MA612L-Partial Differential Equations

Lecture 39: Nonlinear PDE - Local Existence Theorem - Applications

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Applications: quasilinear

Applications: F is quasilinear

Consider the following PDE

$$\begin{cases} F(Du, u, \mathbf{x}) = \mathbf{b}(\mathbf{x}, u).Du(\mathbf{x}) + c(\mathbf{x}, u) = 0 & \mathbf{x} \in \Omega \\ u = g & \text{on} \Gamma \end{cases} \tag{1}$$

Let $\mathbf{x}_0 \in \Gamma$ and the problem has a noncharacteristic boundary. Then

$$\mathbf{b}(\mathbf{x}_0, z_0).\nu(\mathbf{x}_0) \neq 0$$

where $z_0 = g(\mathbf{x}_0)$. We can clearly see that this condition does not depend on $\mathbf{p}_0$. (Remember! We stated this earlier).

Applications: F is quasilinear

Now, we can uniquely solve this equation

$$D_pF(\mathbf{p}_0, z_0, \mathbf{x}_0).\nu(x_0) \neq 0$$

for $\mathbf{q}(\mathbf{y})$ if $\mathbf{y} \in \Gamma$ is near $\mathbf{x}_0$. By the local existence theorem, we can construct a unique solution of (1) in some neighbourhood of $\mathbf{x}_0$.

Remarks

- Unlike the linear case, it is possible that the projected characteristics emanating from distinct points in Γ may intersect outside Ω.
- If such a scenario occurs, we can understand that our local solution ceases to exist within all of Ω.

Applications: F is quasilinear

Consider the following scalar conservation laws (more problems follow!)

$$\begin{cases} u_t + \mathbf{F}'(u).Du = 0 & \text{in } \Omega = \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{2}$$

Let

$$G(Du, u_t, u, \mathbf{x}, t) = u_t + \text{div}\mathbf{F}(u) = u_t + \mathbf{F}'(u).Du = 0$$

Here, $\mathbf{F} : \mathbb{R} \to \mathbb{R}^n, \mathbf{F} = (F_1, F_2, \cdots, F_n)$. Set $t = x_{n+1}$.

Applications: F is quasilinear

We can rewrite the above equation as follows: Let $\mathbf{q} = (\mathbf{p}, p_{n+1})$ and $\mathbf{y} = (\mathbf{x}, t)$, then,

$$G(\mathbf{q}, z, \mathbf{y}) = p_{n+1} + \mathbf{F}'(z).\mathbf{p}$$

Hence

$$D_{\mathbf{q}}G = (\mathbf{F}'(z), 1)$$
$$D_{\mathbf{y}}G = 0$$
$$D_{\mathbf{z}}G = \mathbf{F}''(z).\mathbf{p}$$

We can observe that the noncharacteristic condition is satisfied at each point $\mathbf{y}_0 = (\mathbf{x}_0, 0) \in \Gamma$. Now, characteristic ODE(a) becomes

$$\begin{cases} \dot{x}_i(s) & = F_i'(z(s)), i = 1, 2, \cdots, n \\ \dot{x}_{n+1}(s) & = 1 \end{cases} \tag{3}$$

Applications: F is quasilinear

From above equation, $x_{n+1}(s) = s$ which agrees with $x_{n+1} = t$. In other words, we can identify the parameter s with time t. Therefore, equation (b) in characteristic ODE becomes

$$\dot{z}(s) = 0 \implies z(s) = z_0 = g(\mathbf{x}_0) \tag{4}$$

and

$$\mathbf{x}(s) = \mathbf{F}'(g(\mathbf{x}_0))s + \mathbf{x}_0 \tag{5}$$

Thus the projected characteristic

$$\mathbf{y}(s) = (\mathbf{x}(s), s) = (\mathbf{F}'(g(\mathbf{x}_0))s + \mathbf{x}_0, s)$$

is a straight line along which u is constant.

Applications: F is quasilinear

Crossing Characteristics:

- Suppose the same logic applied to different initial point $z_0 \in \Gamma$ where $g(\mathbf{x}_0) \neq g(z_0)$
- The projected characteristics may intersect at some time $t > 0$.
- $u \equiv g(\mathbf{x}_0)$, projected characteristic through $\mathbf{x}_0$
- $u \equiv g(z_0)$, projected characteristic through z_0
- A contradiction.
- The IVP (2) does not, in general, have a smooth solution existing for all times $t > 0$.

