MA612L-Partial Differential Equations

Lecture 5 : Classifications - II

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Classification of PDEs

PDE

Definition 1 (PDE-Formal Definition)
Let $\Omega \subset \mathbb{R}^n, m \in \mathbb{N}$ and

$$F : \Omega \times \mathbb{R}^p \times \mathbb{R}^{np} \times \mathbb{R}^{n^2 p} \times \cdots \times \mathbb{R}^{n^m p} \to \mathbb{R}^q$$

A system of partial differential equations of order m is defined by the equation

$$F(\mathbf{x}, \mathbf{u}, D\mathbf{u}, D^2\mathbf{u}, \cdots, D^m\mathbf{u}) = \mathbf{0} \tag{1}$$

Here, some m^{th} order derivative of the function $\mathbf{u}$ appears in the system of equations.

Classifications - Recap

1. Number of PDEs - System/Single
2. Higher Order derivative - Order
3. Linear/Nonlinear $\mathcal{L}u = f$, $\sum_{|\alpha| \leq m} a_\alpha(\mathbf{x}) D^\alpha u = f(\mathbf{x})$
4. Quasilinear $\sum_{|\alpha| = m} a_\alpha(\mathbf{x}, u, Du, \cdots D^{m-1}u) D^\alpha u + a_0(\mathbf{x}, u, Du, \cdots D^{m-1}u) = 0$
5. Semilinear $\sum_{|\alpha| = m} a_\alpha(\mathbf{x}) D^\alpha u + a_0(\mathbf{x}, u, Du, \cdots D^{m-1}u) = 0$
6. Almost linear
7. Fully nonlinear $\sum_{|\alpha| \leq m} a_\alpha(\mathbf{x}) D^\alpha u + f(\mathbf{x}, u) = 0$
8. Homogeneous/non-homogeneous $\mathcal{D}(u) = f(\mathbf{x})$

First-order PDEs in Two Variables

The general first-order PDEs in two variables can be written in the form

$$F(x, y, u, u_x, u_y) = 0 \tag{2}$$

A first-order **linear** PDE is of the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u + f(x, y) \tag{3}$$

A first-order **semilinear** PDE is of the form

$$a(x, y)u_x + b(x, y)u_y = c(x, y, u) \tag{4}$$

A first-order **quasilinear** PDE is of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{5}$$

Second-order PDEs in Two Variables

The general second-order PDEs in two variables can be written in the form

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

A second-order **linear** PDE is of the form

$$a_1(x, y)u_{xx} + a_2(x, y)u_{xy} + a_3(x, y)u_{yy} + a_4(x, y)u_x + a_5 u_y + a_6(x, y)u = f(x, y)$$

A second-order **semilinear** PDE is of the form

$$a_1(x, y)u_{xx} + a_2(x, y)u_{xy} + a_3(x, y)u_{yy} = f(x, y, u, u_x, u_y)$$

A second-order **quasilinear** PDE is of the form

$$a_1(x, y, u, u_x, u_y)u_{xx} + a_2(x, y, u, u_x, u_y)u_{xy} + a_3(x, y, u, u_x, u_y)u_{yy} = f(x, y, u, u_x, u_y)$$

Classification

Remarks 1

A few authors classify

- only linear PDE as homogeneous and nonhomogeneous
- nonlinear PDE as semilinear and non-semilinear
- non-semilinear PDE as quasilinear and non-quasilinear/fully nonlinear

Remarks 2

One can prove that

$$\text{Linaer PDE} \subsetneq \text{Semilinear PDE} \subsetneq \text{Quasilinear PDE} \subsetneq \text{PDE}$$

(Prove that the inclusion is strict!)

Exercise

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Exercise 1: Hard

Create a table as in the last lecture and fill in the tick marks accordingly.

