MA612L-Partial Differential Equations

Lecture 9 : Transport and Burger's Equations

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Transport Equation

Transport Equation: Example

Example 1

Solve the transport equation with the following Cauchy data

$$u(x,0) = \begin{cases} x & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

Cauchy data - Initial Condition

The transport equation solution is given by $u(x,t) = f(x - ct)$, therefore,

$$u(x,0) = f(x) = \begin{cases} x & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$$

Hence

$$u(x,t) = f(x-ct) = \begin{cases} x - ct & x - ct \in (0,1) \\ 0 & x - ct \notin (0,1) \end{cases}$$

Transport Equation: Example

or

$$u(x,t) = \begin{cases} x - ct & x \in (ct, ct+1) \\ 0 & x \notin (ct, ct+1) \end{cases}$$

This shows that the initial function moved to the right along the $x-$axis by ct units. The characteristics are given by $x - ct = x(0)$.

Transport Equation: Example

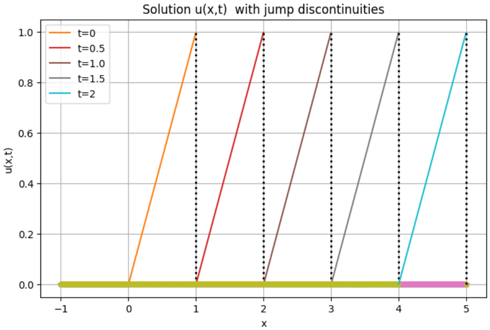


Figure 1: Transport Equation: See the Animation

Transport Equation in $\mathbb{R}^n \times (0, \infty)$

The transport equation in $n-$ dimensional space is given by

$$u_t + b.Du = 0, \mathbf{x} \in \mathbb{R}^n, t > 0$$

where $\mathbf{b} \in \mathbb{R}^n$, $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is the unknown $u = u(\mathbf{x}, t)$.
Fix any point $(\mathbf{x}, t) \in \mathbb{R}^n \times [0, \infty)$ and define

$$z(s) = u(\mathbf{x} + s\mathbf{b}, t + s)$$

Now,

$$\frac{dz}{ds} = Du(\mathbf{x} + s\mathbf{b}, t + s).\mathbf{b} + u_t(\mathbf{x} + s\mathbf{b}, t + s) = 0$$

Hence z is a constant function of s and hence for each point $(\mathbf{x}, t)$, u is constant on the line through $(\mathbf{x}, t)$ with the direction $(\mathbf{b}, 1) \in \mathbb{R}^{n+1}$.

Transport Equation in $\mathbb{R}^n \times (0, \infty)$

The transport equation in $n-$ dimensional space is given by

$$\begin{cases} u_t + b.Du = 0, \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}), x \in \mathbb{R}^n \end{cases} \tag{1}$$

The line parametrically represented by $(\mathbf{x} + s\mathbf{b}, t + s)$ through $(\mathbf{x}, t)$ with the direction $(\mathbf{b}, 1)$ hits the plane $\mathbb{R}^n \times 0$ when $s = -t$ at the point $(\mathbf{x} - t\mathbf{b}, 0)$. Since u is constant on the line and $u(\mathbf{x} - t\mathbf{b}, 0) = g(\mathbf{x} - t\mathbf{b})$, we obtain that

$$u(\mathbf{x}, t) = g(\mathbf{x} - t\mathbf{b}), \mathbf{x} \in \mathbb{R}^n, t \geq 0$$

If $g \notin C^1$, then there is no C^1 solution for (1). In this case, $u(\mathbf{x}, t) = g(\mathbf{x} - t\mathbf{b})$ is said to be a weak solution of (1).

