MA612L-Partial Differential Equations

Lecture 8 : Existence and Uniqueness Theorem

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

August 22, 2025

भारतीय प्रौद्योगिकी संस्थान तिरुपति
TIRUPATI

Existence and Uniqueness Theorem

Recap

Consider the following quasilinear PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{1}$$

where $a, b, c \in C^1(\Omega)$. Let Ω_0 denote the projection of Ω in the $xy-$plane. The integral curves of the characteristic system are given by

$$\frac{dx}{ds} = a(x,y,u), \quad \frac{dy}{ds} = b(x,y,u), \quad \frac{du}{ds} = c(x,y,u) \tag{2}$$

First-Order Quasilinear PDEs

Existence and Uniqueness

Inverse Function Theorem

Theorem 1 (Inverse Function Theorem (Rudin))
Suppose $\mathbf{f}$ is a C^1 mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $\mathbf{f}'(\mathbf{a})$ is invertible for some $\mathbf{a} \in E$ and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then

1. there exist open sets U and V in $\mathbb{R}^n$ such that $\mathbf{a} \in U, \mathbf{b} \in V, \mathbf{f}$ is one-to-one on U and $\mathbf{f}(U) = V$
2. if $\mathbf{g}$ is the inverse of $\mathbf{f}$, defined in V by $\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}, \mathbf{x} \in U$, then $\mathbf{g} \in C^1(V)$.

Let us rewrite this theorem for $\mathbb{R}^2$. This will be used for the existence and uniqueness theorem for quasilinear PDE.

Inverse Function Theorem

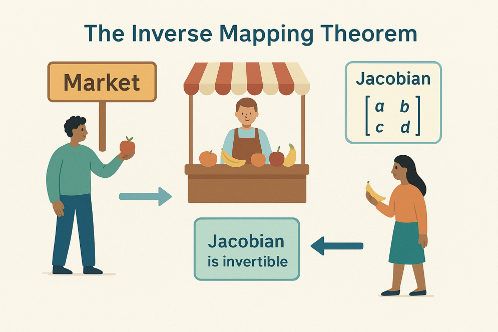

Figure 1: Inverse Function Theorem

Inverse Mapping Theorem

Theorem 2 (Inverse Mapping Theorem)

Let $P_0(s_0,t_0) \in D \subset \mathbb{R}^2_{s,t}$, $Q_0(x_0,y_0) \in D' \subset \mathbb{R}^2_{x,y}$, $\mathbf{\Phi} : D \to D'$, $\mathbf{\Phi} \in C^1(D)$, $\mathbf{\Phi}(P_0) = Q_0$,

$$\mathbf{\Phi} : \begin{cases} x = x(s,t) \\ y = y(s,t) \end{cases}$$

and

$$J\mathbf{\Phi}(P_0) = \frac{\partial(x,y)}{\partial(s,t)}(P_0) = x_s(P_0)y_t(P_0) - x_t(P_0)y_s(P_0) \neq 0$$

Then there exist neighbourhoods U of $P_0 \in D$ and U' of $Q_0 \in D'$ and a mapping $\mathbf{\Phi}^{-1} \in C^1(U')$ such that $\mathbf{\Phi}^{-1}(U') = U$ and

$$J\mathbf{\Phi}^{-1}(Q_0) = (J\mathbf{\Phi}(P_0))^{-1}$$

.

Existence and Uniqueness Theorem

Theorem 3 (Existence and Uniqueness Theorem)
Consider the first-order quasilinear PDE (1) in the domain $\Omega \subset \mathbb{R}^3$ where $a, b, c \in C^1(\Omega)$

$$\Gamma : \begin{cases} x = x_0(s) \\ y = y_0(s) \qquad s \in [0,1] \\ u = u_0(s) \end{cases}$$

is an initial smooth curve in Ω and

$$\frac{dx_0}{ds} b(x_0(s), y_0(s), u_0(s)) - \frac{dy_0}{ds} a(x_0(s), y_0(s), u_0(s)) \neq 0, s \in [0,1]$$

Then there exists at most one solution $u = u(x,y)$ defined in a neighbourhood of the initial curve which satisfies the equation (1) and the initial condition $u_0(s) = u(x_0(s), y_0(s)), s \in [0,1]$.

Existence and Uniqueness Theorem

Proof:Existence

Let us consider the Cauchy problem for the ODE system

$$C : \begin{cases} \frac{dx}{dt} = a(x, y, u) \\ \frac{dy}{dt} = b(x, y, u) \\ \frac{du}{dt} = c(x, y, u) \end{cases}$$

with initial conditions $x(s, 0) = x_0(s), y(s, 0) = y_0(s), u(s, 0) = u_0(s)$. From the existence and uniqueness theorem for ODEs, the problem has a unique solution

$$x = x(s, t), y = y(s, t), u = u(s, t)$$

$t \in [\alpha(s), \beta(s)]$ where α and β are continuous functions.

