MA612L-Partial Differential Equations

Lecture 4 : Classifications

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August 11, 2025

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Preliminaries

Directional Derivative

The derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\boldsymbol{u} = (u_1, u_2)$ is the number

$$(D_{\boldsymbol{u}} f)_{P_0} = \left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0}$$

$$= \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

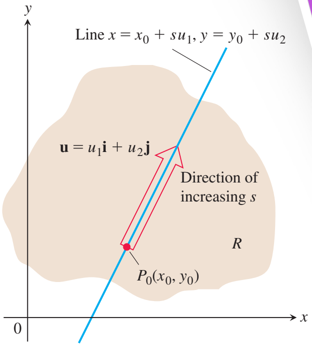

Figure 1: Source: Thomas' Calculus

Interpretation of Directional Derivatives

The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to $\boldsymbol{u}$ intersects the graph S in a curve C. The rate of change of f in the direction of $\boldsymbol{u}$ is the slope of the tangent to C at P in the right-handed system formed by the vectors $\boldsymbol{u}$ and z-axis.

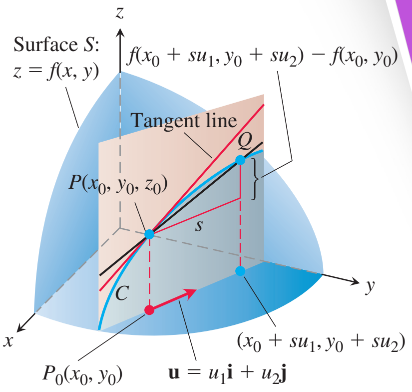

Figure 2: Source: Thomas' Calculus

Partial Derivatives as Directional Derivatives

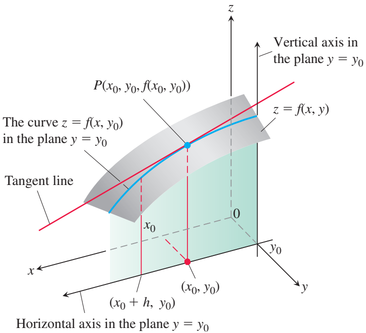

Figure 3: Source: Thomas' Calculus

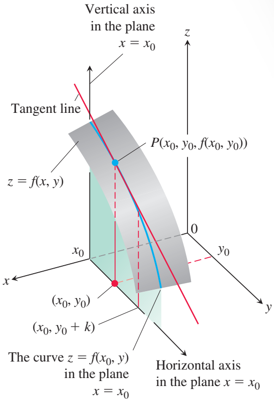

Figure 4: Source: Thomas' Calculus

Gradient

The gradient vector (gradient) of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

obtained by evaluating the partial derivatives of f at P_0.

∇f is read as "gradient of f" or "grad f" or "del f".

Directional Derivative and Gradient

Let $f(x, y)$ be a differentiable function. For the vector $\boldsymbol{u} = (u_1, u_2)$, consider the line

$$x = x_0 + su_1 \qquad y = y_0 + su_2$$

Then by chain rule

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} &= \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{ds}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dx}{ds}\right)\right]_{\boldsymbol{u},P_0} \\
&= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2 \\
&= \left[\left(\frac{\partial f}{\partial x}\right)_{P_0} \vec{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \vec{j}\right] \cdot \left[u_1 \vec{i} + u_2 \vec{j}\right] \\
&= (\nabla f)_{P_0} \cdot \boldsymbol{u}
\end{aligned}$$

Directional Derivative as a Dot Product

Theorem

If $f(x, y)$ be a differentiable function in an open region containing $P_0 = (x_0, y_0)$, then the directional derivative along the unit vector $\boldsymbol{u}$ is

$$\left(\frac{df}{ds}\right)_{\boldsymbol{u},P_0} = (\nabla f)_{P_0} \cdot \boldsymbol{u} = |\nabla f| \cos\theta$$

where θ is the angle between the vectors $\boldsymbol{u}$ and ∇f.

Properties of the directional derivative

- The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\boldsymbol{u}$ is the direction of $\boldsymbol{\nabla} f$. The derivative in this direction is

$$D_{\boldsymbol{u}} f = (\boldsymbol{\nabla} f) \cdot \boldsymbol{u} = |\boldsymbol{\nabla} f| \cos(0) = |\boldsymbol{\nabla} f|$$

- The function f decreases most rapidly when $\cos\theta = -1$ or when $\theta = -\pi$ and $\boldsymbol{u}$ is the direction of $-\boldsymbol{\nabla} f$. The derivative in this direction is

$$D_{\boldsymbol{u}} f = (\boldsymbol{\nabla} f) \cdot \boldsymbol{u} = |\boldsymbol{\nabla} f| \cos(\pi) = -|\boldsymbol{\nabla} f|$$