Transport Equation in $\mathbb{R}^n \times (0, \infty)$

Consider the following inhomogeneous problem

$$\begin{cases} u_t + b.Du = f, \mathbf{x} \in \mathbb{R}^n, t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}), x \in \mathbb{R}^n \end{cases} \quad (2)$$

Using the same $z(s)$, we obtain

$$\frac{dz}{ds} = Du(\mathbf{x} + s\mathbf{b}, t + s).\mathbf{b} + u_t(\mathbf{x} + s\mathbf{b}, t + s) = f(\mathbf{x} + s\mathbf{b}, t + s)$$

$$u(\mathbf{x}, t) - g(\mathbf{x} - t\mathbf{b}) = z(0) - z(-t) = \int_{-t}^{0} \frac{dz}{ds} ds = \int_{-t}^{0} f(\mathbf{x} + s\mathbf{b}, t + s) ds$$

$$u(\mathbf{x}, t) = g(\mathbf{x} - t\mathbf{b}) + \int_0^t f(\mathbf{x} + (s - t)\mathbf{b}, s) ds, x \in \mathbb{R}^n, t \geq 0$$

solves the IVP (2).

Burger’s Equation

Traffic Flow Model

Now, let us derive a quasilinear equation

$$u_t + uu_x = 0$$

using a traffic flow model, which is also known as inviscid Burger's equation[1]. Consider the traffic flow on a single-lane traffic (no overtaking). Let $\rho(x,t)$ denote the density of cars (in vehicles per km) in $x \in \mathbb{R}$ at time $t \geq 0$. The number of cars which are in the interval (a,b) at time t is

$$\int_a^b \rho(x,t)dx$$

Let $v(x,t)$ denote the velocity of the cars in x at time t. The number of cars that pass through x at time t is $\rho(x,t)v(x,t)$.

[1] Also, known as Bateman-Burgers equation, introduced by Harry Bateman in 1915, studied by J. M. Burgers in 1948

Traffic Flow Model

The number of cars in the interval (a, b) changes as per the number of cars that enter or leave this interval.

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = \rho(a,t)v(a,t) - \rho(b,t)v(b,t)$$

If we integrate this equation w.r.to time and make necessary assumptions for ρ, v, we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\int_a^b \frac{\partial}{\partial t}\rho(x,t)dxdt &= \int_{t_1}^{t_2}(\rho(a,t)v(a,t) - \rho(b,t)v(b,t))dt \\ &= -\int_{t_1}^{t_2}\int_a^b \frac{\partial}{\partial x}\rho(x,t)v(x,t)dxdt\end{aligned}$$

Traffic Flow Model

$$\int_{t_1}^{t_2}\int_a^b \frac{\partial}{\partial t}\rho(x,t)dxdt + \int_{t_1}^{t_2}\int_a^b \frac{\partial}{\partial x}\rho(x,t)v(x,t)dxdt = 0$$

$$\int_{t_1}^{t_2}\int_a^b \rho_t + (\rho v)_x dxdt = 0$$

Since $t_1, t_2 > 0, a, b \in \mathbb{R}$ are arbitrary, we conclude that

$$\rho_t + (\rho v)_x = 0, x \in \mathbb{R}, t > 0$$

Now, v also depend on ρ. Assume that v depends only on ρ. If the highway is empty $\rho = 0$, we get $v = v_{max}$. When the traffic is heavy, $v = 0$ and $\rho = \rho_{max}$.

Traffic Flow Model

Consider the linear relation

$$v(\rho) = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right), 0 \leq \rho \leq \rho_{max}$$

Therefore, we obtain that

$$\rho_t + \left[\rho v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)\right]_x = 0, x \in \mathbb{R}, t > 0$$

If we assume

$$v_{max} = 1 \text{ and } u = 1 - \frac{2\rho}{\rho_{max}}$$

we can obtain that

$$u_t + u u_x = 0$$

Traffic Flow Model

$$u_t + uu_x = 0, x \in \mathbb{R}, t > 0$$

is called the inviscid Burgers equation. Consider the initial condition

$$u(x,0) = 1 - \frac{2\rho_0}{\rho_{max}}$$

If $\rho_0 = 0$, $u_0 = 1$ and $\rho_0 = \rho_{max}, u = -1$

Burger's Equation

Example 2

Show that the PDE

$$uu_x + u_y = \frac{1}{2}$$

with initial condition

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = s \\ u = u_0(s) = \frac{s}{4} \end{cases} \quad s \in [0, 1]$$

has a unique solution

Solution: The initial curve is given by

$$\frac{dx_0}{ds}b - \frac{dy_0}{ds} = 1 - \frac{s}{4} \neq 0, \text{ if } s \neq 4$$

