) \tag{7}$$

near x_0, let us differentiate it to find

$$\frac{\partial u}{\partial x_i}(x_0) = \frac{\partial g}{\partial x_i}(x_0), i = 1, 2, \cdots, n-1 \tag{8}$$

Since it should also satisfy our PDE, we should insist $p_0 = ((p_1)_0, (p_2)_0, \cdots, (p_n)_0)$ satisfies the following

$$\begin{cases} (p_i)_0 = \frac{\partial g}{\partial x_i}(x_0), & i = 1, 2, \cdots, n-1 \\ F(p_0, z_0, x_0) = 0 \end{cases} \tag{9}$$

Compatibility Conditions on Boundary Data

Remarks

- (6) and (9) are called compatibility conditions
- A triple $p_0 = ((p_1)_0, (p_2)_0, \cdots, (p_n)_0) \in \mathbb{R}^{2n+1}$ verifying (6) and (9) is admissible.
- z_0 is uniquely determined by the BCs and our choice of x_0
- p_0 satisfying (9) may not exist or may not be unique.

Noncharacteristic Boundary Data

Assumptions:

- $\mathbf{x}_0 \in \Gamma$, that Γ near $\mathbf{x}_0$ lies in the plane $\{x_n = 0\}$
- $(\mathbf{p}_0, z_0, \mathbf{x}_0)$ is admissible

Goal:

Construct a solution u of our PDE in Ω near x_0 by integrating the characteristic ODE with $\mathbf{x}$ intersecting Γ at x_0.

Additional Task:

We need to solve these ODEs for nearby initial points as well, and consequently, if we can perturb $(\mathbf{p}_0, z_0, \mathbf{x}_0)$, but keeping the compatibility conditions.

Noncharacteristic Boundary Data

That is, for a given point $y = (y_1, y_2, \cdots, y_{n-1}, 0) \in \Gamma$ with y close to $\mathbf{x}_0$, we intend to solve the characteristic ODE

$$\begin{cases} (a) \quad \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ (b) \quad \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)).\mathbf{p}(s) \\ (c) \quad \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \tag{10}$$

with initial conditions

$$\begin{cases} \mathbf{p}(0) = \mathbf{p}_0 \\ z(0) = z_0 \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{11}$$

Noncharacteristic Boundary Data

Now, we need to find

$$\mathbf{q}(\mathbf{x}_0) = p_0$$

and

$$(\mathbf{q}(\mathbf{y}), g(\mathbf{y}), \mathbf{y})$$

is admissible. That is,

$$\begin{cases} q_i(\mathbf{y}) = \frac{\partial g}{\partial x_i}(\mathbf{y}), & i = 1, 2, \cdots, n-1 \\ F(\mathbf{q}(\mathbf{y}), g(\mathbf{y}), \mathbf{y}) = 0 \end{cases} \tag{12}$$

hold for all $\mathbf{y} \in \Gamma$ close to $\mathbf{x}_0$

Noncharacteristic Boundary Conditions

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Theorem 1 (Noncharacteristic Boundary Conditions)

There exists a unique solution $\mathbf{q}(.)$ of

$$\mathbf{q}(\mathbf{x}_0) = p_0$$

and

$$\begin{cases} q_i(\mathbf{y}) = \frac{\partial g}{\partial x_i}(\mathbf{y}), & i = 1, 2, \cdots, n-1 \\ F(\mathbf{q}(\mathbf{y}), g(\mathbf{y}), \mathbf{y}) = 0 \end{cases} \tag{13}$$

for all $y \in \Gamma$ sufficiently close to $\mathbf{x}_0$, provided

$$\frac{\partial F}{\partial p_n}(\mathbf{p}_0, z_0, \mathbf{x}_0) \neq 0 \tag{14}$$

We say that admissible triple $(\mathbf{p}_0, z_0, \mathbf{x}_0)$ is noncharacteristic if (14) holds.

Local Solution

Local Solution

Question

Did we achieve our goal? That is, use the characteristic ODE to build a solution u of (1) and (2) at least near Γ

Not really, we are moving towards it. Let us use straightening the boundary, compatibility conditions, and noncharacteristic conditions to achieve this.

Assumptions:

- Select a point $\mathbf{x}_0 \in \Gamma$ and assume that near $\mathbf{x}_0$, the surface Γ is flat lying in the plane $\{x_n = 0\}$.
- $(\mathbf{p}_0, z_0, \mathbf{x}_0)$ is admissible boundary data, which is noncharacteristic

Local Solution

By theorem 1, there exists a function $\mathbf{q}(.)$ such that $\mathbf{p}_0 = \mathbf{q}(\mathbf{x}_0)$ and the triple $(\mathbf{q}(\mathbf{y}), g(\mathbf{y}), \mathbf{y})$ is admissible for all $\mathbf{y}$ sufficiently close to $\mathbf{x}_0$.
For any $\mathbf{y} = (y_1, y_2, \cdots, y_{n-1}, 0)$, we can solve characteristic ODE (10) subject to initial conditions (11).