- Any direction $\boldsymbol{u}$ orthogonal to a gradient $\boldsymbol{\nabla} f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and

$$D_u f = (\boldsymbol{\nabla} f) \cdot \boldsymbol{u} = |\boldsymbol{\nabla} f| \cos(\pi/2) = 0$$

Directional Derivative

Recall the directional derivative definition

Definition 1 (Directional Derivative)
Let $f(x,y)$ be a function defined in a domain $\Omega \subset \mathbb{R}^2$. Let $(x_0,y_0) \in \Omega$. The **directional derivative** of $f(x,y)$ in the direction of a unit vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ at (x_0,y_0) is given by

$$(D_{\mathbf{v}}f)(x_0,y_0) = \left(\frac{df}{ds}\right)_{\mathbf{v}}\Bigg|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+ha,y_0+hb)-f(x_0,y_0)}{h}$$

Here $D_{\mathbf{v}}$ denotes the directional derivative in the direction of $\mathbf{v}$

From Rudin: If $\mathbf{v} = \sum v_i \mathbf{e_i}$, then

$$D_{\mathbf{v}}f(\mathbf{x}) = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}f(\mathbf{x})v_i$$

Gradient

Definition 2 (Gradient)
The vector operator

$$\nabla \equiv \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathbf{e_i}$$

is called the gradient. The gradient of a function $f(x_1, x_2, \cdots, x_n)$ is

$$\nabla f \equiv \text{grad} f := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathbf{e_i}$$

where $f : \mathbb{R}^n \to \mathbb{R}$

In 2D case,

$$\nabla f \equiv \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

Gradient

Remark

$$D_{\mathbf{v}} f|_{(x_0,y_0)} = \text{grad} f|_{(x_0,y_0)}.\mathbf{v}$$

$$D_{\mathbf{v}} f(\mathbf{x}) = \nabla f.\mathbf{v}$$

If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$,

$$D_{\mathbf{v}} f = f_x a + f_y b = (f_x \mathbf{i} + f_y \mathbf{j}).(a\mathbf{i} + b\mathbf{j})$$

If $D_{\mathbf{v}} f = 0$, then f is constant in the direction of the vector $\mathbf{v}$. The gradient of a function at a point is a vector that points in the direction in which the function increases most rapidly.

Divergence

Definition 3 (Divergence)
The divergence of a vector field is the flux per unit time. It is defined as an inner product between the gradient operator and the vector field

$$\nabla . \mathbf{v} \equiv \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$$

where $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$

In 3D, if $v = (v_1, v_2, v_3)$, then

$$\nabla . \mathbf{v} \equiv \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Laplacian

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Definition 4 (Laplacian)

The Laplacian of a scalar-valued function is defined as

$$\Delta f \equiv \nabla^2 f = \nabla.(\nabla f) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

where $f : \mathbb{R}^n \to \mathbb{R}$

Definition 5 (Laplacian of a vector)

The Laplacian of a vector-valued function is defined as

$$\Delta \mathbf{v} \equiv \nabla^2 \mathbf{v} = \nabla.(\nabla \mathbf{v}) = \sum_{i=1}^{n} \frac{\partial^2 v_i}{\partial x_i^2} \mathbf{e_i}$$

where $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$

Boundary Conditions

IVP

Definition 6 (Initial Value Problem)
A partial differential equation subject to certain conditions in the form of initial conditions is known as **initial value problem** or in short IVP. Usually the initial conditions are given as $u(\mathbf{x}, t_0) = f(\mathbf{x})$.

Example 1

$$\begin{cases} u_t - u_x = 0, & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x), & -\infty < x < \infty, t = 0 \end{cases}$$

BVP

Definition 7 (Boundary Value Problem)
A partial differential equation subject to certain conditions in the form of boundary conditions is known as **boundary value problem** or in short BVP. Usually, the boundary conditions are given as the values on the boundary $\partial\Omega$.

Example 2

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in \Omega \\ u(x, y) = \phi(x, y), & (x, y) \in \partial\Omega \end{cases}$$

Dirichlet Boundary Condition

There are three types of boundary conditions usually prescribed (although other conditions, like periodic, inlet, and outlet, are also available)

Dirichlet Boundary Condition: The solution is known at the boundary of the domain, or the values of u are prescribed at each point of the boundary $\partial\Omega$

$$u(x,t) = f(x), x \in \partial\Omega, t > 0$$

Example 3

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x,y) \in \Omega \\ u(x,y) = \phi(x,y), & (x,y) \in \partial\Omega \end{cases}$$

This is also called as **Fixed or Essential Boundary Condition** or boundary conditions of the first kind.

