MA612L-Partial Differential Equations

Lecture 41: Nonlinear PDE - Sobolev Spaces

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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भारतीय प्रौद्योगिकी संस्थान तिरुपति
TIRUPATI

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Hölder Spaces

Motivation

- We know the hierarchy:

$$C^1(\Omega) \subset C^0(\Omega) \subset L^\infty(\Omega)$$

  but how do we measure *how smooth* a continuous function is?
- Consider these two functions on $(0, 1)$:

$$f_1(x) = x, \quad f_2(x) = \sqrt{x}.$$

  Both are continuous, but f_1 is “smoother” near $x = 0$ than f_2.
- We want a way to measure this degree of continuity.

Hölder Continuity

Assume $\Omega \subset \mathbb{R}^n$ is open and $0 < \gamma \leq 1$. We consider the class of Lipschitz continuous functions $u : \Omega \to \mathbb{R}$ which by definition satisfies

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}| \quad (\mathbf{x}, \mathbf{y} \in \Omega)$$

for some constant C. Let us consider the u satisfying the variant of this condition, namely.

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|^\gamma \quad (\mathbf{x}, \mathbf{y} \in \Omega)$$

for some constant C. This function is said to be a Hölder continuous with exponent γ.

Hölder Continuity

Definition (Hölder Continuous Function)
A function $u : \Omega \to \mathbb{R}$ is said to be **Hölder continuous of order** γ, where $0 < \gamma \leq 1$, if there exists $C > 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^\gamma \quad \forall x, y \in \Omega.$$

- $\gamma = 1$: Lipschitz continuous.
- $0 < \gamma < 1$: Hölder continuous but not Lipschitz.
- Example:

$$u(x) = \sqrt{x} \quad \Rightarrow \quad |u(x) - u(y)| \leq |x - y|^{1/2},$$

so $u \in C^{0,1/2}(0, 1)$.

Hölder Norm

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Definition 1 (Hölder norm)

If $u : \Omega \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\Omega)} := \sup_{\mathbf{x} \in \Omega} |u(\mathbf{x})|$$

The γ^{th}-Hölder seminorm of $u : \Omega \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{\substack{\mathbf{x},\mathbf{y} \in \Omega \\ \mathbf{x} \neq \mathbf{y}}} \left\{ \frac{|u(\mathbf{x}) - u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{\gamma}} \right\}$$

and the γ^{th}-Hölder norm is

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [u]_{C^{0,\gamma}(\overline{\Omega})}$$

Hölder Spaces

Definition 2 (Hölder Space)
The Hölder Space

$$C^{k,\gamma}(\overline{\Omega})$$

consists of all functions $u \in C^k(\overline{\Omega})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha| k} [D^\alpha u]_{C^{0,\gamma}(\overline{\Omega})}$$

is finite.

Theorem 1 (Banach Space)
The space of function $C^{k,\gamma}(\overline{\Omega})$ is a Banach space.

Hölder Spaces

Remarks

- Increasing $\alpha \Rightarrow$ stronger smoothness.
- $C^{0,0} = C^0$ (continuous functions).
- $C^{0,1} =$ Lipschitz functions.
- $C^{1,0} = C^1$, and so on.
- Hölder spaces interpolate between C^k and C^{k+1}.
- Hölder spaces measure smoothness in a *pointwise* sense.

Next, we'll see Sobolev spaces measure regularity in an *averaged (integrable)* sense.

Weak Derivatives

Motivation

- Classical calculus requires functions to be differentiable.
- But in PDEs, many natural solutions are not *classically* differentiable.
- Example: $u(x) = |x|$ on $(-1, 1)$
  - Continuous everywhere
  - Not differentiable at $x = 0$
- Still, we can make sense of its derivative *almost everywhere*.
- We need a function space that allows "derivatives in an averaged sense."

Goal

Find a space where both u and its **generalized derivatives** are integrable.

Weak Derivatives

Idea

If u is not differentiable, but there exists $v \in L^2(\Omega)$ such that

$$\int_\Omega u(x)\phi'(x)\,dx = -\int_\Omega v(x)\phi(x)\,dx \quad \forall\, \phi \in \mathcal{D}(\Omega),$$

then v is called the **weak derivative** of u.