Applications: F is quasilinear

Let us remove s from (4) and (5) to obtain an implicit formula for u. For given $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$, we see that since $s = t$,

$$\begin{aligned} u(\mathbf{x}(t), t) = z(t) &= g(\mathbf{x}(t) - t\mathbf{F}'(z_0)) \\ &= g(\mathbf{x}(t) - t\mathbf{F}'(u(\mathbf{x}(t), t))) \end{aligned}$$

$$u = g(\mathbf{x} - t\mathbf{F}'(u))$$

Remember! Burger's equation solution! Transport equation solution! The above equation gives a solution provided

$$1 + tDg(x - t\mathbf{F}'(u)).\mathbf{F}''(u) \neq 0$$

Applications: F is quasilinear

Remarks

For $n = 1$, we need

$$1 + tg'(x - tF'(u))F''(u) \neq 0$$

If $F'' > 0$, but $g' < 0$, then you will get a false statement at some time $t > 0$. This indicates the failure of both the implicit formula and the characteristic method.

Applications: fully nonlinear

Applications: F is fully nonlinear

Remarks
The form of the full characteristic equations is quite complicated for nonlinear PDEs. We discuss only a particular type called the Hamilton-Jacobi equation

The general Hamilton-Jacobi PDE is given by

$$G(Du, u_t, u, \mathbf{x}, t) = u_t + H(Du, \mathbf{x}) = 0 \tag{6}$$

Similar to the above discussion, we have

$$G(\mathbf{q}, z, \mathbf{y}) = p_{n+1} + H(\mathbf{p}, \mathbf{x})$$

Applications: F is fully nonlinear

Hence

$$D_{\mathbf{q}}G = (D_p H(\mathbf{p}, x), 1)$$
$$D_{\mathbf{y}}G = D_x H(\mathbf{p}, x)$$
$$D_{\mathbf{z}}G = 0$$

Now, characteristic ODE (c) becomes

$$\begin{cases} \dot{x}_i(s) = \frac{\partial H}{\partial p_i}(\mathbf{p}(s), \mathbf{x}(s)), i = 1, 2, \cdot, n \\ \dot{x}_{n+1}(s) = 1 \end{cases} \tag{7}$$

Applications: F is fully nonlinear

Now, characteristic ODE (a) becomes

$$\begin{cases} \dot{p}_i(s) = -\frac{\partial H}{\partial x_i}(\mathbf{p}(s), \mathbf{x}(s)), i = 1, 2, \cdot, n \\ \dot{p}_{n+1}(s) = 0 \end{cases} \tag{8}$$

Now, from characteristic ODE (b) becomes

$$\begin{aligned} \dot{z}(s) &= D_{\mathbf{p}} H(\mathbf{p}(s), \mathbf{x}(s)).\mathbf{p}(s) + p^{n+1}(s) \\ &= D_{\mathbf{p}} H(\mathbf{p}(s), \mathbf{x}(s)).\mathbf{p}(s) - H(\mathbf{p}(s), \mathbf{x}(s)).\mathbf{p}(s) \end{aligned}$$

Applications: F is fully nonlinear

Therefore, the characteristic equation for the Hamilton-Jacobi equation is given by

$$\begin{cases} \dot{p}(s) = -D_{\mathbf{x}}H(\mathbf{p}(s), \mathbf{x}(s)) \\ \dot{z}(s) = D_{\mathbf{p}}H(\mathbf{p}(s), \mathbf{x}(s)).\mathbf{p}(s) + p^{n+1}(s) \\ \dot{x}(s) = D_{\mathbf{p}}H(\mathbf{p}(s), \mathbf{x}(s)) \end{cases} \tag{9}$$

The equation below is called Hamilton's equations.

$$\begin{cases} \dot{p}(s) = -D_{\mathbf{x}}H(\mathbf{p}(s), \mathbf{x}(s)) \\ \dot{x}(s) = D_{\mathbf{p}}H(\mathbf{p}(s), \mathbf{x}(s)) \end{cases} \tag{10}$$

Once we solve Hamilton's equations, solving the Hamilton-Jacobi equation becomes trivial. In general, the Hamilton-Jacobi equation does not have a smooth solution u lasting for all time $t > 0$

Scalar Conservation Laws

Scalar Conservation Laws

The initial value problem for scalar conservation laws in one space dimension

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{11}$$

Here $F, g : \mathbb{R} \to \mathbb{R}$ are known and u is unknown. In general, there does not exist a smooth solution (11) for all times $t > 0$. Now, let us devise a way to interpret a less regular function u, somehow solving this IVP. The idea is to multiply the PDE by a smooth function v and then integrate by parts, thereby transferring the derivatives onto v.