Notations

$$\begin{cases} \mathbf{p}(s) = \mathbf{p}(\mathbf{y}, s) = \mathbf{p}(y_1, y_2 \cdots, y_{n-1}, s) \\ z(s) = z(\mathbf{y}, s) = \mathbf{p}(y_1, y_2 \cdots, y_{n-1}, s) \\ \mathbf{x}(s) = \mathbf{x}(\mathbf{y}, s) = \mathbf{x}(y_1, y_2 \cdots, y_{n-1}, s) \end{cases} \tag{15}$$

Local Invertibility

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Theorem 2 (Local Invertibility)
Assume we have the noncharacteristic condition

$$F_{p_n}(\mathbf{p}_0, z_0, \mathbf{x}_0) \neq 0.$$

Then there exist

1. an open interval $I \subset \mathbb{R}$ such that $0 \in I$
2. a neighbourhood W of $\mathbf{x}_0$ in $\Gamma \subset \mathbb{R}^{n-1}$
3. a neighbourhood V of $\mathbf{x}_0$ in $\mathbb{R}^n$ such that for each $\mathbf{x} \in V$, there exist unique $s \in I, \mathbf{y} \in W$ such that

$$\mathbf{x} = \mathbf{x}(\mathbf{y}, s)$$

The mapping $x \to s, y$ is C^2.

Local Invertibility

Proof: We have $(\mathbf{x}_0, 0) = \mathbf{x}_0$. By the Inverse Mapping theorem, we can obtain the result if $\det D\mathbf{x}(\mathbf{x}_0, 0) \neq 0$. For,

$$\mathbf{x}(\mathbf{y}, 0) = (\mathbf{y}, 0), \mathbf{y} \in \Gamma$$

and so if $i = 1, 2, \cdots, n-1$

$$\frac{\partial x^j}{\partial y_i}(\mathbf{x}_0, 0) = \begin{cases} \delta_{ij} & j = 1, 2, \cdots, n-1 \\ 0 & j = n \end{cases}$$

From (10)(c), we have

$$\frac{\partial x^j}{\partial s}(\mathbf{x}_0, 0) = F_{p_j}(\mathbf{p}_0, z_0, \mathbf{x}_0)$$

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Local Invertibility

Proof: We have $(\mathbf{x}_0, 0) = \mathbf{x}_0$. By the Inverse Mapping theorem, we can obtain the result if $\det D\mathbf{x}(\mathbf{x}_0, 0) \neq 0$. For,

$$\mathbf{x}(\mathbf{y}, 0) = (\mathbf{y}, 0), \mathbf{y} \in \Gamma$$

and so if $i = 1, 2, \cdots, n-1$

$$\frac{\partial x^j}{\partial y_i}(\mathbf{x}_0, 0) = \begin{cases} \delta_{ij} & j = 1, 2, \cdots, n-1 \\ 0 & j = n \end{cases}$$

From (10)(c), we have

$$\frac{\partial x^j}{\partial s}(\mathbf{x}_0, 0) = F_{p_j}(\mathbf{p}_0, z_0, \mathbf{x}_0)$$

Local Invertibility

Thus,

$$D\mathbf{x}(\mathbf{x}_0, 0) = \begin{pmatrix} 1 & 0 & \cdots & 0 & F_{p_1}(\mathbf{p}_0, z_0, \mathbf{x}_0) \\ 0 & 1 & \cdots & 0 & F_{p_2}(\mathbf{p}_0, z_0, \mathbf{x}_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & F_{p_n}(\mathbf{p}_0, z_0, \mathbf{x}_0) \end{pmatrix}_{n\times n}$$

Therefore, when $F_{p_n}(\mathbf{p}_0, z_0, \mathbf{x}_0) \neq 0$, $D\mathbf{x}(\mathbf{x}_0, 0) \neq 0$. Hence the proof.

Local Solution

Remarks

By above theorem, for each $x \in \Omega$, we can locally uniquely solve the equation

$$\begin{cases} \mathbf{x} = \mathbf{x}(\mathbf{y}, s) \\ \text{for } \mathbf{y} = \mathbf{y}(\mathbf{x}), s = s(\mathbf{x}) \end{cases} \tag{16}$$

Now, let us define for each $\mathbf{x} \in \Omega$ and $s, \mathbf{y}$ as in (16)

$$\begin{cases} u(\mathbf{x}) := z(y(\mathbf{x}), s(\mathbf{x})) \\ \mathbf{p}(\mathbf{x}) := \mathbf{p}(y(\mathbf{x}), s(\mathbf{x})) \end{cases} \tag{17}$$

By the above equations, we can locally weave together the solutions of the characteristic ODE into a solution of the PDE.

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in