MA612L-Partial Differential Equations

Lecture 6 : Method of Characteristics

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Transport Equation

The Ant and the Dove Story

One quiet morning, an ant was collecting food along the edge of a river. Suddenly, it slipped and fell into the flowing water. The river began to carry it downstream. A dove, watching from a nearby tree, noticed the ant struggling. Acting quickly, the dove dropped a leaf into the water just ahead of the ant.

The leaf floated with the river's current. The ant climbed onto the leaf and was carried safely to the riverbank. The ant survived, not because the water stopped, but because something carried it safely across the stream.

Ant and Dove

The Ant and the Dove Story

This story serves as a metaphor for the **transport equation** in physics and mathematics:

- The **river flow** is the *velocity field* (constant in this case).
- The **leaf** is the carrier — like a parcel of mass or temperature.
- The **ant** represents a quantity (e.g., pollutant, energy) that is being transported.

Mathematical Derivation-1D Transport Equation

Let $u(x, t)$ represent the quantity (e.g., temperature, concentration) being transported along a 1D line, where:

- x is the spatial coordinate,
- t is time,
- c is the constant velocity of transport (e.g., river flow speed).

Now, let $u(x, t)$ denote the concentration of the pollut**ant** in kg/m (unit mass per unit length) at time t. The amount of pollut**ant** in the interval $[a, b]$ at time t is then

$$\int_a^b u(x,t)dx$$

Transport Equation - Derivation

Due to conservation of mass, the above quantity must be equal to the amount of the pollut**ant** after some time δt. After the time δt, the pollut**ant** would have flown to the interval $[a + c\delta t, b + c\delta t]$, thus the conservation of mass gives

$$\int_a^b u(x,t)dx = \int_{a+c\delta t}^{b+c\delta t} u(x, t+\delta t)dx$$

Differentiating with respect to b we get

$$u(b,t) = u(b + c\delta t, t + \delta t)$$

Transport Equation - Derivation

$$u(b,t) = u(b + c\delta t, t + \delta t)$$

This equation asserts that the concentration at the point b at time t is equal to the concentration at the point $b + c\delta t$ at time $t + \delta t$, which is to be expected as the water containing the pollutant particles flows with constant speed. Since b is arbitrary, we can replace it with x. So, it becomes

$$u(x,t) = u(x + c\delta t, t + \delta t)$$

Transport Equation - Derivation

Now,

$$u(x + c\delta t, t + \delta t) - u(x, t) = 0 \implies \frac{u(x + c\delta t, t + \delta t) - u(x, t)}{\delta t} = 0$$

When $\delta t \to 0$ and taking $v = (c, 1)$, we obtain that

$$D_v u = 0 \implies (c, 1).(u_x, u_t) = 0 \implies cu_x + u_t = 0$$

First-Order linear PDEs

with constant Coefficients

First-Order linear PDEs with Const. Coeff.

Consider the general first-order linear PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2 \tag{1}$$

- We are expecting an $u(x,y)$ such that $u \in C^1(\Omega)$, and u satisfies (1).
- We expect $u = u(x,y)$ is an explicit function of x and y or an implicit form of the solution $U(x,y,u) = 0$ which is known as **integral surface or solution surface.**

Let us look at how to solve the problem when a and b are constants, that is,

$$au_x + bu_y = 0 \tag{2}$$

where a and b are constants such that $a^2 + b^2 \neq 0$. Equation (2) is often called the convection or advection or transport equation.

Geometric View Point to get Solution

Equation (2) can be rewritten in the dot product in $\mathbb{R}^2$ as follows

$$(a, b).(u_x, u_y) = 0 \quad \text{or} \quad (a, b).\nabla u = 0 \tag{3}$$

What does equation (3) represent geometrically? Let $\mathbf{v} = (a, b)$, then (3) can be rewritten as

$$D_{\mathbf{v}} u(x, y) = 0 \tag{4}$$

where $D_{\mathbf{v}}\mathbf{u}$ denotes the directional derivative of $\mathbf{u}$ in the direction of the vector $\mathbf{v}$. Therefore, the solution (4) must be constant in the direction of the vector $\mathbf{v} = (a, b)$ and hence for (3) and (2).

