MA612L-Partial Differential Equations

Lecture 40: Nonlinear PDE - Riemann Problem

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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Riemann’s Problem

Scalar Conservation Laws

The initial value problem for scalar conservation laws in one space dimension

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{1}$$

Lipschitz continuous

Assume $\Omega \subset \mathbb{R}^n$ is open. We consider the class of Lipschitz continuous functions $u : \Omega \to \mathbb{R}$ which by definition satisfies

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}| \quad (\mathbf{x}, \mathbf{y} \in \Omega)$$

for some constant C.
The above condition implies u is continuous, and uniform modulus of continuity.

Riemann's Problem

The IVP with a piecewise-constant initial function

$$g(x) = \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases} \quad (2)$$

is called Riemann's problem for the scalar conservation law (1). Here, $u_l, u_r \in \mathbb{R}$ are the left and right initial states $u_l \neq u_r$.
Assume that F is uniformly convex and C^2 and

$$G = (F')^{-1}$$

Solution of Riemann's Problem

Theorem 1 (Solution of Riemann's Problem)

1. If $u_l > u_r$, the unique entropy solution of the Riemann problem (1), (2) is

$$u(x,t) = \begin{cases} u_l & \text{if } \frac{x}{t} < \sigma \\ u_r & \text{if } \frac{x}{t} > \sigma \end{cases} \tag{3}$$

$$\text{where} \qquad \sigma := \frac{F(u_l) - F(u_r)}{u_l - u_r} \tag{4}$$

2. If $u_l < u_r$, the unique entropy solution of the (1), (2) is

$$u(x,t) = \begin{cases} u_l & \text{if } \frac{x}{t} < F'(u_l) \\ G\left(\frac{x}{t}\right) & \text{if } F'(u_l) < \frac{x}{t} < F'(u_r) \\ u_r & \text{if } \frac{x}{t} > F'(u_r) \end{cases} \tag{5}$$

here $x \in \mathbb{R}, t > 0$.

Solution of Riemann's Problem

Proof: Assume $u_l > u_r$. Clearly, u defined in (3) and (4) is an integral solution of our PDE. In particular, since

$$\sigma = \frac{[[F(u)]]}{[[u]]},$$

the R-H condition holds. Further

$$F'(u_r) < \sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r} = \fint_{u_r}^{u_l} F'(r)dr < F'(u_l)$$

Hence, the entropy condition holds. By the Uniqueness of the entropy solution theorem, we obtain the unique entropy solution.

Solution of Riemann's Problem

Assume that $u_l < u_r$. We must first check that u defined by (5) solves the conservation law in the region

$$\left\{ F'(u_l) < \frac{x}{t} < F'(u_r) \right\}$$

Question

When the function of the form

$$u(x,t) = v\left(\frac{x}{t}\right)$$

solve (1)?

Solution of Riemann's Problem

We compute

$$
\begin{aligned}
u_t + F(u)_x &= u_t + F'(u)u_x \\
&= -v'\left(\frac{x}{t}\right)\frac{x}{t^2} + F'(v)v'\left(\frac{x}{t}\right)\frac{1}{t} \\
&= v'\left(\frac{x}{t}\right)\frac{1}{t}\left[F'(v) - \frac{x}{t}\right]
\end{aligned}
$$

Assuming that v' never vanishes, we find

$$F'\left(v\left(\frac{x}{t}\right)\right) = \frac{x}{t}$$

Hence

$$u(x,t) = v\left(\frac{x}{t}\right) = G\left(\frac{x}{t}\right)$$

solves the conservation law.

Solution of Riemann's Problem

Now

$$v\left(\frac{x}{t}\right) = u_l$$

provided

$$\frac{x}{t} = F'(u_l)$$

and similarly

$$v\left(\frac{x}{t}\right) = u_r$$

provided

$$\frac{x}{t} = F'(u_r)$$

Solution of Riemann's Problem

Consequently, we see that the rarefaction wave u defined by (5) is continuous in $\mathbb{R} \times (0, \infty)$ and is a solution of the PDE $u_t + F(u)_x = 0$ in each of its regions of definition. It is easy to check that u is thus an integral solution of (1), (2). Further, assume that G is Lipschitz continuous, and we have

$$u(x+z,t) - u(x,t) = G\left(\frac{x+z}{t}\right) - G\left(\frac{x}{t}\right) \leq \frac{\text{Lip}(G)z}{t} \tag{6}$$

if

$$F'(u_l)t < x < x+z < F'(u_r)t$$

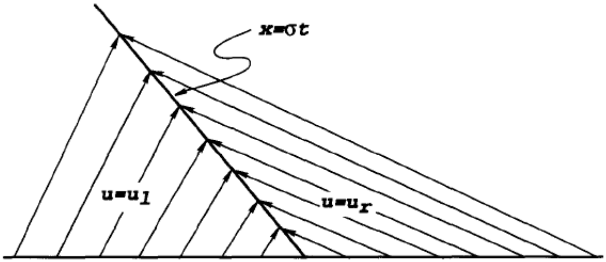
This implies that u satisfies the entropy condition and hence by the uniqueness of entropy solution theorem proves this theorem.