Example 2

Let

$$u(x) = \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 < x < 2. \end{cases} \quad \Rightarrow \quad v(x) = \begin{cases} 1, & 0 < x \le 1, \\ 0, & 1 < x < 2. \end{cases}$$

Then v is the weak derivative of u.

Weak Derivatives

Remember:
Let $\mathcal{D} = C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions $\phi : \Omega \to \mathbb{R}$ with compact support in Ω. We call ϕ a test function. Let $u \in C^1(\Omega)$. Then if $\phi \in \mathcal{D}$, by integration by parts, we have

$$\int_\Omega u\phi_{x_i} d\mathbf{x} = -\int_\Omega u_{x_i}\phi d\mathbf{x}, \quad (i = 1, 2, \cdots, n)$$

Let $u \in C^k(\Omega)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multiindex of order $|\alpha| = \sum_{i=1}^n \alpha_i = k$. Then if $\phi \in \mathcal{D}$, by integration by parts, we have

$$\int_\Omega uD^\alpha \phi d\mathbf{x} = (-1)^{|\alpha|} \int_\Omega D^\alpha u\phi d\mathbf{x}$$

Weak Derivatives

Definition 3 (Weak Derivatives)
Suppose $u, v \in L^1_{loc}$ and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, which is written as

$$D^\alpha u = v$$

if

$$\int_\Omega u D^\alpha \phi d\mathbf{x} = (-1)^{|\alpha|} \int_\Omega v \phi d\mathbf{x}$$

for all test functions $\phi \in C_c^\infty(\Omega)$

Weak Derivatives

Theorem 3 (Uniqueness of Weak Derivatives)
A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero

Proof:
Let $u, v_1, v_2 \in L^1_{loc}(\Omega)$ satisfy

$$\int_\Omega u D^\alpha \phi d\mathbf{x} = (-1)^{|\alpha|} \int_\Omega v_1 \phi d\mathbf{x} = (-1)^{|\alpha|} \int_\Omega v_2 \phi d\mathbf{x} \quad \forall \phi \in \mathcal{D}$$

The rest of the proof follows immediately.

Weak Derivatives

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Example 4

Let $n = 1, \Omega = (0, 2)$ and

$$u = \begin{cases} x, & 0 < x \leq 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

Define

$$v = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & 1 \leq x < 2 \end{cases}$$

Then $u' = v$ in a weak sense.

Weak Derivatives

For, let $\phi \in \mathcal{D}(\Omega)$. Then we need to prove that

$$\int_0^2 u\phi' dx = -\int_0^2 v\phi dx$$

This follows immediately as

$$\int_0^2 u\phi' dx = \int_0^1 x\phi' dx + \int_1^2 \phi' dx = [x\phi]_0^1 - \int_0^1 \phi dx + \phi(2) - \phi(1)$$

$$= \phi(1) - \int_0^1 \phi dx + \phi(2) - \phi(1) = -\int_0^2 v\phi dx$$

Weak Derivatives

Example 5

Let $n = 1, \Omega = (0, 2)$ and

$$u = \begin{cases} x, & 0 < x \leq 1 \\ 2, & 1 \leq x < 2 \end{cases}$$

For this, u' does not exist in the weak sense.

Weak Derivatives

Suppose there exists v such that

$$\int_0^2 u\phi' dx = -\int_0^2 v\phi dx$$

for all $\phi \in \mathcal{D}$, then

$$-\int_0^2 v\phi dx = \int_0^2 u\phi' dx = \int_0^1 x\phi' dx + 2\int_1^2 \phi' dx = -\int_0^1 \phi dx - \phi(1)$$

Choose a sequence $\{\phi_m\}_{m=1}^{\infty}$ of smooth functions satisfying

$$0 \leq \phi_m \leq 1, \forall x \text{ and } m, \phi_m(1) = 1, \forall m, \phi_m(x) \to 0 \forall x \neq 1$$

Weak Derivatives

Using this ϕ_m, we obtain

$$1 = \lim_{m\to\infty} \phi_m(1) = \lim_{m\to\infty} \left[\int_0^2 v\phi_m dx - \int_0^1 \phi_m dx \right] = 0$$

It is a contradiction.

Sobolev Spaces

Sobolev Spaces

Fix $1 \leq p \leq \infty$ and k be a nonnegative integer.

