MA612L-Partial Differential Equations

Lecture 7 : Method of Characteristics - II

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MA612L PARTIAL DIFFERENTIAL EQUATIONS

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First-Order Quasilinear PDEs

Recap

Consider the following quasilinear PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{1}$$

where $a, b, c \in C^1(\Omega)$. Let Ω_0 denote the projection of Ω in the $xy-$plane. The integral curves of the characteristic system are given by

$$\frac{dx}{ds} = a(x,y,u), \quad \frac{dy}{ds} = b(x,y,u), \quad \frac{du}{ds} = c(x,y,u) \tag{2}$$

The characteristic system can also be rewritten as

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}$$

This is an autonomous system of ODEs. If $a, b, c \in C^1(\Omega)$, then by the existence and uniqueness theorem of ODEs, through each point $P_0(x_0, y_0, u_0) \in \Omega$ passes exactly one characteristic curve Γ_0. The solutions of the characteristic system are called the characteristic curves of the quasilinear PDE (1).

First-Order Quasilinear PDEs

Theorem 1

Let the characteristic curve

$$\Gamma_0 : \begin{cases} x = x_0(s) \\ y = y_0(s) \quad\quad s \in I \\ u = u_0(s) \end{cases}$$

intersect the integral surface S at the point $P_0(x_0, y_0, u_0) \in \Omega$. Then $\Gamma_0 \subset S$ which means

$$u_0(s) = u(x_0(s), y_0(s)), s \in I$$

First-Order Quasilinear PDEs

Proof:

Let $U(s) = u_0(s) - u(x_0(s), y_0(s))$. As $P_0(x_0, y_0, u_0) \in S \cap \Gamma_0, \exists s_0 \in I \ni$

$$x_0 = x_0(s_0), y_0 = y_0(s_0), u_0 = u_0(s_0), U(s_0) = 0$$

Now

$$\begin{aligned}
\frac{dU}{ds} &= \frac{du_0}{ds} - u_x \frac{dx_0}{ds} - u_y \frac{dy_0}{ds} \\
&= c(x_0(s), y_0(s), u_0(s)) - a(x_0(s), y_0(s), u_0(s))u_x - b(x_0(s), y_0(s), u_0(s))u_y \\
&= c(x_0(s), y_0(s), U(s) + u(s)) - a(x_0(s), y_0(s), U(s) + u(s))u_x \\
&\quad -b(x_0(s), y_0(s), U(s) + u(s))u_y \\
&= f(s, U)
\end{aligned}$$

First-Order Quasilinear PDEs

Proof (contd):

The above equation is an ODE with initial condition $U(s_0) = 0$. Since a, b, c are continuously differentiable, and the function u defining the surface $u = u(x, y)$ is assumed to be continuously differentiable, the function $f(s, U)$ is locally Lipschitz w.r.to U. Since $U(s) \equiv 0$ is a solution of the ODE. Therefore, by the uniqueness theorem for the Cauchy problem or IVP for ODE, it follows that

$$U \equiv 0 \implies u_0(s) - u(x_0(s), y_0(s)) = 0, s \in I$$

First-Order Quasilinear PDEs

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Theorem 2

Let $D \subset \Omega_0$ and $S : u = u(x, y)$ be a surface in $\mathbb{R}^3$ where $u : D \to \mathbb{R}$ and $u \in C^1(D)$. Then the following statements are equivalent

1. The surface S is an integral surface of the equation (1)
2. The surface S is the union of characteristic curves of the equation (1)

Corollary 1

Let S_1 and S_2 be two integral surfaces that $P \in S_1 \cap S_2$. Then some part of the characteristics passing through P lies on both S_1 and S_2

Corollary 2

If two integral surfaces intersect without touching and the intersection is a curve Γ, then Γ is a characteristic curve.