Neumann Boundary Condition

Neumann Boundary Condition: The derivative of the solution is given in a direction at the boundary of the domain or the values of the normal derivative of u are prescribed at each point of the boundary $\partial\Omega$

$$\frac{\partial u}{\partial \mathbf{n}} = \mathbf{n}.\nabla u = f(x), x \in \partial\Omega$$

Here $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$

Example 4

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = \psi(x, y), & (x, y) \in \partial\Omega \end{cases}$$

This is also called as **Natural Boundary Condition** or boundary conditions of the second kind.

Robin Boundary Condition

Robin Boundary Condition: It is a linear combination of Dirichlet and Neumann boundary conditions or when the values of a linear combination of u and its normal derivative are prescribed at each point of the boundary $\partial\Omega$

$$\alpha \frac{\partial u}{\partial \mathbf{n}} + \beta u(\mathbf{x}) = f(x), x \in \partial\Omega$$

Example 5

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in \Omega \\ \alpha \frac{\partial u}{\partial \mathbf{n}} + \beta u = g(x, y), & (x, y) \in \partial\Omega \end{cases}$$

This is also called as **impedance or convective boundary condition** or boundary conditions of the third kind.

Classification of PDEs

Why Classification

- Based on the number of properties, we can group families of similar equations
- In fact, a few researchers see no advantage in the classification process
- Some classifications are given a few branding, like Navier-Stokes, Heat Equation, etc
- Some classifications help to identify or guess, or predict the properties of solutions of PDEs in that class.
- Some classification helps to identify the allowable initial and boundary conditions
- A few classification helps to select an effective numerical method
- Classifications are done using characteristics, order, linearity, and so on.

PDE

Definition 8 (PDE-Formal Definition)

Let $\Omega \subset \mathbb{R}^n, m \in \mathbb{N}$ and

$$F : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \mathbb{R}^{n^2 p} \times \cdots \times \mathbb{R}^{n^m p} \to \mathbb{R}^q$$

A system of partial differential equations of order m is defined by the equation

$$F(\mathbf{x}, \mathbf{u}, D\mathbf{u}, D^2\mathbf{u}, \cdots, D^m\mathbf{u}) = \mathbf{0} \tag{1}$$

Here, some m^{th} order derivative of the function $\mathbf{u}$ appears in the system of equations.

Classification-I - System

Based on the number of equations, we can classify PDEs.

Definition 9
If a PDE (1) consists of more than one equation, it is called a system of PDEs. Otherwise, it is called a single PDE or a scalar PDE, or simply PDE.

Exercise 1:

Classify all PDEs given in our last class into a system of PDEs and a single PDE

Classification-II - Order

Based on the highest order derivative, we can classify PDEs.

Definition 10

If the highest order derivative appearing in the PDE is m, then such PDEs are classified as m^{th} order PDEs.

Exercise 2:

Find the order of PDEs of all PDEs discussed in our last class.

Classification-III - Linear/Nonlinear

Through algebra, we can also classify PDEs. In algebra, we categorize algebraic equations as linear and nonlinear equations. To define linearity, let us rewrite the equation (1) as

$$\mathcal{L}u = f \tag{2}$$

where $\mathcal{L}$ is an operator which assigns u a new function $\mathcal{L}u$. Here f is a function of x only.

Definition 11

The operator $\mathcal{L}$ is called linear if

$$\mathcal{L}(\alpha u + \beta v) = \alpha \mathcal{L}u + \beta \mathcal{L}v \tag{3}$$

for any function u and v and constants α and β.

Classification-III - Linear

Definition 12

If the operator $\mathcal{L}$ in (2) is linear, then the PDE is called a linear PDE. Equivalently, an m^{th}-order PDE is linear if it can be written as

$$\sum_{|\alpha| \le m} a_\alpha(\mathbf{x}) D^\alpha u = f(\mathbf{x}) \tag{4}$$

Here a_α's are functions of $\mathbf{x}$ only.

Example 6

1. $u_t + u_x = 0$
2. $u_{xx} + u_{yy} = 0$
3. $u_t + x^2 u_x = 0$

Classification-III - Nonlinear

Definition 13

If the operator $\mathcal{L}$ in (2) is not linear (or equivalently, it can't be written in the form of (4)), then the PDE is called a nonlinear PDE.

Example 7

1. $u_t + uu_x = 0$
2. $u_x^2 + u_y^2 = 0$

Exercise 3:

Identify the list of linear and nonlinear PDEs from all PDEs discussed in our last class.