Geometric View Point to get Solution

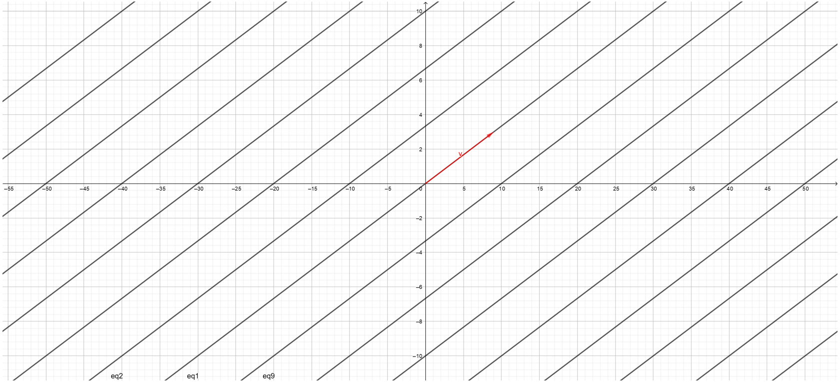
A line parallel to the vector $\mathbf{v} = (a, b)$ and through a point (x_1, y_1) is given by (from School Mathematics on Geometry)

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}$$

$$\implies bx - ay = c$$

Any line parallel to the vector $\mathbf{v}$ has the equation $bx - ay = c$. Since $(b, -a).(a, b) = 0$, $(b, -a)$ is a normal vector to the lines parallel to $\mathbf{v}$. Since c is an arbitrary constant, $bx - ay = c$ determines the particular line in the family of parallel lines. These lines are called **characteristic lines** for the equation (2). Refer to the Figure in the next slide.

Characteristic Lines



Solutions

Since $u(x, y)$ is constant in the direction of $\mathbf{v}$ and hence along the lines $bx - ay = c$. The line containing the point (x, y) is determined by $bx - ay = c$. Therefore, u will depend only on $bx - ay$, that is,

$$u(x, y) = f(bx - ay)$$

where f is an arbitrary function. You can verify that $au_x + bu_y = 0$.
If we compare this with the transport equation

$$cu_x + u_t = 0$$

we can see that $a = c, b = 1, x = x, y = t$, therefore

$$u(x, t) = f(x - ct)$$

Method of Characteristics

$$au_x + bu_y = 0 \tag{5}$$

The underlying concept of the Method of Characteristics is to convert the PDEs into a system of ODEs. In order to have an ODE, let us eliminate one of the partial derivatives in the equation (5). Since the directional derivative vanishes in the direction of $v = (a, b)$, let us transform the coordinate system such that its x-axis is parallel to v. Let (ξ, η) be the transformed coordinate system. That is,

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

In this coordinate system we have,

$$(\xi, \eta) = ((x, y).(a, b), (x, y).(b, -a)) = (ax + by, bx - ay)$$

Method of Characteristics

$$\xi(x,y) = ax + by$$

$$\eta(x,y) = bx - ay$$

Let us convert (5) in (ξ, η) coordinate system

$$u_x = u_\xi \xi_x + u_\eta \eta_x = a u_\xi + b u_\eta$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = b u_\xi - a u_\eta$$

$$\implies a u_x + b u_y = (a^2 + b^2) u_\xi = 0$$

Since $a^2 + b^2 \neq 0$, we have

$$u_\xi = 0$$

$$\implies u(\xi, \eta) = f(\eta) \implies u(x,y) = f(bx - ay)$$

Characteristics

We obtained the same solution as we obtained from our geometrical view.

- This method is called **method of characteristics**
- The coordinates given below are called **characteristic coordinates**

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

Definition 1 (Method of Characteristics)
The reduction of a PDE to an ODE along its characteristics is called the method of characteristics.

Recall
In our Lecture-2, we mentioned that $u_x + u_y = 0$ has a general solution of the form $u = f(x - y)$.

First-Order Quasilinear PDEs

First-Order Quasilinear PDEs

Consider the following quasilinear PDE

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \tag{6}$$

where $a, b, c \in C^1(\Omega)$. Let Ω_0 denote the projection of Ω in the $xy-$plane. As we did earlier, the solution of (6) defines an integral surface.

Definition 2 (Integral Surface)
Let $D \subset \Omega_0$ and $u : D \to \mathbb{R}$ be a solution of the equation (6). The surface S represented by $u = u(x, y)$ in the Euclidean space (x, y, u) is called an integral surface corresponding to a given solution u.