First-Order Quasilinear PDEs

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Exercise 1: Theorems and Corollary

1. Prove theorem (2). $(1) \implies (2)$ follows from Theorem (1).
2. Prove corollary (1)
3. Prove corollary (2)

First-Order linear PDEs

with Constant Coefficients

First-Order linear PDEs with Const. Coeff.

Consider the following PDE

$$au_x + bu_y + cu = 0 \tag{3}$$

Use the same change of coordinates

$$\xi(x,y) = ax + by, \eta(x,y) = bx - ay$$

then the corresponding canonical form is given by

$$(a^2 + b^2)u_\xi + cu = 0$$

This can be solved using standard ODE methods.

First-Order linear PDEs with Const. Coeff.

$$(a^2+b^2)u_\xi + cu = 0 \implies u_\xi + \frac{c}{a^2+b^2}u = 0$$

The solution is given by

$$u(\xi,\eta) = e^{-\frac{c\xi}{a^2+b^2}} f(\eta)$$

$$u(x,y) = e^{-\frac{c(ax+by)}{a^2+b^2}} f(bx-ay)$$

First-Order linear PDEs with Const. Coeff.

Consider the following PDE

$$au_x + bu_y + cu = f(x, y) \tag{4}$$

Use the same change of coordinates

$$\xi(x, y) = ax + by, \eta(x, y) = bx - ay$$

then the corresponding canonical form is given by

$$(a^2 + b^2)u_\xi + cu = f(\xi, \eta)$$

This can be solved using standard ODE methods.

First-Order linear PDEs with Const. Coeff.

$$(a^2+b^2)u_\xi + cu = f(\xi,\eta) \implies u_\xi + \frac{c}{a^2+b^2}u = \frac{f(\xi,\eta)}{a^2+b^2}$$

The solution is given by

$$u(\xi,\eta) = e^{-\frac{c\xi}{a^2+b^2}}\left(g(\eta) + \int \frac{f(\xi,\eta)}{a^2+b^2}e^{-\frac{c\xi}{a^2+b^2}}\right)$$

To find the solution in terms of (x,y), first do the integration in ξ in the above formula and substitute the values of ξ and η.

First-Order linear PDEs with Const. Coeff.

Example 3
Solve:

$$-2u_x - 4u_y + 5u = e^{x+3y}$$

Solution: $a = -2, b = 4, c = 5, f = e^{x+3y}, a^2 + b^2 = 20$

$$u(x,y) = e^{-\frac{2x-4y}{4}} \left(g(4x+2y) + \frac{e^{\frac{2x+16y}{4}}}{15} \right)$$

First-Order linear PDEs

with Variable Coefficients

First-Order linear PDEs with Var. Coeff.

Consider the following PDE

$$a(x,y)u_x + b(x,y)u_y = 0 \tag{5}$$

Consider a curve with geometric representation $x = x(s), y = y(s)$ given by the ODE

$$\frac{dx}{ds} = a(x,y), \frac{dy}{ds} = b(x,y)$$

then the tangent direction of the curve is given by

$$\left(\frac{dx}{ds}, \frac{dy}{ds}\right) = (a(x,y), b(x,y))$$

This can be solved using standard ODE methods.

First-Order linear PDEs with Var. Coeff.

Consider the following problem

$$u(x(s), y(s)) \implies \frac{du}{ds} = u_x \frac{dx}{ds} + u_y \frac{dy}{ds} = a(x,y)u_x + b(x,y)u_y$$

$$\frac{du}{ds} = \nabla . \left(\frac{dx}{ds}, \frac{dy}{ds} \right) = \nabla .(a(x,y), b(x,y))$$

$$D_v u = 0, v = (a(x,y), b(x,y))$$

If u satisfies (5), then

$$\frac{du}{ds} = 0$$

That is $u(x,y)$ is constant in the direction (a,b) at (x,y)

First-Order linear PDEs with Var. Coeff.

If $a(x, y) \neq 0$, then we obtain

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

This is called the characteristic equation of (5). The solutions of these characteristic equations are called characteristic curves of (5). This method is called the method of characteristics.