Scalar Conservation Laws

Assume $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is smooth with compact support. We call v a test function. Hence, we obtain

$$0 = \int_0^\infty \int_{-\infty}^\infty (u_t + F(u)_x) v dx dt$$

$$= - \int_0^\infty \int_{-\infty}^\infty u v_t dx dt + \int_{-\infty}^\infty u v dx|_{t=0} - \int_0^\infty \int_{-\infty}^\infty F(u) v_x dx dt$$

$$\int_0^\infty \int_{-\infty}^\infty (u v_t + F(u) v_x) dx dt + \int_{-\infty}^\infty g v dx|_{t=0} = 0 \tag{12}$$

This equality is true if u is a smooth solution of (11), but the resulting formula has meaning even if u is only bounded.

Integral Solution

Definition 1 (Integral Solution)
We say that $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an integral solution of (11), provided equality (12) holds for each test function v.

Suppose in some open region $V \subset \mathbb{R} \times (0, \infty)$ that u is smooth on either side of a smooth curve C. Let V_l be the part of V on the left of the curve and V_r be the part of V on the right. Assume that u is an integral solution of (11), and that u and its first derivatives are uniformly continuous in V_l and in V_r.

Integral Solution

Choose v with a compact support in V_l. Then (12) becomes

$$0 = \int_0^\infty \int_{-\infty}^\infty (uv_t + F(u)v_x)dxdt = -\int_0^\infty \int_{-\infty}^\infty [u_t + F(u)_x]vdxdt \tag{13}$$

the integration by parts being justified since u is a C^1 in V_l and v vanishes near the boundary of V_l. The identity (13) holds for all test functions v with compact support in V_l and so

$$u_t + F(u)_x = 0 \quad \text{in} \quad V_l \tag{14}$$

Similarly,

$$u_t + F(u)_x = 0 \quad \text{in} \quad V_r \tag{15}$$

Integral Solution

Select a test function v with compact support in V, but which does not necessarily vanish along the curve C. Again

$$0 = \int_0^\infty \int_{-\infty}^\infty (uv_t + F(u)v_x)dxdt = \iint_{V_l} [uv_t + F(u)v_x]dxdt + \iint_{V_r} [uv_t + F(u)v_x]dxdt$$

$$\iint_{V_l} [uv_t + F(u)v_x]dxdt = -\iint_{V_l} [uv_t + F(u)_x]vdxdt + \int_C (u_l\nu^2 + F(u_l)\nu^1)vdl$$

$$= \int_C (u_l\nu^2 + F(u_l)\nu^1)vdl$$

Here, $\nu = (\nu^1, \nu^2)$ is the unit normal to the curve C, pointing from V_l into V_r.

Integral Solution

Similarly, we obtain

$$\iint\limits_{V_r} [uv_t + F(u)v_x]dxdt = -\int\limits_C (u_r\nu^2 + F(u_r)\nu^1)vdl$$

By adding the above two equations, we obtain that

$$\int\limits_C ((u_l - u_r)\nu^2 + [F(u_l) - F(u_r)]\nu^1)vdl = 0$$

Since this is true for all test functions v, we have

$$(u_l - u_r)\nu^2 + [F(u_l) - F(u_r)]\nu^1 = 0$$

along C.

Integral Solution

Suppose C is represented parametrically as $\{(x,t) : x = s(t)\}$ for some smooth function $s(.) : [0,\infty) \to \mathbb{R}$. We can then take

$$\nu = (\nu^1, \nu^2) = \left(\frac{1}{\sqrt{1+\dot{s}^2}}, -\frac{\dot{s}}{\sqrt{1+\dot{s}^2}} \right)$$

Consequently, implies

$$F(u_l) - F(u_r) = \dot{s}(u_l - u_r)$$

in V along the curve C.