First-Order Quasilinear PDEs

The normal to this surface at the point $P(x, y, u)$ is the vector $(u_x, u_y, -1)$ and $v = (a(x, y), b(x, y), u(x, y))$. v is also known as the **Characteristic vector field** of the equation (6). Let us define the characteristic curves as

$$\Gamma : \begin{cases} x = x(s) \\ y = y(s) \quad\quad s \in I \\ u = u(s) \end{cases}$$

As we did earlier, the integral curves of the characteristic system are given by

$$\frac{dx}{ds} = a(x, y, u), \quad \frac{dy}{ds} = b(x, y, u), \quad \frac{du}{ds} = c(x, y, u) \tag{7}$$

First-Order Quasilinear PDEs

The characteristic system can be rewritten as

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}$$

This is an autonomous system of ODEs. If $a, b, c \in C^1(\Omega)$, then by the existence and uniqueness theorem of ODEs, through each point $P_0(x_0, y_0, u_0) \in \Omega$ passes exactly one characteristic curve Γ_0. The solutions of the characteristic system (7) are called the characteristic curves of the quasilinear equation (6).

Definition 3 (Monge's Equation)
The above characteristic equations are also known as Monge's Equations

Monge's Curves

Definition 4 (Monge's Curves)
The solution of the above Monge's[1] equations form a 2-parameter family of curves in the (x, y, u)-space are called Monge curves. The projection of Monge's curves on $xy-$ plane are two parameter family of characteristic curves.

Definition 5 (Monge's direction)
The characteristic direction (a, b, c) is also called Monge's direction.

[1] Gaspard Monge is known as the father of differential geometry.

First-Order Quasilinear PDEs

Theorem 1

Let the characteristic curve

$$\Gamma_0 : \begin{cases} x = x_0(s) \\ y = y_0(s) \quad & s \in I \\ u = u_0(s) \end{cases}$$

intersect the integral surface S at the point $P_0(x_0, y_0, u_0) \in \Omega$. Then $\Gamma_0 \subset S$ which means

$$u_0(s) = u(x_0(s), y_0(s)), s \in I$$

First-Order Quasilinear PDEs

Proof:

Let $U(s) = u_0(s) - u(x_0(s), y_0(s))$. As $P_0(x_0, y_0, u_0) \in S \cap \Gamma_0, \exists s_0 \in I \ni$

$$x_0 = x_0(s_0), y_0 = y_0(s_0), u_0 = u_0(s_0), U(s_0) = 0$$

Now

$$\begin{aligned} \frac{dU}{ds} &= \frac{du_0}{ds} - u_x \frac{dx_0}{ds} - u_y \frac{dy_0}{ds} \\ &= c(x_0(s), y_0(s), u_0(s)) - a(x_0(s), y_0(s), u_0(s))u_x - b(x_0(s), y_0(s), u_0(s))u_y \\ &= c(x_0(s), y_0(s), U(s) + u(s)) - a(x_0(s), y_0(s), U(s) + u(s))u_x \\ &\quad -b(x_0(s), y_0(s), U(s) + u(s))u_y \\ &= f(s, U) \end{aligned}$$

First-Order Quasilinear PDEs

Proof (contd):
The above equation is an ODE with initial condition $U(s_0) = 0$. Since a, b, c are continuously differentiable, and the function u defining the surface $u = u(x, y)$ is assumed to be continuously differentiable, the function $f(s, U)$ is locally Lipschitz w.r.to U. Since $U(s) \equiv 0$ is a solution of the ODE. Therefore, by the uniqueness theorem for the Cauchy problem or IVP for ODE, it follows that

$$U \equiv 0 \implies u_0(s) - u(x_0(s), y_0(s)) = 0, s \in I$$

First-Order Quasilinear PDEs

Theorem 2

Let $D \subset \Omega_0$ and $S : u = u(x, y)$ be a surface in $\mathbb{R}^3$ where $u : D \to \mathbb{R}$ and $u \in C^1(D)$. Then the following statements are equivalent

1. The surface S is an integral surface of the equation (6)
2. The surface S is the union of characteristic curves of the equation (6)

Corollary 1

Let S_1 and S_2 be two integral surfaces that $P \in S_1 \cap S_2$. Then some part of the characteristics passing through P lies on both S_1 and S_2

Corollary 2

If two integral surfaces intersect without touching and the intersection is a curve Γ, then Γ is a characteristic curve.

First-Order Quasilinear PDEs

Exercise 1: Theorems and Corollary

1. Prove theorem (2). $(1) \implies (2)$ follows from Theorem (1).
2. Prove corollary (1)
3. Prove corollary (2)

First-Order linear PDEs

with Constant Coefficients

First-Order linear PDEs with Const. Coeff.