1. $u_{xxx} - 4u_{xxyy} + u_{yyzz} = 0$
2. $u_x^2 u_{tt} - 0.5u = 1 - u^2$
3. $u_{tt}u_{xxx} - u_x u_{ttt} = x^2 + t^2$
4. $e^{u_{xtt}} - u_{xt}u_{xxx} + u^2 = 0$
5. $2\cos(xt)u_t - xe^t u_x - 9u = e^t \sin x$
6. $uu_t + u^2 u_x + u = e^x$
7. $\sqrt{1 + x^2y^2}u_{xyy} - \cos(xy^3)u_{xxy} + e^{-y^3}u_x - (5x^2 - 2xy + 3y^2)u = 0$

Classification of Second-Order Semilinear PDEs

Classification of Second-Order Semilinear PDEs

The most general case of second-order Semilinear PDEs in two independent variables is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = H \tag{6}$$

where the coefficients A, B and C are functions of x and y and do not vanish simultaneously (why?). Assume that $u \in C^2(\Omega)$. The above equation can also be rewritten as

$$A(x, y)u_{xx} + B(x, y)u_{xy} + C(x, y)u_{yy} = \Phi(x, y, u, u_x, u_y) \tag{7}$$

This equation is somehow similar to a conic section. Therefore, the classification is similar to it.

Classification of Second-Order Semilinear PDEs

Depending on the discriminant $B^2 - 4AC$, they are classified as hyperbolic, elliptic, and parabolic PDEs. They physically represent propagation, steady state or equilibrium process, or diffusion process.

- Hyperbolic:
    - The transport of some physical quantity, such as fluids or waves
    - $B^2 - 4AC > 0$
- Parabolic:
    - Describes evolutionary phenomena that lead to a steady state
    - $B^2 - 4AC = 0$
- Elliptic:
    - Special state of a system. Related to the minimum of the energy.
    - $B^2 - 4AC < 0$

Classification of Second-Order Semilinear PDEs

Note that a given PDE may be one type at a specific point, but another type at some other point. Consider the following Tricomi equation

$$u_{xx} + xu_{yy} = 0 \tag{8}$$

Then $B = 0, A = 1, C = x$. Hence $B^2 - 4AC = -4x$

- Hyperbolic:
  - $B^2 - 4AC > 0 \implies x < 0$
- Parabolic:
  - $B^2 - 4AC = 0 \implies x = 0$
- Elliptic:
  - $B^2 - 4AC < 0 \implies x > 0$

Classification of Second-Order Semilinear PDEs

Definition 2 (Hyperbolic/Elliptic/Parabolic)
A PDE is said to be hyperbolic or elliptic, or parabolic in a region Ω if the PDE is hyperbolic or elliptic or parabolic at each point in Ω

Examples 1
Classify the following PDEs as elliptic or hyperbolic, or parabolic

- $u_{tt} - c^2 u_{xx} = 0$
- $u_t - c^2 u_{xx} = 0$
- $u_{xx} + u_{yy} = 0$
- $(1 - M_\infty^2) u_{xx} + u_{yy} = 0$
- $u_{xx} + x^2 u_{yy} = 0$
- $u_{xx} - x u_{yy} = 0$

Classification of Second-Order Linear PDEs: n-Variables

Classification of Second-Order Linear PDEs: n-Variables

Consider the general second-order Linear PDEs in $n > 1$ independent variables.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0 \tag{9}$$

where a_{ij}, b_i, c, d are functions of $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $u = u(x_1, x_2, \cdots, x_n)$. It can also be rewritten as,

$$\begin{bmatrix} u_{x_1} & u_{x_2} & \cdots & u_{x_n} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{x_2} \\ \cdots \\ u_{x_n} \end{bmatrix} + \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \begin{bmatrix} u_{x_1} \\ u_{x_2} \\ \cdots \\ u_{x_n} \end{bmatrix} + cu + d = 0$$

Classification of Second-Order Linear PDEs: n-Variables

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \nabla u = \begin{bmatrix} u_{x_1} \\ u_{x_2} \\ \cdots \\ u_{x_n} \end{bmatrix}, b^T = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