Classification-IV - Quasilinear

We have categorized PDEs as linear and nonlinear already. The PDEs can be further categorized based on the linearity of different derivatives. For example,

- Quasilinear
- Non-Quasilinear or Fully nonlinear

Definition 14

The equation (1) of order m is called quasilinear if it is linear in the derivatives of order m with coefficients that depend on the independent variables and derivatives of the unknown function of order strictly less than m.

Classification-IV - Quasilinear

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Definition 15

Equivalently, an m^{th} order PDE is **quasilinear** if it can be written in the form

$$\sum_{|\alpha|=m} a_\alpha(\mathbf{x}, u, Du, \cdots D^{m-1}u) D^\alpha u + a_0(\mathbf{x}, u, Du, \cdots D^{m-1}u) = 0 \qquad (5)$$

Here a_α's are functions of $\mathbf{x}$ and derivatives of the unknown function of order less than m.

Definition 16

An m^{th} order PDE is called **fully nonlinear** if it is not linear in the derivatives of order m. Equivalently, a PDE that is not quasilinear is called a fully nonlinear PDE.

Quasilinear

Example,

$$u_x + uu_y = 0$$

For,

$$(c_1u_{1x} + c_2u_{2x}) + u(c_1u_{1y} + c_2u_{2y}) = c_1u_{1x} + c_2u_{2x} + c_1uu_{1y} + c_2uu_{2y}$$

Example 8

1. $u_x + uu_y = 0$ is quasilinear
2. $u_t + a(u)u_x = 0$ is quasilinear
3. $u_x^2 + u_y^2 = 0$ is not quasilinear. It is fully nonlinear
4. $div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$ is fully nonlinear
5. $u_t + u_x^2 - u = \cos(xt)$ is fully nonlinear

Classification-V - Semilinear

Quasilinear PDEs are further categorized into

- Semilinear
- Non-semilinear

Definition 17

A quasilinear PDE of order m is called a semilinear PDE if the coefficients of derivatives of order m are functions of the independent variables alone.

Classification-V - Semilinear

Definition 18

A quasilinear PDE of order m is called a **semilinear PDE** if the coefficients of derivatives of order m are functions of the independent variables alone. Equivalently

$$\sum_{|\alpha|=m} a_\alpha(\mathbf{x}) D^\alpha u + a_0(\mathbf{x}, u, Du, \cdots D^{m-1}u) = 0 \tag{6}$$

Here a_α's are functions of $\mathbf{x}$ alone.

Example 9

1. $u_t + u_x + u^2 = 0$ is semilinear
2. $u_t + u_{xxx} + uu_x = 0$ is semilinear
3. $xu_x + yu_y = u$ is semilinear
4. $u_t + uu_x = 0$ is not semilinear

Classification-VI - Almost linear

Definition 19

An m^{th} order semilinear PDE is called almost linear if it can be written in the form

$$\sum_{|\alpha|\leq m} a_\alpha(\mathbf{x}) D^\alpha u + f(\mathbf{x}, u) = 0 \tag{7}$$

Here a_α's are function of $\mathbf{x}$ alone or if it is of the form

$$\mathcal{L}u = f(x, u) \tag{8}$$

where $f(x, u)$ is a nonlinear function with respect to u and $\mathcal{L}$ is a linear operator.

Example 10

1. $u_t + u_x + u^2 = 0$ is almost linear
2. $xu_x + yu_y = u$ is almost linear

Classification-VII - In/Homogeneous

Suppose (1) can be written in the following form

$$\mathcal{D}(u) = f(\mathbf{x}) \tag{9}$$

Definition 20

If $f \equiv 0$ in (9), then the PDE is called a homogeneous PDE. If $f \neq 0$, then the PDE is an inhomogeneous PDE[1].

Example 11

1. $u_t + uu_x = 0$ is homogeneous
2. $2u_y - 5u^3 = x$ is inhomogeneous
3. $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = f(r, \theta)$ is inhomogeneous if $f \neq 0$

[1] In some textbooks it is also called nonhomogeneous PDE. Also, many textbooks usually classify only linear PDE as homogeneous and nonhomogeneous

Examples

Example 12

PDE	O	Lin	AL	Sem	Qua	HG	FNL
$u_t + u_x + u^2 = 0$	1	✗	✓	✓	✓	✓	✗
$u_{xx} + u_{yy} = 0$	2	✓	✓	✓	✓	✓	✗
$u_x^2 + u_y^2 = x^2 + y^2$	1	✗	✗	✗	✗	✗	✓
$u_x + 5u = x^2 y$	1	✓	✓	✓	✓	✗	✗

O - Order, Lin - Linear, AL - Almost linear, Sem - Semilinear, Qua - Quasilinear, HG - Homogeneous, FNL - Fully nonlinear.

Thanks

Doubts and Suggestions

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