MA612L-Partial Differential Equations

Lecture 38: Nonlinear PDE - Local Existence Theorem

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Local Existence Theorem

Noncharacteristic Boundary Data

$$F(Du, u, x) = 0 \text{ in } \Omega$$

$$u = g \text{ on } \Gamma$$

$$\begin{cases} (a) \quad \dot{\mathbf{p}}(s) = -D_x F(\mathbf{p}(s), z(s), \mathbf{x}(s)) - D_z F(\mathbf{p}(s), z(s), \mathbf{x}(s))\mathbf{p}(s) \\ (b) \quad \dot{z}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)).\mathbf{p}(s) \\ (c) \quad \dot{\mathbf{x}}(s) = D_p F(\mathbf{p}(s), z(s), \mathbf{x}(s)) \end{cases} \tag{1}$$

with initial conditions

$$\begin{cases} \mathbf{p}(0) = \mathbf{p}_0 \\ z(0) = z_0 \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} \tag{2}$$

Assumptions:

- $\mathbf{x}_0 \in \Gamma$, that Γ near $\mathbf{x}_0$ lies in the plane $\{x_n = 0\}$
- $(\mathbf{p}_0, z_0, \mathbf{x}_0)$ is admissible

Noncharacteristic Boundary Conditions

Theorem 1 (Noncharacteristic Boundary Conditions)

There exists a unique solution $\mathbf{q}(.)$ of

$$\mathbf{q}(\mathbf{x}_0) = p_0$$

and

$$\begin{cases} q_i(\mathbf{y}) = \frac{\partial g}{\partial x_i}(\mathbf{y}), & i = 1, 2, \cdots, n-1 \\ F(\mathbf{q}(\mathbf{y}), g(\mathbf{y}), \mathbf{y}) = 0 \end{cases} \tag{3}$$

for all $y \in \Gamma$ sufficiently close to $\mathbf{x}_0$, provided

$$\frac{\partial F}{\partial p_n}(\mathbf{p}_0, z_0, \mathbf{x}_0) \neq 0 \tag{4}$$

We say that admissible triple $(\mathbf{p}_0, z_0, \mathbf{x}_0)$ is noncharacteristic if (4) holds.

Local Solution

Remarks

By above theorem, for each $x \in \Omega$, we can locally uniquely solve the equation

$$\begin{cases} \mathbf{x} = \mathbf{x}(\mathbf{y}, s) \\ \text{for } \mathbf{y} = \mathbf{y}(\mathbf{x}), s = s(\mathbf{x}) \end{cases} \tag{5}$$

Now, let us define for each $\mathbf{x} \in \Omega$ and $s, \mathbf{y}$ as in (5)

$$\begin{cases} u(\mathbf{x}) := z(y(\mathbf{x}), s(\mathbf{x})) \\ \mathbf{p}(\mathbf{x}) := \mathbf{p}(y(\mathbf{x}), s(\mathbf{x})) \end{cases} \tag{6}$$

By the above equations, we can locally weave together the solutions of the characteristic ODE into a solution of the PDE.

Local Existence Theorem

Theorem 2
The function u defined by

$$u(\mathbf{x}) := z(y(\mathbf{x}), s(\mathbf{x}))$$

is C^2 and solves the PDE

$$F(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) = 0 \quad (\mathbf{x} \in \Omega)$$

with the boundary condition

$$u(\mathbf{x}) = g(\mathbf{x}) \quad (\mathbf{x} \in \Gamma \cap \Omega)$$

Local Existence Theorem

Claim 1: Local Solutions

Fix $\mathbf{y} \in \Gamma$ close to $\mathbf{x}_0$. Then we can solve the characteristic ODE (1) and (2) for

$$\begin{cases} \mathbf{p}(s) = \mathbf{p}(\mathbf{y}, s) = \mathbf{p}(y_1, y_2 \cdots, y_{n-1}, s) \\ z(s) = z(\mathbf{y}, s) = \mathbf{z}(y_1, y_2 \cdots, y_{n-1}, s) \\ \mathbf{x}(s) = \mathbf{x}(\mathbf{y}, s) = \mathbf{x}(y_1, y_2 \cdots, y_{n-1}, s) \end{cases} \tag{7}$$

Claim 2: There exists f such that

$$f(\mathbf{y}, s) := F(\mathbf{p}(\mathbf{y}, s), z(\mathbf{y}, s), \mathbf{x}(\mathbf{y}, s)) = 0, \quad s \in \mathbb{R} \tag{8}$$

if $\mathbf{y}$ is sufficiently close to $\mathbf{x}_0$.