First-Order linear PDEs with Var. Coeff.

Consider the following PDE

$$a(x,y)u_x + b(x,y)u_y = f(x,y) \tag{6}$$

Then

$$u(x(s),y(s)) \implies \frac{du}{ds} = u_x\frac{dx}{ds} + u_y\frac{dy}{ds} = a(x,y)u_x + b(x,y)u_y$$

If u satisfies (6), then

$$\frac{du}{ds} = f(x(s),y(s))$$

and

$$\frac{du}{dx} = \frac{f(x(s),y(s))}{a(x(s),y(s))}$$

This equation is called the compatibility condition.

First-Order linear PDEs with Var. Coeff.

Consider the following PDE

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y) \tag{7}$$

Then

$$u(x(s),y(s)) \implies \frac{du}{ds} = u_x\frac{dx}{ds} + u_y\frac{dy}{ds} = a(x,y)u_x + b(x,y)u_y$$

If u satisfies (7), then

$$\frac{du}{ds} + c(x(s),y(s)) = f(x(s),y(s)) \tag{8}$$

First-Order linear PDEs with Var. Coeff.

When $c \equiv 0$, then we obtain

$$\frac{du}{ds} = f(x(s), y(s))$$

and

$$\frac{du}{dx} = \frac{f(x(s), y(s))}{a(x(s), y(s))}$$

Equation (8) can also be written as

$$\frac{du}{ds} + c(s) = f(s) \tag{9}$$

First-Order linear PDEs with Var. Coeff.

Since we know how to solve (9) from ODE courses, we can write the solution as

$$u(s) = e^{-\int c(s)ds}\left(C + \int f(s)e^{\int c(s)ds}\right)ds \tag{10}$$

If $u(s_0)$ is prescribed, then the value of the solution of u along the entire characteristic curve can be completely determined. If Γ is a curve passing through all initial points of the integral surface, it is called as **initial curve** on the integral surface.

Remarks

Suppose Γ is an initial curve given by

$$x(s_0) = x_0(s), y(s_0) = y_0(s), u(s_0) = u_0(s) = u(x_0(s), y_0(s)), s \in I$$

then every value of s fixes a point on Γ through which a unique characteristic curve passes.

First-Order linear PDEs with Var. Coeff.

Now, let us again consider the following problem

$$a(x,y)u_x + b(x,y)u_y = 0 \tag{11}$$

Now, we try to find the canonical form of this equation. As we did before for constant coefficients, let use change of coordinates

$$\xi = \xi(x,y), \eta = \eta(x,y)$$

Then

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

Upon substitution and simplification, we obtain

$$(a\xi_x + b\xi_y)u_\xi + (a\eta_x + b\eta_y)u_\eta = 0$$

First-Order linear PDEs with Var. Coeff.

In the method of characteristics, our aim is to bring it to an ODE. So, let us say we require the coefficient of η become zero, so that the above equation will turn into be ODE in ξ. That is, we require

$$a\eta_x + b\eta_y = 0$$

WLOG, let us assume $a \neq 0$ (locally), then

$$\eta_x + \frac{b}{a}\eta_y = 0$$

Now, for curves that have slope $\frac{dy}{dx} = \frac{b}{a}$, we have

$$\frac{d}{dx}(\eta(x, y(x))) = \eta_x + \eta_y \frac{dy}{dx} = \eta_x + \frac{b}{a}\eta_y = 0$$

First-Order linear PDEs with Var. Coeff.

Along these characteristic curves, we have $\eta(x, y) = C$. Hence, the Jacobian becomes

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \eta_y \neq 0$$

Hence, we obtain the following canonical form

$$a(\xi, \eta) u_\xi = 0 \implies u_\xi = 0 \implies u = f(\eta)$$

First-Order linear PDEs with Var. Coeff.