Consider the following PDE

$$au_x + bu_y + cu = 0 \tag{8}$$

Use the same change of coordinates

$$\xi(x, y) = ax + by, \eta(x, y) = bx - ay$$

then the corresponding canonical form is given by

$$(a^2 + b^2)u_\xi + cu = 0$$

This can be solved using standard ODE methods.

First-Order linear PDEs with Const. Coeff.

$$(a^2 + b^2)u_\xi + cu = 0 \implies u_\xi + \frac{c}{a^2 + b^2}u = 0$$

The solution is given by

$$u(\xi, \eta) = e^{-\frac{c\xi}{a^2+b^2}} f(\eta)$$

$$u(x, y) = e^{-\frac{c(ax+by)}{a^2+b^2}} f(bx - ay)$$

First-Order linear PDEs with Const. Coeff.

Consider the following PDE

$$au_x + bu_y + cu = f(x, y) \tag{9}$$

Use the same change of coordinates

$$\xi(x, y) = ax + by, \eta(x, y) = bx - ay$$

then the corresponding canonical form is given by

$$(a^2 + b^2)u_\xi + cu = f(\xi, \eta)$$

This can be solved using standard ODE methods.

First-Order linear PDEs with Const. Coeff.

$$(a^2+b^2)u_\xi + cu = f(\xi,\eta) \implies u_\xi + \frac{c}{a^2+b^2}u = \frac{f(\xi,\eta)}{a^2+b^2}$$

The solution is given by

$$u(\xi,\eta) = e^{-\frac{c\xi}{a^2+b^2}}\left(g(\eta) + \int \frac{f(\xi,\eta)}{a^2+b^2}e^{-\frac{c\xi}{a^2+b^2}}\right)$$

To find the solution in terms of (x, y), first do the integration in ξ in the above formula and substitute the values of ξ and η.

First-Order linear PDEs with Const. Coeff.

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Example 3

Solve:

$$-2u_x - 4u_y + 5u = e^{x+3y}$$

Solution: $a = -2, b = 4, c = 5, f = e^{x+3y}, a^2 + b^2 = 20$

$$u(x, y) = e^{-\frac{2x-4y}{4}} \left(g(4x + 2y) + \frac{e^{\frac{2x+16y}{4}}}{15} \right)$$

First-Order linear PDEs

with Variable Coefficients

First-Order linear PDEs with Var. Coeff.

Consider the following PDE

$$a(x,y)u_x + b(x,y)u_y = 0 \tag{10}$$

Consider a curve with geometric representation $x = x(s), y = y(s)$ given by the ODE

$$\frac{dx}{ds} = a(x,y), \frac{dy}{ds} = b(x,y)$$

then the tangent direction of the curve is given by

$$\left(\frac{dx}{ds}, \frac{dy}{ds}\right) = (a(x,y), b(x,y))$$

This can be solved using standard ODE methods.

First-Order linear PDEs with Var. Coeff.

Consider the following probkem

$$u(x(s), y(s)) \implies \frac{du}{ds} = u_x \frac{dx}{ds} + u_y \frac{dy}{ds} = a(x, y)u_x + b(x, y)u_y$$

$$\frac{du}{ds} = \nabla . \left(\frac{dx}{ds}, \frac{dy}{ds} \right) = \nabla . (a(x, y), b(x, y))$$

$$D_v u = 0, v = (a(x, y), b(x, y))$$

If u satisfies (10), then

$$\frac{du}{ds} = 0$$

That is $u(x, y)$ is constant in the direction (a, b) at (x, y)

First-Order linear PDEs with Var. Coeff.

If $a(x, y) \neq 0$, then we obtain

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

This is called characteristic equation of (10). The solutions of these characteristic equations are called characteristic curves of (10). This method is called the method of characteristics.

First-Order linear PDEs with Var. Coeff.

Consider the following PDE

$$a(x,y)u_x + b(x,y)u_y = f(x,y) \tag{11}$$

Then

$$u(x(s),y(s)) \implies \frac{du}{ds} = u_x\frac{dx}{ds} + u_y\frac{dy}{ds} = a(x,y)u_x + b(x,y)u_y$$

If u satisfies (10), then

$$\frac{du}{ds} = f(x(s),y(s))$$

and

$$\frac{du}{dx} = \frac{f(x(s),y(s))}{a(x(s),y(s))}$$

This equation is called the compatibility condition.

First-Order linear PDEs with Var. Coeff.