Existence and Uniqueness Theorem

Proof (contd..): Existence

Define

$$D = \{(s,t) : s \in [0,1], t \in [\alpha(s), \beta(s)]\} \subset \Omega'$$

D is the projection of Ω in $xy-$ plane. Also, define $\mathbf{\Phi}$ as in the inverse mapping theorem, then

$$J\mathbf{\Phi}|_{t=0} = \frac{dx_0}{ds}b - \frac{dy_0}{ds}a \neq 0$$

By Inverse Mapping Theorem, there exists a unique mapping $\mathbf{\Phi}^{-1} : D' \to D$

$$\mathbf{\Phi}^{-1} : \begin{cases} s = s(x,y) \\ t = t(x,y) \end{cases}$$

defined in a neighbourhood N' of Γ' where Γ' is a projection of Γ in the $xy-$ plane.

Existence and Uniqueness Theorem

Proof (contd..):Existence

Consider

$$u = u(s(x,y), t(x,y)) = \phi(x,y)$$

$$\begin{aligned}
\implies a\phi_x + b\phi_y &= a(u_s s_x + u_t t_x) + b(u_s s_y + u_t t_y) \\
&= u_s(as_x + bs_y) + u_t(at_x + bt_y) \\
&= u_s(x_t s_x + y_t s_y) + u_t(x_t t_x + y_t t_y) \\
&= u_s(ds/dt) + u_t(dt/dt) \\
&= u_s.0 + u_t.1 \\
&= c
\end{aligned}$$

Also, $\phi(x_0(s), y_0(s)) = u(s(x_0(s), y_0(s)), t(x_0(s), y_0(s))) = u(s, 0) = u_0(s)$

Existence and Uniqueness Theorem

Proof (contd..): Uniqueness

Suppose ϕ_1 and ϕ_2 are two distinct solutions satisfying the initial conditions. Let $S_1 = \phi_1(x, y), S_2 = \phi_2(x, y)$ be the corresponding integral surfaces. Consider the system of ODEs ($i = 1, 2$)

$$\begin{cases} \frac{dx}{dt} = a(x, y, \phi_i(x, y)) & x(s, 0) = x_0(s) \\ \frac{dy}{dt} = b(x, y, \phi_i(x, y)) & y(s, 0) = y_0(s) \end{cases}$$

Then we find solutions $(x_1(s, t), y_1(s, t))$ and $(x_2(s, t), y_2(s, t))$. Therefore, $(x_1(s, t), y_1(s, t), \phi_1(s, t))$ and $(x_2(s, t), y_2(s, t), \phi_2(s, t))$ are solutions of the system (C). Therefore, by the uniqueness theorem for ODEs, $(x_1(s, t), y_1(s, t), \phi_1(s, t))$ and $(x_2(s, t), y_2(s, t), \phi_2(s, t))$ coincide in the common domain of definition. It follows that the characteristics Γ_1 and Γ_2 starting from the point $P(x_0(s), y_0(s), u_0(s))$ also coincide.

Existence and Uniqueness Theorem

Remarks

- The theorem states that whenever the data curve is not tangential to a characteristic curve and the functions $a, b, c \in C^1(\Omega)$, the solution exists and is unique.
- The condition

$$T(s) \equiv \frac{dx_0}{ds} b(x_0(s), y_0(s), u_0(s)) - \frac{dy_0}{ds} a(x_0(s), y_0(s), u_0(s)) \neq 0, s \in [0, 1]$$

  is called as **transversality condition.**
- The geometrical interpretation: the projection of the characteristics curves to the $xy-$ plane passing through the point $x_0(s), y_0(s), u_0(s)$ intersects the projection of the initial curve Γ non-tangentially.
- What will happen if this condition fails? **Neither existence nor uniqueness is guaranteed**

Existence and Uniqueness Theorem

Remarks

Let $\vec{T}$ denote the tangent vector to the initial curve Γ

$$\vec{T} = \left(\frac{dx_0}{ds}, \frac{dy_0}{ds} \right)$$

Let $\vec{C}$ denote the projection of the characteristic direction in the xy plane

$$\vec{C} = (a, b)$$

Then the transversality condition is $T = \vec{T} \times \vec{C} \neq 0$. It means the initial curve Γ is not tangent to the characteristic direction in the xy plane

Geometric Summary of Existence and Uniqueness Theorem

Remarks

- The PDE's solution propagates along characteristic curves in 3D (x, y, u)-space.
- The initial curve Γ is used to start the propagation.
- The transversality condition ensures that we can uniquely "trace out" characteristic curves from each point on Γ
- If the initial curve is tangent to the characteristic direction, information can "pile up" or be under-determined, leading to non-uniqueness or non-existence.

Physical Meaning of Existence and Uniqueness Theorem

Remarks

- Imagine tracing flow lines (characteristics) through a ribbon Γ
- If the flow cuts across the ribbon transversely, you can propagate information uniquely.
- If the flow runs along the ribbon (tangent), then the information is not enough to determine a unique path — like trying to draw streamlines when your initial condition is along the stream.