Consider the following PDE

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y) \tag{12}$$

then the corresponding canonical form is given by

$$a(\xi, \eta)u_\xi + c(\xi, \eta)u = f(\xi, \eta)$$

This can be solved using standard ODE methods.

First-Order linear PDEs with Var. Coeff.

If

$$\mu(\xi,\eta) = e^{\int \frac{c(\xi,\eta)}{a(\xi,\eta)} d\xi}$$

is the integrating factor of

$$a(\xi,\eta)u_\xi + c(\xi,\eta)u = f(\xi,\eta)$$

then the solution is given by

$$u(\xi,\eta) = \frac{1}{\mu(\xi,\eta)}\left(\int \mu(\xi,\eta)\frac{f(\xi,\eta)}{a(\xi,\eta)}d\xi + u_0(\xi,\eta)\right)$$

First-Order linear PDEs with Var. Coeff.

Example 4

Solve the following PDE

$$u_x + yu_y = 0$$

Solution: The char. Eqn is given by

$$\frac{dy}{dx} = \frac{y}{1} \implies y = Ce^x$$

$$\xi = x, \eta = ye^{-x}$$

$$\implies u(\xi, \eta) = f(\eta), u(x, y) = f(ye^{-x})$$

(Check!!)

First-Order linear PDEs with Var. Coeff.

Example 5

Solve the following PDE

$$xu_x - yu_y + y^2u = y^2$$

Solution:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\xi = x, \eta = xy, x = \xi, y = \frac{\eta}{\xi}$$

$$u(x, y) = f(xy)e^{y^2/2} + 1$$

(Check!!)

First-Order Quasilinear PDEs

Existence and Uniqueness

Inverse Function Theorem

Theorem 6 (Inverse Function Theorem (Rudin))
Suppose $\mathbf{f}$ is a C^1 mapping of an open set $E \subset \mathbb{R}^n$, $\mathbf{f}'(\mathbf{a})$ is invertible for some $\mathbf{a} \in E$ and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then

1. there exist open sets U and V in $\mathbb{R}^n$ such that $\mathbf{a} \in U, \mathbf{b} \in V, \mathbf{f}$ is one-to-one on U and $\mathbf{f}(U) = V$
2. if $\mathbf{g}$ is the inverse of $\mathbf{f}$, defined in V by $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}, \mathbf{x} \in U$, then $\mathbf{g} \in C^1(V)$.

Let us rewrite this theorem for $\mathbb{R}^2$. This will be used for the existence and uniqueness theorem for quasilinear PDE.

Inverse Mapping Theorem

Theorem 7 (Inverse Mapping Theorem)

Let $P_0(s_0, t_0) \in D \subset \mathbb{R}^2_{s,t}$, $Q_0(x_0, y_0) \in D' \subset \mathbb{R}^2_{x,y}$, $\mathbf{\Phi} : D \to D'$, $\mathbf{\Phi} \in C^1(D)$, $\mathbf{\Phi}(P_0) = Q_0$,

$$\mathbf{\Phi} : \begin{cases} x = x(s,t) \\ y = y(s,t) \end{cases}$$

and

$$J\mathbf{\Phi}(P_0) = \frac{\partial(x,y)}{\partial(s,t)}(P_0) = x_s(P_0)y_t(P_0) - x_t(P_0)y_s(P_0) \neq 0$$

Then there exist neighbourhoods U of $P_0 \in D$ and U' of $Q_0 \in D'$ and a mapping $\mathbf{\Phi}^{-1} \in C^1(U')$ such that $\mathbf{\Phi}^{-1}(U') = U$ and

$$J\mathbf{\Phi}^{-1}(Q_0) = (J\mathbf{\Phi}(P_0))^{-1}$$

.