Burger's Equation

The characteristic system is given by

$$C : \begin{cases} \frac{dx}{dt} = u \\ \frac{dy}{dt} = 1 \\ \frac{du}{dt} = \frac{1}{2} \end{cases}$$

with initial conditions $x(s,0) = y(s,0) = s, u(s,0) = s/4$. Upon solving, we obtain

$$\begin{cases} x(s,t) = s + st/4 + t^2/4 \\ y(s,t) = s + t \\ u(s,t) = s/4 + t/2 \end{cases}$$

Burger's Equation

Rewriting s, t in terms of x, y, we obtain

$$\begin{cases} s = \frac{4x-y^2}{4-y} \\ t = \frac{4(y-x)}{4-y} \end{cases}$$

and the unique solution to the problem is

$$u = \frac{8y - 4x - y^2}{4(4-y)}, y = s \neq 4$$

Rarefaction

Rarefaction

$$u_t + uu_x = 0, x \in \mathbb{R}, t > 0$$

Suppose

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases} \tag{3}$$

The characteristic system is given by

$$C : \begin{cases} \frac{dx}{ds} = u \\ \frac{dt}{ds} = 1 \\ \frac{du}{ds} = 0 \end{cases} \tag{4}$$

$$\frac{du}{dt} = 0 \implies u(x(t),t) = c \implies c = u(x(0),0),$$

$$\frac{dx}{dt} = u \implies \frac{dx}{dt} = u(x(0),0) \implies x(t) = u(x(0),0)t + D$$

Rarefaction

$$x(t) = u(x(0), 0)t + D \implies x(t) = u(x(0), t)t + x(0)$$

Using the initial conditions, this will become

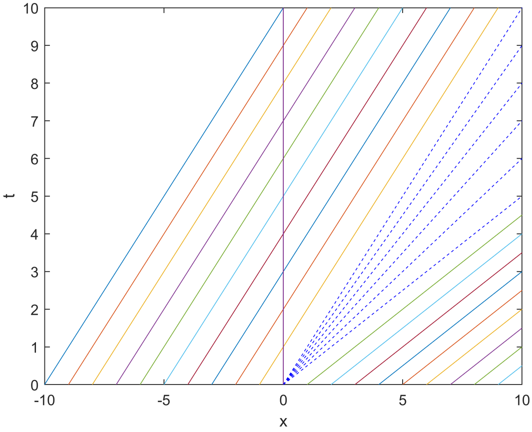
$$x(t) = \begin{cases} t + x(0) & \text{if } x(0) < 0 \\ 2t + x(0) & \text{if } x(0) > 0 \end{cases}$$

Solving for t, we have

$$t = \begin{cases} x - x(0) & \text{if } x(0) < 0 \\ \frac{1}{2}(x - x(0)) & \text{if } x(0) > 0 \end{cases} \tag{5}$$

Rarefaction

The characteristic lines corresponding to the initial condition (3). These lines are two families of characteristic lines with different slopes.



Rarefaction

Remarks

1. The waves originating at $x(0) > 0$ move to the right faster than the waves originating waves at points $x(0) < 0$
2. Increasing gap is formed between the faster moving wave front and the slower one
3. There are no characteristic lines from either of the two families (5) passing through the origin, since there is a jump discontinuity at $x = 0$ in the initial condition (3).

Rarefaction

Imagine that there are infinitely many characteristics originating from the origin with slopes ranging between $\frac{1}{2}$ and 1. The proper way to see this is to notice that in the case of $x(0) = 0$ implies that

$$u = \frac{x}{t} \quad \text{if } t < x < 2t$$

This type of waves, which arise from decompression or **rarefaction** of the medium due to the increasing gap formed between the wave fronts traveling at different speeds, are called **rarefaction waves**. Putting all the pieces together, we can write the solution of Burger's equation satisfying the initial condition as follows

$$u(x,t) = \begin{cases} 1 & \text{if } x < t \\ \frac{x}{t} & \text{if } t < x < 2t \\ 2 & \text{if } x > 2t \end{cases} \tag{6}$$

Shock Waves

Shock waves

- It is the complete opposite phenomenon of rarefaction.
- Here, it has faster moving from left to right, catching up to a slower wave.