Local Existence Theorem

By the compatibility condition, we have

$$f(\mathbf{y},0) := F(\mathbf{p}(\mathbf{y},0), z(\mathbf{y},0), \mathbf{x}(\mathbf{y},0)) = F(\mathbf{q}(\mathbf{y}), g(\mathbf{y}), \mathbf{y}) = 0 \tag{9}$$

Further,

$$\begin{aligned}
\frac{\partial f}{\partial s}(\mathbf{y},s) &= \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}\dot{p}_j + \frac{\partial F}{\partial z}\dot{z} + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\dot{\mathbf{x}}_j \\
&= \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}\left(-\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial z}p_j\right) + \frac{\partial F}{\partial z}\left(\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}p_j\right) + \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\left(\frac{\partial F}{\partial p_j}\right) \\
&= 0
\end{aligned}$$

By solving $f_s(\mathbf{y},s) = 0$ and $f(\mathbf{y},0) = 0$, claim 2 follows.

Local Existence Theorem

Therefore, from local invertibility, we have

$$F(\mathbf{p}(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) = 0 \quad (\mathbf{x} \in \Omega)$$

To prove our theorem, it is enough to prove that

$$\mathbf{p}(\mathbf{x}) = Du(\mathbf{x}), \quad (\mathbf{x} \in \Omega)$$

Claim 3:

$$\frac{\partial z}{\partial s}(\mathbf{y}, s) = \sum_{j=1}^{n} p_j(\mathbf{y}, s)\frac{\partial x_j}{\partial s}(\mathbf{y}, s), \quad \text{for } \mathbf{y} \in W, s \in I \tag{10}$$

By using the characteristic ODE (1)(b) and (1)(c), claim 3 follows immediately.

Local Existence Theorem

Claim 4:

$$\frac{\partial z}{\partial y_i}(\mathbf{y}, s) = \sum_{j=1}^{n} p_j(\mathbf{y}, s)\frac{\partial x_j}{\partial y_i}(\mathbf{y}, s), \quad i = 1, 2, \cdots, n-1 \tag{11}$$

Fix $\mathbf{y} \in \Gamma, i \in \{1, 2, \cdots, n-1\}$ set

$$r_i(s) = \frac{\partial z}{\partial y_i}(\mathbf{y}, s) - \sum_{j=1}^{n} p_j(\mathbf{y}, s)\frac{\partial x_j}{\partial y_i}(\mathbf{y}, s) \tag{12}$$

Claim 4a: It suffices to prove that $r_i(s) = 0, \forall s \in \mathbb{R}, i = 1, 2, \cdots, n-1.$
Using noncharacteristic boundary conditions theorem, equation (3), we can obtain that

$$r_i(0) = \frac{\partial g}{\partial x_i}(\mathbf{y}) - q_i(\mathbf{y}) = 0$$

Local Existence Theorem

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Now

$$\dot{r}_i(s) = \frac{\partial^2 z}{\partial y_i \partial s} - \sum_{j=1}^{n} \left[\frac{\partial p_j}{\partial s} \frac{\partial x_j}{\partial y_i} + p_j \frac{\partial^2 x_j}{\partial y_i \partial s} \right] \tag{13}$$

Similarly,

$$\frac{\partial^2 z}{\partial s \partial y_i} = \sum_{j=1}^{n} \left[\frac{\partial p_j}{\partial y_i} \frac{\partial x_j}{\partial s} + p_j \frac{\partial^2 x_j}{\partial y_i \partial s} \right] \tag{14}$$

Hence,

$$\dot{r}_i(s) = \sum_{j=1}^{n} \left[\frac{\partial p_j}{\partial y_i} \frac{\partial x_j}{\partial s} - \frac{\partial p_j}{\partial s} \frac{\partial x_j}{\partial y_i} \right] = \sum_{j=1}^{n} \left[\frac{\partial p_j}{\partial y_i} \left(\frac{\partial F}{\partial p_j} \right) - \left(-\frac{\partial F}{\partial x_j} - \frac{\partial F}{\partial z} p_j \right) \frac{\partial x_j}{\partial y_i} \right]$$

Local Existence Theorem

From claim 2, we have

$$F(\mathbf{p}(\mathbf{y},s),z(\mathbf{y},s),\mathbf{x}(\mathbf{y},s))=0 \implies \sum_{j=1}^{n}\frac{\partial F}{\partial p_j}\frac{\partial p_j}{\partial y_i}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y_i}+\sum_{j=1}^{n}\frac{\partial F}{\partial x_j}\frac{\partial x_j}{\partial y_i}=0 \tag{15}$$