Solution of Riemann's Problem

Remarks

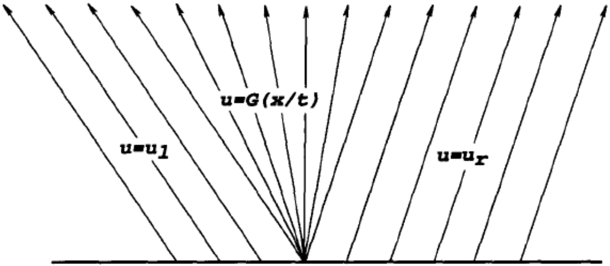
- If $u_l > u_r$, then u_l and u_r are separated by shock wave with constant speed σ.
- If $u_l < u_r$, then u_l and u_r are separated a rarefaction wave.
- This theorem shows the power of the uniqueness assertion and the Lax-Oleinik formula.

Riemann's Problem



Shock wave solving Riemann's problem for $u_l > u_r$

Riemann's Problem



Rarefaction wave solving Riemann's problem for $u_l < u_r$

Legendre Transform

Motivation and Intuition

- Suppose you have a convex function $f(x)$.
- You can describe it by its **value at each** x, or by its **slope** $p = f'(x)$.
- The **Legendre Transform** switches the variable:

$$\text{Position } (x) \longleftrightarrow \text{Slope } (p)$$

- It provides a new function $f^*(p)$ that measures how f behaves as the slope varies.

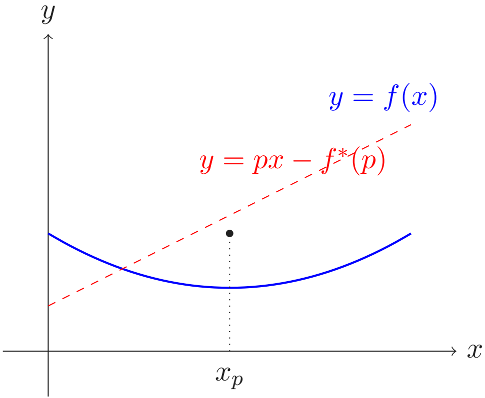
Definition

For a convex function $f : \mathbb{R} \to \mathbb{R}$, the **Legendre Transform** (or convex conjugate) is defined as

$$\boxed{f^*(p) = \sup_{x \in \mathbb{R}}(px - f(x))}.$$

- p is the slope of the tangent line.
- $f^*(p)$ gives the intercept of that tangent line.
- It measures the "cost" or "energy" associated with slope p.

Geometric Interpretation



$$f^*(p) = px_p - f(x_p)$$

- x_p is where the line of slope p touches $f(x)$.

Analytic Relations

If f is differentiable and convex:

$$p = f'(x), \qquad f^*(p) = px - f(x).$$

- The transform is **involutive**: $(f^*)^* = f$.
- The variables are dual:

$$x = (f^*)'(p).$$

- f and f^* are "mirror images" across the line $y = x$ in slope space.

Example 1: Quadratic Function

$$f(x) = \frac{1}{2}x^2.$$

Compute:

$$f^*(p) = \sup_x \left(px - \frac{1}{2}x^2 \right).$$

Differentiate: $p - x = 0 \Rightarrow x = p$.

$$\boxed{f^*(p) = \frac{1}{2}p^2.}$$

$\Rightarrow f^* = f.$ The quadratic function is *self-dual*.

Example 2: Exponential Function

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$$f(x) = e^x.$$

Then

$$f^*(p) = \sup_x (px - e^x).$$

Differentiating: $p - e^x = 0 \Rightarrow x = \ln p$ (for $p > 0$).

$$\boxed{f^*(p) = p \ln p - p, \quad p > 0.}$$

This appears in **information theory and entropy** formulas.

Physical Interpretation

In Classical Mechanics:

$$H(p) = \text{Hamiltonian: energy as a function of momentum},$$
$$L(v) = H^*(v) = \sup_p (pv - H(p)) \quad \text{(Lagrangian)}.$$

- The Legendre transform converts momentum p to velocity v.
- It connects the **Hamiltonian** and **Lagrangian** formulations.

Application: Hamilton–Jacobi PDEs

For the Hamilton–Jacobi equation

$$h_t + H(Dh) = 0, \qquad h(x,0) = h_0(x),$$

define the **Legendre transform** $L = H^*$.
Then the **Hopf–Lax formula** gives the solution:

$$h(x,t) = \inf_{y \in \mathbb{R}} \left\{ h_0(y) + tL\left(\frac{x-y}{t}\right) \right\}.$$

This expresses the evolution in terms of the **Lagrangian cost function** L.