R-H Condition

$$\begin{cases} [[u]] = u_l - u_r = \text{jump in } u \text{ across the curve } C \\ [[F(u)]] = F(u_l) - F(u_r) = \text{jump in } F(u) \\ \sigma = \dot{s} = \text{speed of the curve } C \end{cases} \tag{16}$$

Hence

$$[[F(u)]] = \sigma[[u]] \tag{17}$$

along the discontinuity curve. This is the Rankine-Hugoniot condition.

Shocks

R-H Condition

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Example 1

Consider the IVP for Burger's equation

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{18}$$

with initial data

$$g(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ 1 - x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x \geq 1 \end{cases} \tag{19}$$

R-H Condition

Then, the solution is given by

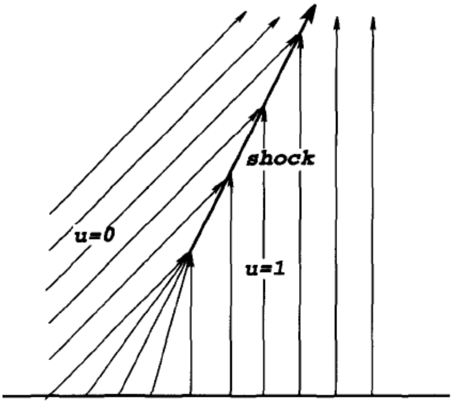
$$u(x,t) = \begin{cases} 1 & \text{if } x \leq t, 0 \leq t \leq 1 \\ \frac{1-x}{1-t} & \text{if } t \leq x \leq 1, 0 \leq t \leq 1 \\ 0 & \text{if } x \geq 1, 0 \leq t \leq 1 \end{cases} \tag{20}$$

Let us set $s(t) = \frac{1+t}{2}$ and write

$$u(x,t) = \begin{cases} 1 & \text{if } x < s(t) \\ 0 & \text{if } s(t) < x \end{cases} \tag{21}$$

if $t \geq 1$.

Entropy Condition



Formation of a shock

R-H Condition

Now, along the curve parameterized by $s(.), u_l = 1$, we have

$$u_r = 0, F(u_l) = \frac{1}{2}u_l^2 = \frac{1}{2}$$

$$F(u_r) = 0$$

Thus

$$[[F(u)]] = \frac{1}{2} = \sigma[[u]]$$

Entropy Condition

Rarefaction

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Example 2

Consider the IVP for Burger's equation

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{22}$$

with initial data

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{23}$$

Using the method of characteristics, we can obtain a solution to this problem. Let us set

$$u_1(x,t) = \begin{cases} 0 & \text{if } x < \frac{t}{2} \\ 1 & \text{if } x > \frac{t}{2} \end{cases} \tag{24}$$

Rarefaction

Hence, it is easy to check that the R-H condition holds and u is an integral solution. However, we create another solution by writing

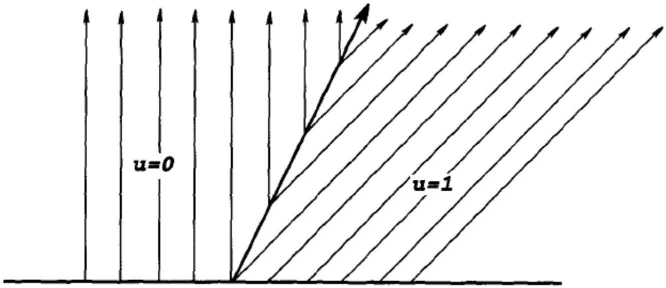
$$u_2(x,t) = \begin{cases} 1 & \text{if } x > t \\ \frac{x}{t} & \text{if } 0 < x < t \\ 0 & \text{if } x < 0 \end{cases} \tag{25}$$

This function is called the Rarefaction wave. Note that it is a continuous integral solution. Therefore, the integral solutions are not in general unique.