Hence, we obtain

$$\begin{aligned}\dot{r}_i(s) &= -\frac{\partial F}{\partial z}\frac{\partial z}{\partial y_i}+\sum_{j=1}^{n}\frac{\partial F}{\partial z}p_j\frac{\partial x_j}{\partial y_i} \\ &= -\frac{\partial F}{\partial z}\left[\frac{\partial z}{\partial y_i}-\sum_{j=1}^{n}p_j\frac{\partial x_j}{\partial y_i}\right]=-\frac{\partial F}{\partial z}r_i(s)\end{aligned}$$

Local Existence Theorem

By solving $\dot{r}_i(s) = -F_z r_i(s)$ and $r_i(0) = 0$, claim 4a and claim 4 follows.
Claim 5: If we claim the following, the proof completes.

$$\mathbf{p}(\mathbf{x}) = Du(\mathbf{x}), \quad (\mathbf{x} \in \Omega)$$

Now, by (6) for $j = 1, 2, \cdots, n$, we have

$$\begin{aligned}
\frac{\partial u}{\partial x_j} &= \frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j} = \left(\sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial s}\right)\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1}\left(\sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial y_i}\right)\frac{\partial y_i}{\partial x_j} \\
&= \sum_{k=1}^{n} p_k \left(\frac{\partial x_k}{\partial s}\frac{\partial s}{\partial x_j} + \sum_{i=1}^{n-1} \frac{\partial x_k}{\partial y_i}\frac{\partial y_i}{\partial x_j}\right) = \sum_{k=1}^{n} p_k \frac{\partial x_k}{\partial x_j} \\
&= \sum_{k=1}^{n} p_k \delta_{jk} = p_j
\end{aligned}$$

Hence the proof.

Applications: Linear

Applications: F is linear

Consider the following PDE

$$\begin{cases} F(Du, u, \mathbf{x}) = \mathbf{b}(\mathbf{x}).Du(\mathbf{x}) + c(\mathbf{x})u(\mathbf{x}) = 0 & \mathbf{x} \in \Omega \\ u = g & \text{on } \Gamma \end{cases} \tag{16}$$

Let $\mathbf{x}_0 \in \Gamma$ and the problem has a noncharacteristic boundary. Then

$$\mathbf{b}(\mathbf{x}_0).\nu(\mathbf{x}_0) \neq 0$$

We can clearly see that this condition does not depend on z_0 or $\mathbf{p}_0$. (Remember! We stated this earlier).

Applications: F is linear

Now, we can uniquely solve this equation

$$D_pF(\mathbf{p}_0, z_0, \mathbf{x}_0).\nu(x_0) \neq 0$$

for $\mathbf{q}(\mathbf{y})$ if $\mathbf{y} \in \Gamma$ is near $\mathbf{x}_0$. By the local existence theorem, we can construct a unique solution of (16)

Remarks

- For proving the theorem, we have used all characteristic ODEs, but it is not necessary to use them if we know the solution exists
- Instead, we can use the reduced system, which does not involve $\mathbf{p}(.)$
- Since we have a unique solution for the IVP, the projected characteristics $\mathbf{x}(.)$ emanate from distinct points on Γ cannot cross

Applications: F is linear

Let us consider three different cases of trajectories of the ODE

$$\dot{\mathbf{x}}(s) = \mathbf{b}(\mathbf{x}(s)) \tag{17}$$

Case 1:
Assumptions:

- The vector field $\mathbf{b}$ vanishes within Ω only at one point.
- Let the point be the origin $\mathbf{0}$
- $\mathbf{b}.\nu < 0$ on Γ

Applications: F is linear

Question

Can we solve the following IVP?

$$\begin{cases} \mathbf{b}(\mathbf{x}).Du(\mathbf{x}) = 0 & \mathbf{x} \in \Omega \\ u = g & \text{on } \Gamma \end{cases} \tag{18}$$

By the local existence theorem, there exists a unique u defined near Γ. Further,

$$u(\mathbf{x}(s)) \equiv u(\mathbf{x}(0)) = g(\mathbf{x}_0)$$

for each solution of the ODE (17) with $\mathbf{x}(0) = \mathbf{x}_0 \in \Gamma$.

Remarks

- Unless g is constant, this solution can't be smoothly continued to all of Ω
- Any smooth solution of (18) is constant on trajectories of (17)

Applications: F is linear

Case 2:
Assumptions:

- Each trajectory of the ODE (17) enters and exits Ω exactly once somewhere through the set

$$\Gamma : \{\mathbf{x} \in \partial\Omega : \mathbf{b}(\mathbf{x}).\nu(\mathbf{x}) < 0\}$$

- Exception for the above assumption on characteristic points where it touches the boundary.