Summary

- Legendre transform: $f^*(p) = \sup_x (px - f(x))$
- Converts "position" to "slope" representation.
- $f^{**} = f$ for convex functions.
- Links Hamiltonian (H) and Lagrangian (L) in mechanics and PDEs.
- Key ingredient in the Hopf–Lax formula for nonlinear PDEs.

Legendre Transform = Geometry of Convex Duality.

Lax-Oleinik Formula

Lax-Oleinik Formula

This formula obtains a weak solution of the IVP (1) with the assumption that F is uniformly convex. WLOG, $F(0) = 0$. Suppose $g \in L^{\infty}(\mathbb{R})$ and define

$$h(x) := \int_0^x g(y)dy \quad x \in \mathbb{R} \tag{7}$$

Define the Hopf-Lax formula

$$w(x,t) := \min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\} \quad (x \in \mathbb{R}, t > 0) \tag{8}$$

where

$$L = F^*$$

Lax-Oleinik Formula

Thus, w is the unique, weak solution of this IVP for the Hamilton-Jacobi equation

$$\begin{cases} w_t + F(w_x) = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ w = h & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{9}$$

If w is smooth, if we differentiate the PDE and its initial condition w.r.t. x to deduce

$$\begin{cases} w_{xt} + F(w_x)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ w_x = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases} \tag{10}$$

Note that w is not in general smooth.

Lax-Oleinik Formula

If w is differentiable a.e., then

$$u(x,t) := \frac{\partial}{\partial x}\left[\min_{y\in\mathbb{R}}\left\{tL\left(\frac{x-y}{t}\right)+h(y)\right\}\right] \tag{11}$$

is defined for a.e. (x,t). This allows a weak solution of the IVP. Since F is convex, F' is strictly increasing and onto, we can write

$$G := (F')^{-1}$$

for the inverse of F'

Lax-Oleinik Formula

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Theorem 2 (Lax-Oleinik Formula)
Assume $F : \mathbb{R} \to \mathbb{R}$ is smooth uniformly convex and $g \in L^\infty(\mathbb{R})$.

1. For each time $t > 0$, there exists for all but at most countably many values of $x \in \mathbb{R}$ a unique point $y(x, t)$ such that

$$\min_{y \in \mathbb{R}} \left\{ tL\left(\frac{x-y}{t}\right) + h(y) \right\} = tL\left(\frac{x - y(x,t)}{t}\right) + h(y(x,t)) \tag{12}$$

2. The mapping $x \to y(x, t)$ is nondecreasing
3. For each time $t > 0$, the function u defined by (11) is

$$u(x,t) = G\left(\frac{x - y(x,t)}{t}\right) \tag{13}$$

for a.e. x. In particular, the formula (13) holds for a.e. $(x, t) \in \mathbb{R}^n \times (0, \infty)$

Lax-Oleinik Formula

The Proof of the above theorem is left as an exercise. Refer to Evans' Book.

Definition 1 (Lax-Oleinik Formula)
The equation below is called the Lax-Oleinik formula for the solution (1)

$$u(x,t) = G\left(\frac{x - y(x,t)}{t}\right) \tag{14}$$

Lax-Oleinik Formula

Theorem 3 (Lax-Oleinik Formula as Integral Solution)
With the assumptions of the above theorem, the function defined by (14) is an integral solution of the IVP (1).

Theorem 4 (One-side jump estimate)
With the assumptions of the above theorem, there exists a constant C such that the function u defined by the (14) satisfies the inequality (entropy condition)

$$u(x+z,t) - u(x,t) \leq \frac{C}{t} z \qquad (15)$$

for all $t > 0$ and $x, z \in \mathbb{R}$, $z > 0$.

Weak Solution

The proof of the above theorems is left as an exercise again.

Definition 2 (Weak Solution)
We say that a function $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is an entropy solution of the IVP (1) provided

$$\int_0^\infty \int_{-\infty}^\infty uv_t + F(u)v_x dxdt + \int_{-\infty}^\infty gvdx|_{t=0} = 0 \tag{16}$$

for all test functions $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ with compact support, and

$$u(x+z,t) - u(x,t) \leq C\left(1 + \frac{1}{t}\right) z \tag{17}$$

for some constant $C \geq 0$ and a.e. $x, z \in \mathbb{R}, t > 0$ with $z > 0$.

Uniqueness of Entropy Solution

Theorem 5 (Uniqueness of Entropy Solution)
Assume F is convex and smooth. Then there exists upto a set of measure zero at most one entropy solution of (1).

The proof of this theorem is again an exercise.

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in