Question

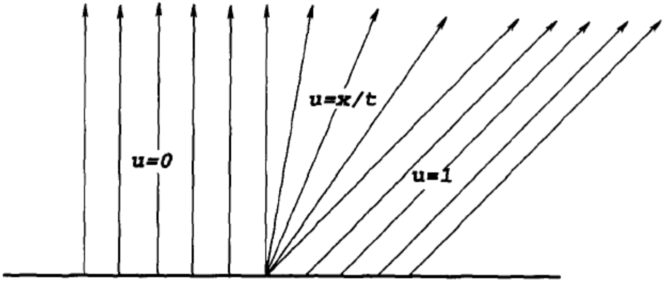
Any other criterion to ensure uniqueness?

Entropy Condition



A “nonphysical” shock

Entropy Condition



Rarefaction wave

Entropy Condition

Remember our conservation law discussion in the quasilinear case. We discussed that the general scalar conservation law of the form

$$u_t + F(u)_x = 0$$

the solution u, whenever smooth, takes the constant value $z_0 = g(\mathbf{x}_0)$ along the projected characteristic

$$\mathbf{y}(s) = (F'(g(\mathbf{x}_0))s + \mathbf{x}_0, s) \quad (s \geq 0) \tag{26}$$

Entropy Condition

- We know that we will typically encounter crossing characteristics and resultant discontinuities in the solution if we move forward in time.
- If we start at some point in $\mathbb{R}^n \times (0, \infty)$ and go backwards along a characteristic, we will not cross any others.
- That is, if we consider piecewise-smooth integral solutions with the property that if we move backwards in t along any characteristic, we will not encounter any lines of discontinuity for u.
- Suppose at some point on a curve C of discontinuities that u has distinct left and right limits u_l and u_r and that a characteristic from the left and the right hit C at this point.

Hence, we obtain the entropy condition.

$$F'(u_l) > \sigma > F'(u_r)$$

Entropy Condition

- In the thermodynamic principle that physical entropy cannot decrease as time goes forward
- A curve of discontinuity is called a shock provided both R-H identity and entropy inequalities hold.

Suppose F is uniformly convex, that is $F'' \geq \theta > 0$ for some constant θ, F' is strictly increasing, then entropy condition is equivalent to

$$u_l > u_r$$

along any shock curve.

Entropy Condition

Example 3

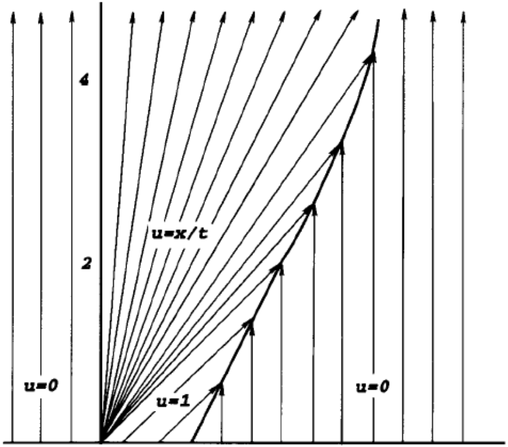
Consider the IVP for Burger's equation again with different initial conditions

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases} \tag{27}$$

For $0 \leq t \leq 2$, we may combine the analyses to find

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 \leq x \leq t \\ 1 & \text{if } t \leq x \leq 1 + \frac{1}{2} \\ 0 & \text{if } x > 1 + \frac{t}{2} \end{cases} \tag{28}$$

Entropy Condition



Entropy Condition

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For time $t \geq 2$, we expect the shock wave parametrized by $s(.)$ to continue, with $u = \frac{x}{t}$ to the left of $s(.)$, $u = 0$ to the right. That is, compatible with the entropy condition. Let us calculate the behaviour of the shock curve by applying the R-H jump condition. Now,

$$[[u]] = \frac{s(t)}{t}, \quad [[F(u)]] = \frac{1}{2}\left(\frac{s(t)}{t}\right)^2, \quad \sigma = \dot{s}(t) \tag{29}$$

along the shock curve $t \geq 0$. Thus

$$[[F(u)]] = \sigma[[u]] \implies \dot{s}(t) = \frac{s(t)}{2t} \quad (t \geq 2)$$

Further, $s(2) = 2$ and so we can solve this ODE to find $s(t) = \sqrt{2t}, t \geq 2$.

Entropy Condition

Hence, we obtain,

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{t} & \text{if } 0 \leq x \leq \sqrt{2t}, t \geq 2 \\ 0 & \text{if } x > \sqrt{2t} \end{cases} \tag{30}$$

Thanks

Doubts and Suggestions

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