Then (9) can be written as

$$(\nabla u)^T A(\nabla u) + b^T \nabla u + cu + d = 0 \tag{10}$$

Assume that A is symmetric. If not, we can always find a symmetric matrix H such that it can be rewritten as

$$(\nabla u)^T H(\nabla u) + b^T \nabla u + cu + d = 0 \tag{11}$$

Classification of Second-Order Linear PDEs: n-Variables

Consider the transformation

$$\xi = Q\mathbf{x} \tag{12}$$

where Q is an arbitrary $n \times n$ matrix. Now,

$$\xi = \sum_{i=1}^{n} q_{ij} x_j$$

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i}$$

$$\frac{\partial^2}{\partial x_i \partial x_j} = \sum_{k,l=1}^{n} \frac{\partial^2}{\partial \xi_k \partial \xi_l} \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_l}{\partial x_j}$$

Classification of Second-Order Linear PDEs: n-Variables

Therefore, equation (9) can be written as

$$\sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} q_{kj} a_{ij} q_{i} j \right) \frac{\partial^2 u}{\partial \xi_k \partial \xi_l} + \text{lowe-order terms} = 0$$

The coefficient matrix of the terms $\frac{\partial^2 u}{\partial \xi_k \partial \xi_l}$ is given by $Q^T AQ$. That is,

$$(q_{kj} a_{ij} q_{ij}) = Q^T AQ \tag{13}$$

Since A is a real symmetric matrix, there exists an orthogonal matrix P such that

$$P^T AP = \Lambda$$

where Λ is a diagonal matrix with elements as eigenvalues of A.

Classification of Second-Order Linear PDEs: n-Variables

Choose Q as an orthogonal matrix such that

$$Q^T A Q = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & \cdots & \lambda_n \end{bmatrix}, \tag{14}$$

Since A is a real symmetric matrix, λ_i's are always real.

- Elliptic: If $\lambda_i \neq 0, \forall i$ and sign$(\lambda_i) =$ sign$(\lambda_1), \forall\ i \neq 1$
- Hyperbolic: If $\lambda_i \neq 0, \forall i,$ sign$(\lambda_i) =$ sign$(\lambda_2), \forall\ i \neq 1, 2$ and sign$(\lambda_2) \neq$ sign(λ_1)
- Parabolic: If $\exists i$ such that $\lambda_i = 0$

Classification of Second-Order Linear PDEs: n-Variables

- Elliptic: All eigenvalues are non-zero and have the same sign (all positive or all negative)
- Hyperbolic: All eigenvalues are non-zero, and exactly one eigenvalue has a different sign from the others
- Parabolic: Any of the eigenvalues is zero

Can we extend this to semilinear PDEs also?

Classification of Second-Order Linear PDEs: n-Variables

Example 2

Classify the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0 \tag{15}$$

The coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

A is already diagonal. Also, eigenvalues of A are 1. $\lambda_i \neq 0$ and all eigenvalues are positive. Therefore, the Laplace equation is elliptic.

Classification of Second-Order Linear PDEs: n-Variables

Example 3

Classify the two-dimensional Wave equation

$$u_{tt} - c^2(u_{xx} + u_{yy}) = 0 \tag{17}$$

The coefficient matrix is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -c^2 & 0 \\ 0 & 0 & -c^2 \end{bmatrix} \tag{18}$$

A is already diagonal. Also, eigenvalues of A are $1, -c^2$ and $-c^2$. $\lambda_i \neq 0$ and all eigenvalues are negative except one. Therefore, the wave equation is a hyperbolic PDE.

Classification of Second-Order Linear PDEs: n-Variables

Example 4

Classify the two-dimensional Heat equation

$$u_t - c^2(u_{xx} + u_{yy}) = 0 \tag{19}$$

The coefficient matrix is given by

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -c^2 & 0 \\ 0 & 0 & -c^2 \end{bmatrix} \tag{20}$$

A is already diagonal. Also, eigenvalues of A are 0, $-c^2$ and $-c^2$. One eigenvalue is 0. Therefore, it is a parabolic PDE.

Classification of Second-Order Linear PDEs: n-Variables

Exercise 2: Verify

Verify all 2-variables cases discussed in this lecture with a 2×2 matrix, with this matrix concept of classification.

Thanks

Doubts and Suggestions

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