Existence and Uniqueness Theorem

Theorem 8 (Existence and Uniqueness Theorem)
Consider the first-order quasilinear PDE (1) in the domain $\Omega \subset \mathbb{R}^3$ where $a, b, c \in C^1(\Omega)$

$$\Gamma : \begin{cases} x = x_0(s) \\ y = y_0(s) \\ u = u_0(s) \end{cases} \quad s \in [0,1]$$

is an initial smooth curve in Ω and

$$\frac{dx_0}{ds} b(x_0(s), y_0(s), u_0(s)) - \frac{dy_0}{ds} a(x_0(s), y_0(s), u_0(s)) \neq 0, s \in [0,1]$$

Then there exists at most one solution $u = u(x,y)$ defined in a neighbourhood of the initial curve which satisfies the equation (1) and the initial condition $u_0(s) = u(x_0(s), y_0(s)), s \in [0,1]$.

Existence and Uniqueness Theorem

Proof:Existence

Let us consider the Cauchy problem for the ODE system

$$C : \begin{cases} \frac{dx}{dt} = a(x, y, u) \\ \frac{dy}{dt} = b(x, y, u) \\ \frac{du}{dt} = c(x, y, u) \end{cases}$$

with initial conditions $x(s, 0) = x_0(s), y(s, 0) = y_0(s), u(s, 0) = u_0(s)$. From the existence and uniqueness theorem for ODEs, the problem has a unique solution

$$x = x(s, t), y = y(s, t), u = u(s, t)$$

$t \in [\alpha(s), \beta(s)]$ where α and β are continuous functions.

Existence and Uniqueness Theorem

Proof (contd..): Existence

Define

$$D = \{(s,t) : s \in [0,1], t \in [\alpha(s), \beta(s)]\} \subset \Omega'$$

D is the projection of Ω in $xy-$ plane. Also, define Φ as in the inverse mapping theorem, then

$$J\Phi|_{t=0} = \frac{dx_0}{ds}b - \frac{dy_0}{ds}a \neq 0$$

By Inverse Mapping Theorem, there exists a unique mapping $\Phi^{-1} : D' \to D$

$$\mathbf{\Phi}^{-1} : \begin{cases} s = s(x,y) \\ t = t(x,y) \end{cases}$$

defined in a neighbourhood N' of Γ' where Γ' is a projection of Γ in the $xy-$ plane.

Existence and Uniqueness Theorem

Proof (contd..):Existence

Consider

$$u = u(s(x,y), t(x,y)) = \phi(x,y)$$

Then

$$\begin{aligned} a\phi_x + b\phi_y &= a(u_s s_x + u_t t_x) + b(u_s s_y + u_t t_y) \\ &= u_s(a s_x + b s_y) + u_t(a t_x + b t_y) \\ &= u_s(x_t s_x + y_t s_y) + u_t(x_t t_x + y_t t_y) \\ &= u_s.0 + u_t.1 \\ &= c \end{aligned}$$

Also,

$$\phi(x_0(s), y_0(s)) = u(s(x_0(s), y_0(s)), t(x_0(s), y_0(s))) = u(s,0) = u_0(s)$$

Existence and Uniqueness Theorem

Proof (contd..): Uniqueness

Suppose ϕ_1 and ϕ_2 are two distinct solutions satisfying the initial conditions. Let $S_1 = \phi_1(x, y), S_2 = \phi_2(x, y)$ be the corresponding integral surfaces. Consider the system of ODEs ($i = 1, 2$)

$$\begin{cases} \frac{dx}{dt} = a(x, y, \phi_i(x, y)) & x(s, 0) = x_0(s) \\ \frac{dy}{dt} = b(x, y, \phi_i(x, y)) & y(s, 0) = y_0(s) \end{cases}$$

Then we find solutions $(x_1(s,t), y_1(s,t))$ and $(x_2(s,t), y_2(s,t))$. Therefore, $(x_1(s,t), y_1(s,t), \phi_1(s,t))$ and $(x_2(s,t), y_2(s,t), \phi_2(s,t))$ are solutions of the system (C). Therefore, by the uniqueness theorem for ODEs, $(x_1(s,t), y_1(s,t), \phi_1(s,t))$ and $(x_2(s,t), y_2(s,t), \phi_2(s,t))$ coincide in the common domain of definition. It follows that the characteristics Γ_1 and Γ_2 starting from the point $P(x_0(s), y_0(s), u_0(s))$ also coincide.