For this case, we can find a smooth solution of (18). Set u to be constant along each flow line.

Applications: F is linear

Case 3:
Assumptions:

- A few trajectories of the ODE (17) enter and exit Ω exactly once somewhere through the set, and a few do not obey it

$$\Gamma : \{\mathbf{x} \in \partial\Omega : \mathbf{b}(\mathbf{x}).\nu(\mathbf{x}) < 0\}$$

For this case, we can define u to be constant along the characteristics. Then u may be discontinuous at a few points. In this case, the local existence theory may fail near those characteristic points.

Applications: quasilinear

Applications: F is quasilinear

Consider the following PDE

$$\begin{cases} F(Du,u,\mathbf{x}) = \mathbf{b}(\mathbf{x},u).Du(\mathbf{x}) + c(\mathbf{x},u) = 0 & \mathbf{x} \in \Omega \\ u = g & \text{on} \Gamma \end{cases} \tag{19}$$

Let $\mathbf{x}_0 \in \Gamma$ and the problem has noncharacteristic boundary. Then

$$\mathbf{b}(\mathbf{x}_0, z_0).\nu(\mathbf{x}_0) \neq 0$$

where $z_0 = g(\mathbf{x}_0)$. We can clearly see that this condition does not depend on $\mathbf{p}_0$. (Remember! We stated this earlier).

Applications: F is quasilinear

Now, we can uniquely solve this equation

$$D_p F(\mathbf{p}_0, z_0, \mathbf{x}_0).\nu(x_0) \neq 0$$

for $\mathbf{q}(\mathbf{y})$ if $\mathbf{y} \in \Gamma$ is near $\mathbf{x}_0$. By the local existence theorem, we can construct a unique solution of (19) in some neighbourhood of $\mathbf{x}_0$.

Remarks

- Unlike the linear case, it is possible that the projected characteristics emanating from distinct points in Γ may intersect outside Ω.
- If such a scenario occurs, we can understand that our local solution ceases to exist within all of Ω.

Applications: F is quasilinear

Consider the following scalar conservation laws (more problems follow!)

$$\begin{cases} u_t + \mathbf{F}'(u).Du = 0 & \text{in } \Omega = \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \quad (20)$$

Let

$$G(Du, u_t, u, \mathbf{x}, t) = u_t + \text{div}\mathbf{F}(u) = u_t + \mathbf{F}'(u).Du = 0$$

Here, $\mathbf{F} : \mathbb{R} \to \mathbb{R}^n, \mathbf{F} = (F_1, F_2, \cdots, F_n)$. Set $t = x_{n+1}$.

Applications: F is quasilinear

We can rewrite the above equation as follows: Let $\mathbf{q} = (\mathbf{p}, p_{n+1})$ and $\mathbf{y} = (\mathbf{x}, t)$, then,

$$G(\mathbf{q}, z, \mathbf{y}) = p_{n+1} + \mathbf{F}'(z).\mathbf{p}$$

Hence

$$D_{\mathbf{q}}G = (\mathbf{F}'(z), 1)$$
$$D_{\mathbf{y}}G = 0$$
$$D_{\mathbf{z}}G = \mathbf{F}''(z).\mathbf{p}$$

We can observe that the noncharacteristic condition is satisfied at each point $\mathbf{y}_0 = (\mathbf{x}_0, 0) \in \Gamma$. Now, characteristic ODE(a) becomes

$$\begin{cases} \dot{x}_i(s) & = F_i'(z(s)), i = 1, 2, \cdot, n \\ \dot{x}_{n+1}(s) & = 1 \end{cases} \tag{21}$$

Applications: F is quasilinear

From above equation, $x_{n+1}(s) = s$ which agrees with $x_{n+1} = t$. In other words, we can identify the parameter s with time t. Therefore, equation (b) in characteristic ODE becomes

$$\dot{z}(s) = 0 \implies z(s) = z_0 = g(\mathbf{x}_0) \tag{22}$$

and

$$\mathbf{x}(s) = \mathbf{F}'(g(\mathbf{x}_0))s + \mathbf{x}_0 \tag{23}$$

Thus the projected characteristic

$$\mathbf{y}(s) = (\mathbf{x}(s), s) = (\mathbf{F}'(g(\mathbf{x}_0))s + \mathbf{x}_0, s)$$

is a straight line along which u is constant.