Consider the following PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y) \tag{12}$$

Then

$$u(x(s),y(s)) \implies \frac{du}{ds} = u_x\frac{dx}{ds} + u_y\frac{dy}{ds} = a(x,y)u_x + b(x,y)u_y$$

If u satisfies (10), then

$$\frac{du}{ds} + c(x(s),y(s)) = f(x(s),y(s)) \tag{13}$$

First-Order linear PDEs with Var. Coeff.

When $c \equiv 0$, then we obtain

$$\frac{du}{ds} = f(x(s), y(s))$$

and

$$\frac{du}{dx} = \frac{f(x(s), y(s))}{a(x(s), y(s))}$$

Equation (13) can also be written as

$$\frac{du}{ds} + c(x(s), y(s)) = f(x(s), y(s)) \tag{14}$$

First-Order linear PDEs with Var. Coeff.

Since we know how to solve (14) from ODE courses, we can write the solution as

$$u(s) = e^{-\int c(s)ds}\left(C + \int f(s)e^{\int c(s)ds}\right)ds \tag{15}$$

If $u(s_0)$ is prescribed, then the value of the solution of u along the entire characteristic curve can be completely determined. If Γ is a curve passing through all initial points of the integral surface, it is called as **initial curve** on the integral surface.

Remarks

Suppose Γ is an initial curve given by

$$x(s_0) = x_0(s), y(s_0) = y_0(s), u(s_0) = u_0(s) = u(x_0(s), y_0(s)), s \in I$$

then every value of s fixes a point on Γ through which a unique characteristic curve passes.

First-Order linear PDEs with Var. Coeff.

Now, let us again consider the following problem

$$a(x,y)u_x + b(x,y)u_y = 0 \tag{16}$$

Now, we try to find the canonical form of this equation. As we did before for constant coefficients, let use change of coordinates

$$\xi = \xi(x,y), \eta = \eta(x,y)$$

Then

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

Upon substitution and simplification, we obtain

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta = 0$$

First-Order linear PDEs with Var. Coeff.

In the method of characteristics, our aim is to bring it to an ODE. So, let us say we require the coefficient of η becomes zero, so that the above equation will turn into be ODE in ξ. That is, we require

$$a\eta_x + b\eta_y = 0$$

WLOG, let us assume $a \neq 0$ (locally), then

$$\eta_x + \frac{b}{a}\eta_y = 0$$

Now, for curves that have slope $\frac{dy}{dx} = \frac{b}{a}$, we have

$$\frac{d}{dx}(\eta(x, y(x))) = \eta_x + \eta_y \frac{dy}{dx} = \eta_x + \frac{b}{a}\eta_y = 0$$

First-Order linear PDEs with Var. Coeff.

Along these characteristic curves, we have $\eta(x, y) = C$. Hence, the Jacobian becomes

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \eta_y \neq 0$$

Hence, we obtain the following canonical form

$$a(\xi, \eta)u_\xi = 0 \implies u_\xi = 0 \implies u = f(\eta)$$

First-Order linear PDEs with Var. Coeff.

Consider the following PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y) \tag{17}$$

then the corresponding canonical form is given by

$$a(\xi,\eta)u_\xi + c(\xi,\eta)u = f(\xi,\eta)$$

This can be solved using standard ODE methods.

First-Order linear PDEs with Var. Coeff.

If

$$\mu(\xi,\eta) = e^{\int \frac{c(\xi,\eta)}{a(\xi,\eta)} d\xi}$$

is the integrating factor of

$$a(\xi,\eta)u_\xi + c(\xi,\eta)u = f(\xi,\eta)$$

then the solution is given by

$$u(\xi,\eta) = \frac{1}{\mu(\xi,\eta)} \left(\int \mu(\xi,\eta) \frac{f(\xi,\eta)}{a(\xi,\eta)} d\xi + u_0(\xi,\eta) \right)$$

First-Order linear PDEs with Var. Coeff.

Example 4

Solve the following PDE

$$u_x + yu_y = 0$$

Solution: The char. Eqn is given by

$$\frac{dy}{dx} = \frac{y}{1} \implies y = Ce^x$$

$$\xi = x, \eta = ye^{-x}$$

$$\implies u(\xi, \eta) = f(\eta), u(x, y) = f(ye^{-x})$$

(Check!!)

First-Order linear PDEs with Var. Coeff.

Example 5
Solve the following PDE

$$xu_x - yu_y + y^2u = y^2$$

Solution:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\xi = x, \eta = xy, x = \xi, y = \frac{\eta}{\xi}$$

$$u(x, y) = f(xy)e^{y^2/2} + 1$$

(Check!!)

Thanks

Doubts and Suggestions

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