Existence and Uniqueness Theorem

Remarks

- The theorem states that whenever the data curve is not tangential to a characteristic curve and the functions $a, b, c \in C^1(\Omega)$, the solution exists and is unique.
- The condition

$$T(s) \equiv \frac{dx_0}{ds} b(x_0(s), y_0(s), u_0(s)) - \frac{dy_0}{ds} a(x_0(s), y_0(s), u_0(s)) \neq 0, s \in [0, 1]$$

  is called as transversality condition.
- The geometrical interpretation: the projection of the characteristics curves to the $xy-$ plane passing through the point $x_0(s), y_0(s), u_0(s)$ intersects the projection of the initial curve Γ non-tangentially.
- What will happen if this condition fails? **Neither existence nor uniqueness is guaranteed**

Examples

Example 9

Show that

$$yu_x - xu_y = 0$$

with

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = 0 \\ u = u_0(s) = s^2 \end{cases}$$

has a circular paraboloid as the unique solution.

Solution: The characteristic equations are

$$C : \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \\ \frac{du}{dt} = 0 \end{cases}$$

Examples

$$\frac{dy}{dx} = -\frac{x}{y} \implies ydy + xdx = 0 \implies x^2 + y^2 = C$$

The characteristic curves are circles with centre (0,0), and the general solution is

$$u(x, y) = f(x^2 + y^2)$$

It is given that $x_0(s) = s, y_0(s) = 0$ and $u_0(s) = s^2$. This is a parabola $u = x^2, y = 0$ in the $xu-$ plane. Also, from transversality condition $T(s) = -s \neq 0$. Therefore, by the existence and uniqueness theorem, we have a unique solution. Using the initial condition, we obtain that

$$s^2 = f(s^2) \implies f(x) = x \implies u = x^2 + y^2$$

$u = x^2 + y^2$ is a circular paraboloid.

Examples

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Example 10

Show that

$$yu_x - xu_y = 0$$

with

$$\Gamma : \begin{cases} x = x_0(s) = \cos s \\ y = y_0(s) = \sin s \\ u = u_0(s) = \sin s \end{cases}$$

has no solution.

Solution: As in the previous example $u = f(x^2 + y^2)$. However, $T(s) = 0$. Also, the initial curve is the ellipse $x^2 + y^2 = 1, u = y$. If $u = f(x^2 + y^2)$ is a solution, then on the circle $x^2 + y^2 = 1$, we obtain $u = f(1)$ a constant, which contradicts with $u = y$. Therefore, no solution exists.

Examples

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Example 11

Show that

$$yu_x - xu_y = 0$$

with

$$\Gamma : \begin{cases} x = x_0(s) = \cos s \\ y = y_0(s) = \sin s \\ u = u_0(s) = 1 \end{cases}$$

has infinitely many solutions.

Solution: As in the previous example $u = f(x^2 + y^2)$. However, $T(s) = 0$ Also, the initial curve is the circle $x^2 + y^2 = 1, u = 1$. If $u = f(x^2 + y^2)$ is a solution, then on the circle $x^2 + y^2 = 1$, we obtain $u = f(1) = 1$ which is possible for any function such that $f(\omega) = \omega^n$, here ω is the n^{th} root of unity. Therefore, there are infinitely many solutions in this case.

Exercise

Exercise 2: Existence and Uniqueness

Consider the following PDE

$$u_x = cu$$

Determine for which of the following Cauchy (Initial value) data, the PDE has a unique or no or infinitely many solution(s).

1. $u(x,0) = e^{cx}$
2. $u(0,y) = 0$
3. $u(x,0) = \sin x$

Thanks

Doubts and Suggestions

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