Definition 4 (Sobolev Spaces)
The Sobolev space $W^{k,p}(\Omega)$ consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Remarks

- If $p = 2$, we get
$$H^k(\Omega) = W^{k,2}(\Omega), \quad (k = 0, 1, \cdots)$$
- H is used because it is a Hilbert space.
- $H^0(\Omega) = L^2(\Omega)$

Sobolev Spaces

Definition 5 (Essential Supremum)

Let $f : \Omega \to \mathbb{R}$ be measurable. The **essential supremum** of f is

$$\operatorname*{ess\,sup}_{x\in\Omega} f(x) = \inf\{M \in \mathbb{R} : f(x) \le M \text{ for a.e. } x \in \Omega\}.$$

It is the smallest upper bound that f does not exceed, except on a set of measure zero.

Example 6

If

$$f(x) = \begin{cases} 0, & x \neq 0, \\ 100, & x = 0, \end{cases}$$

then $\sup f = 100$ but $\operatorname{ess\,sup} f = 0$.

Sobolev Spaces

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Definition 6 (Sobolev Norm)
If $u \in W^{k,p}(\Omega)$, its norm is defined by

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\ \sum_{|\alpha| \leq k} \operatorname{ess\,sup}_{x \in \Omega} |D^\alpha u(x)|, & p = \infty. \end{cases}$$

Intuition and Importance

Norm (Energy Norm)

$$\|u\|_{H^1}^2 = \int_\Omega \left(|u|^2 + |\nabla u|^2\right) dx.$$

- $H^1(\Omega)$ is also called the **energy space**.
- If u represents displacement, $|\nabla u|^2$ measures stored elastic energy.
- Finite energy $\Leftrightarrow u, \nabla u \in L^2(\Omega)$.
- Hence Sobolev spaces are the natural setting for weak (variational) solutions of PDEs.

Intuition and Importance

Example: Poisson Equation

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \phi \in H_0^1(\Omega).$$

This makes sense even when u'' doesn't exist classically!

Geometric Picture and Hierarchy

- Sobolev spaces extend the notion of differentiability:

$$C_0^1(\Omega) \subset H_0^1(\Omega) \subset L^2(\Omega).$$

- They are complete under the H^1-norm, hence Hilbert spaces.
- Sobolev Embedding (1D intuition):

$$H^1(0,1) \hookrightarrow C^{0,1/2}(0,1),$$

  meaning H^1 functions are automatically continuous.
- Thus, Sobolev spaces **bridge pure and applied analysis**, connecting geometry, energy, and weak solutions.

Sobolev Spaces

Definition 7 (Convergence)
Let $\{u_m\}_{m=1}^{\infty}, u \in W^{k,p}(\Omega)$. We say that u_m converges to u in $W^{k,p}(\Omega)$, written

$$u_m \to u \text{ in } W^{k,p}(\Omega)$$

if

$$\lim_{m\to\infty} \|u_m - u_n\|_{W^{k,p}(\Omega)} = 0$$

We write

$$u_m \to u \text{ in } W^{k,p}_{loc}(\Omega)$$

to mean

$$u_m \to u \text{ in } W^{k,p}(V)$$

for each $V \subset\subset \Omega$

Here $V \subset\subset \Omega$ denotes that V is compactly contained in Ω

Sobolev Spaces

Definition 8 (Closure)
We denote by $W_0^{k,p}(\Omega)$ the closure of $\mathcal{D}$ in $W^{k,p}(\Omega)$

It claims that $u \in W_0^{k,p}(\Omega)$ if and only if there exists function $u_m \in \mathcal{D}(\Omega)$ such that $u_m \to u$. We interpret $W_0^{k,p}(\Omega)$ as comprising those function $u \in W^{k,p}(\Omega)$ such that

$$D^\alpha u = 0 \quad \text{on} \quad \partial\Omega, \forall|\alpha| \leq k-1$$

Remarks

- $H_0^k(\Omega) = W_0^{k,2}(\Omega)$
- If $n = 1$ and Ω is an open interval in $\mathbb{R}$, then $u \in W^{1,p}(\Omega)$ if and only if u equals a.e. an absolutely continuous function whose derivative (which exists a.e.) belongs to $L^p(\Omega)$.
- In general, a function can belong to a Sobolev space, and yet be discontinuous and/or unbounded.