Applications: F is quasilinear

Crossing Characteristics:

- Suppose the same logic applied to different initial point $z_0 \in \Gamma$ where $g(\mathbf{x}_0) \neq g(z_0)$
- The projected characteristics may intersect at some time $t > 0$.
- $u \equiv g(\mathbf{x}_0)$, projected characteristic through $\mathbf{x}_0$
- $u \equiv g(z_0)$, projected characteristic through z_0
- A contradiction.
- The IVP (20) does not, in general, have a smooth solution existing for all times $t > 0$.

Applications: F is quasilinear

Let us remove s from (22) and (23) to obtain an implicit formula for u. For given $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$, we see that since $s = t$,

$$\begin{aligned} u(\mathbf{x}(t), t) = z(t) &= g(\mathbf{x}(t) - t\mathbf{F}'(z_0)) \\ &= g(\mathbf{x}(t) - t\mathbf{F}'(u(\mathbf{x}(t), t))) \end{aligned}$$

$$u = g(\mathbf{x} - t\mathbf{F}'(u))$$

Remember! Burger's equation solution! Transport equation solution! The above equation gives a solution provided

$$1 + tDg(x - t\mathbf{F}'(u)).\mathbf{F}''(u) \neq 0$$

Applications: F is quasilinear

Remarks

For $n = 1$, we need

$$1 + tg'(x - tF'(u))F''(u) \neq 0$$

If $F'' > 0$, but $g' < 0$, then you will get a false statement at some time $t > 0$. This indicates the failure of both the implicit formula and the characteristic method.

Applications: fully nonlinear

Applications: F is fully nonlinear

Remarks
The form of the full characteristic equations is quite complicated for nonlinear PDEs. We discuss only a particular type called the Hamilton-Jacobi equation

The general Hamilton-Jacobi PDE is given by

$$G(Du, u_t, u, \mathbf{x}, t) = u_t + H(Du, \mathbf{x}) = 0 \tag{24}$$

Similar to the above discussion, we have

$$G(\mathbf{q}, z, \mathbf{y}) = p_{n+1} + H(\mathbf{p}, \mathbf{x})$$

Applications: F is fully nonlinear

Hence

$$D_{\mathbf{q}}G = (D_pH(\mathbf{p}, x), 1)$$
$$D_{\mathbf{y}}G = D_xH(\mathbf{p}, x)$$
$$D_{\mathbf{z}}G = 0$$

Now, characteristic ODE (c) becomes

$$\begin{cases} \dot{x}_i(s) = \frac{\partial H}{\partial p_i}(\mathbf{p}(s), \mathbf{x}(s)), i = 1, 2, \cdot, n \\ \dot{x}_{n+1}(s) = 1 \end{cases} \tag{25}$$

Applications: F is fully nonlinear

Now, characteristic ODE (a) becomes

$$\begin{cases} \dot{p}_i(s) = -\frac{\partial H}{\partial x_i}(\mathbf{p}(s), \mathbf{x}(s)), i = 1, 2, \cdot, n \\ \dot{p}_{n+1}(s) = 0 \end{cases} \tag{26}$$

Now, from characteristic ODE (b) becomes

$$\begin{aligned} \dot{z}(s) &= D_{\mathbf{p}} H(\mathbf{p}(s), \mathbf{x}(s)).\mathbf{p}(s) + p^{n+1}(s) \\ &= D_{\mathbf{p}} H(\mathbf{p}(s), \mathbf{x}(s)).\mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{x}(s)).\mathbf{p}(s) \end{aligned}$$

Applications: F is fully nonlinear

Therefore, the characteristic equation for the Hamilton-Jacobi equation is given by

$$\begin{cases} \dot{p}(s) = -D_{\mathbf{x}}H(\mathbf{p}(s), \mathbf{x}(s)) \\ \dot{z}(s) = D_{\mathbf{p}}H(\mathbf{p}(s), \mathbf{x}(s)).\mathbf{p}(s) + p^{n+1}(s) \\ \dot{x}(s) = D_{\mathbf{p}}H(\mathbf{p}(s), \mathbf{x}(s)) \end{cases} \tag{27}$$

The equation below is called Hamilton's equations.

$$\begin{cases} \dot{p}(s) = -D_{\mathbf{x}}H(\mathbf{p}(s), \mathbf{x}(s)) \\ \dot{x}(s) = D_{\mathbf{p}}H(\mathbf{p}(s), \mathbf{x}(s)) \end{cases} \tag{28}$$

Once we solve Hamilton's equations, solving the Hamilton-Jacobi equation becomes trivial. In general, the Hamilton-Jacobi equation does not have a smooth solution u lasting for all time $t > 0$

Thanks

Doubts and Suggestions

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