Existence and Uniqueness Theorem

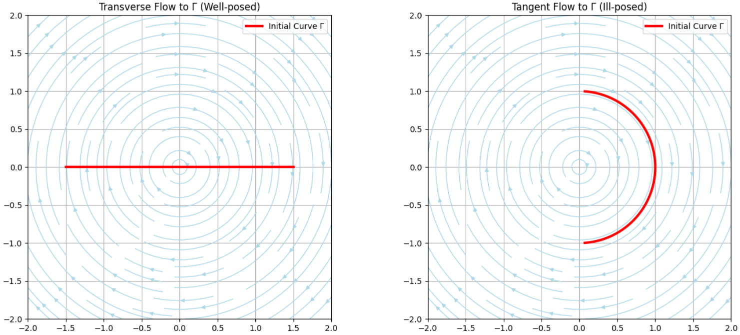

Figure 2: Transversality Condition

Existence and Uniqueness Theorem

Remarks

- Imagine the initial curve Γ as a ribbon laid out in the xy-plane.
- The characteristic directions are like the direction of wind or streamlines.
- If the wind crosses the ribbon **transversely**, each point on the ribbon gives rise to a unique trajectory — the information propagates smoothly.
- If the wind flows **along** the ribbon (tangentially), then the trajectories are ambiguous — information either piles up (non-uniqueness) or doesn't propagate (lack of existence).
- The transversality condition ensures that characteristics "launch" cleanly from the initial data curve.

Existence and Uniqueness Theorem

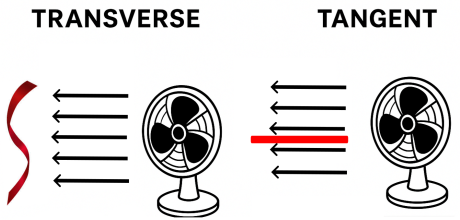

Figure 3: Existence and Uniqueness Theorem

Examples

Example 4

Show that

$$yu_x - xu_y = 0$$

with

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = 0 \\ u = u_0(s) = s^2 \end{cases}$$

$s \in \mathbb{R} \setminus \{0\}$ has a circular paraboloid as the unique solution.

Solution: The characteristic equations are

$$C : \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \\ \frac{du}{dt} = 0 \end{cases}$$

Examples

$$\frac{dy}{dx} = -\frac{x}{y} \implies ydy + xdx = 0 \implies x^2 + y^2 = C$$

The characteristic curves are circles with centre (0,0), and the general solution is

$$u(x, y) = f(x^2 + y^2)$$

It is given that $x_0(s) = s, y_0(s) = 0$ and $u_0(s) = s^2$. This is a parabola $u = x^2, y = 0$ in the $xu-$ plane. Also, from the transversality condition $T(s) = -s \neq 0$. Therefore, by the existence and uniqueness theorem, we have a unique solution. Using the initial condition, we obtain that

$$s^2 = f(s^2) \implies f(x) = x \implies u = x^2 + y^2$$

$u = x^2 + y^2$ is a circular paraboloid.

Example Simulation

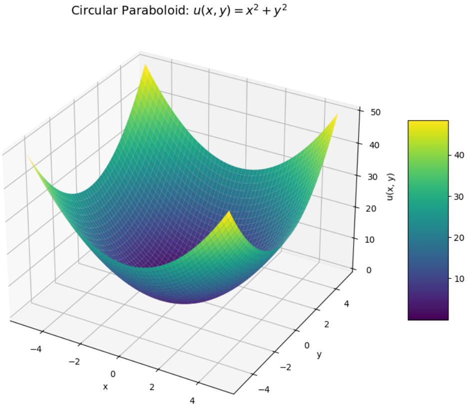

Figure 4: Circular Paraboloid

Examples

Example 5

Show that

$$yu_x - xu_y = 0$$

with

$$\Gamma : \begin{cases} x = x_0(s) = \cos s \\ y = y_0(s) = \sin s \\ u = u_0(s) = \sin s \end{cases}$$

has no solution.

Solution: As in the previous example $u = f(x^2 + y^2)$. However, $T(s) = 0$. Also, the initial curve is the ellipse $x^2 + y^2 = 1, u = y$. If $u = f(x^2 + y^2)$ is a solution, then on the circle $x^2 + y^2 = 1$, we obtain $u = f(1)$ a constant, which contradicts with $u = y$. Therefore, no solution exists.

Examples

Example 6
Show that

$$yu_x - xu_y = 0$$

with

$$\Gamma : \begin{cases} x = x_0(s) = \cos s \\ y = y_0(s) = \sin s \\ u = u_0(s) = 1 \end{cases}$$

has infinitely many solutions.

Solution: As in the previous example $u = f(x^2 + y^2)$. However, $T(s) = 0$ Also, the initial curve is the circle $x^2 + y^2 = 1, u = 1$. If $u = f(x^2 + y^2)$ is a solution, then on the circle $x^2 + y^2 = 1$, we obtain $u = f(1) = 1$ which is possible for any function such that $f(\omega) = \omega^n$, here ω is the n^{th} root of unity. Therefore, there are infinitely many solutions in this case.

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in