Sobolev Spaces

Example 7

Let $\Omega = B^0(0,1)$, the open unit ball in $\mathbb{R}^n$ and

$$u(\mathbf{x}) = |\mathbf{x}|^{-\alpha}, \quad \mathbf{x} \in \Omega, \mathbf{x} \neq 0$$

For which values of $\alpha > 0, n, p$ does $u \in W^{1,p}(\Omega)$?
Note that u is smooth away from 0.

$$u_{x_i}(\mathbf{x}) = \frac{-\alpha x_i}{|\mathbf{x}|^{\alpha+2}}, \quad \mathbf{x} \neq 0$$

and

$$|D^\alpha u(\mathbf{x})| = \frac{|\alpha|}{|\mathbf{x}|^{\alpha+1}}, \quad \mathbf{x} \neq 0$$

Let $\phi \in \mathcal{D}$ and fix $\epsilon > 0$.

Sobolev Spaces

Then

$$\int\limits_{\Omega - B(0,\epsilon)} u\phi_{x_i} d\mathbf{x} = -\int\limits_{\Omega - B(0,\epsilon)} u_{x_i}\phi d\mathbf{x} + \int\limits_{\partial B(0,\epsilon)} u\phi\nu^i dS$$

Now, if $\alpha + 1 < n$, $|Du(\mathbf{x})| \in L^1(\Omega)$. In this case,

$$\left|\int\limits_{\partial B(0,\epsilon)} u\phi\nu^i dS\right| \leq \|\phi\|_{L^\infty} \int\limits_{\partial B(0,\epsilon)} \epsilon^{-\alpha}\phi\nu^i dS \leq C\epsilon^{n-1-\alpha} \to 0$$

Hence

$$\int\limits_{\Omega} u\phi_{x_i} d\mathbf{x} = -\int\limits_{\Omega} u_{x_i}\phi d\mathbf{x} \quad \forall \phi \in \mathcal{D}(\Omega), 0 \leq \alpha < n-1$$

Sobolev Spaces

Further,

$$|D^\alpha u(\mathbf{x})| = \frac{|\alpha|}{|\mathbf{x}|^{\alpha+1}} \in L^p(\Omega) \quad \text{if and only if} \quad (\alpha+1)p < n$$

Therefore,

$$u \in W^{1,p}(\Omega) \quad \text{if and only if} \quad \alpha < \frac{n-p}{p}$$

In particular,

$$u \notin W^{1,p}(\Omega) \quad \text{for each} \quad p \geq n$$

Sobolev Spaces

Example 8

Let $r_k{}_{k=1}^{\infty}$ be a countable dense subset of $\Omega = B^0(0,1)$. Write

$$u(\mathbf{x}) = \sum_{k=1}^{\infty} \frac{1}{2^k} |x - r_k|^{-\alpha}, \quad x \in \Omega$$

Then

$$u \in W^{1,p}(\Omega) \quad \text{if and only if} \quad \alpha < \frac{n-p}{p}$$

If $0 < \alpha < \frac{n-p}{p}$, we can observe that $u \in W^{1,p}(\Omega)$ and yet is unbounded on each open subset of Ω.

Sobolev Spaces

Theorem 9 (Properties of Weak Derivatives)

Assume $u, v \in W^{k,p}(\Omega)$, $|\alpha| \leq k$. Then

1. $\forall \alpha, \beta$ with $|\alpha| + |\beta| \leq k$,

$$D^\alpha u \in W^{k-|\alpha|,p}(\Omega) \text{ and } D^\beta(D^\alpha u) = D^\alpha(D^\beta u),$$

2. For each $\lambda, \mu \in \mathbb{R}$,

$$\lambda u + \mu v \in W^{k,p}(\Omega), \text{ and } D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v, |\alpha| \leq k$$

3. If V is an open set of Ω, then $u \in W^{k,p}(V)$
4. If $\zeta \in \mathcal{D}$, then $\zeta u \in W^{k,p}(\Omega)$ and

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u$$

Sobolev Spaces

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Theorem 10 (Sobolev Spaces as function spaces)
For each $k = 1, 2, \cdots$, and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.

Theorem 11 (Trace Theorem)
Assume Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

1. $Tu = u|_{\partial\Omega}$ is $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$
2. $$\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

   for each $u \in W^{1,p}(\Omega)$ with constant C depending only on p and Ω.

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in