Consider the following initial condition for Burger's equation

$$u(x,0) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{7}$$

Shock waves

The characteristic lines are

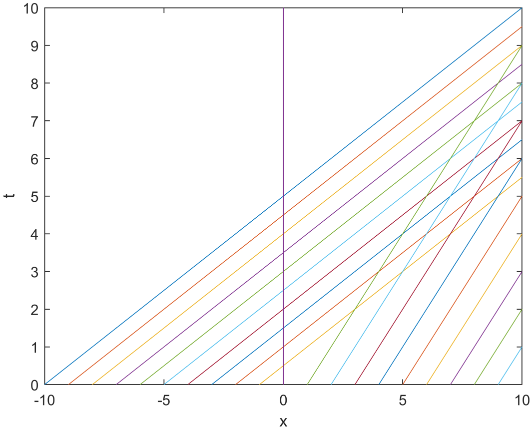
$$x(t) = \begin{cases} 2t + x(0) & \text{if } x(0) < 0 \\ t + x(0) & \text{if } x(0) > 0 \end{cases}$$

Solving for t, we have

$$t = \begin{cases} \frac{1}{2}(x - x(0)) & \text{if } x(0) < 0 \\ x - x(0) & \text{if } x(0) > 0 \end{cases} \tag{8}$$

Shock wave

The characteristic lines corresponding to the initial condition (7). These lines are two families of characteristic lines with different slopes.



Shock wave

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Remarks

1. The characteristic lines originating at $x(0) < 0$ have smaller slope compared the characteristic lines originating from $x(0) > 0$
2. Characteristics from two families intersect
3. It leads to a problem as we can't trace back the correct characteristics to an initial value
4. At the intersection points, u becomes multivalued
5. This phenomenon is called **shock waves**
6. The faster moving wave catches up to the slower moving wave to form a multivalued wave.

Shock wave

There are number of examples for shock waves

Examples 3

Examples

1. Moving shock - Balloon bursting, Shock tube
2. Detonation wave - TNT explosive or high explosive
3. Bow shock - Space Shuttle return, bullets
4. Attached shock - Supersonic wedges
5. Normal shock (at 90°) - Oblique Shock - Bow Shock, R-H

Supernova, an asteroid hitting Earth's atmosphere. Let us see one more theorem and see nonlinear PDEs again with Charpit's methods, R-H condition, and Riemann problem in the later part of our course.

Exercises

Exercise 1: Shock Waves and Rarefaction

Solve the Burger's equation for the following initial data

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - x & \text{if } x \in (0,1) \\ 0 & \text{if } x > 1 \end{cases}$$

and then for

$$u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

General Solutions for Quasilinear PDEs

General Solution

Consider the quasilinear PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{9}$$

Suppose that $P(x,y,u) \in \Omega, v \neq 0$ The characteristic curve

$$\Gamma : \begin{cases} x = x(s) \\ y = y(s) \\ u = u(s) \end{cases}$$

can be represented as the intersection of two surfaces

$$\begin{aligned} \Gamma &= S_1 \cap S_2 \\ S_1 &: \phi(x,y,u) = C_1 \\ S_2 &: \psi(x,y,u) = C_2 \end{aligned} \tag{10}$$

for which n_ϕ and n_ψ are linearly independent at each P.

General Solution

Here $n_\phi = n_\phi(\phi_x, \phi_y, \phi_u)$ and $n_\psi = n_\psi(\psi_x, \psi_y, \psi_u)$.

Definition 1 (First Integral)
A continuously differentiable function $\phi(x, y, u)$ is said to be a first integral of (9) if it is constant on characteristic curves.

Definition 2 (Functionally Independent)
The first two integrals $\phi(x, y, u)$ and $\psi(x, y, u)$ of (9) are functionally independent if

$$rank \begin{bmatrix} \phi_x & \phi_y & \phi_u \\ \psi_x & \psi_y & \psi_u \end{bmatrix} = 2$$

that is, if n_ϕ and n_ψ are linearly independent.

General Solution

Suppose $\phi(x,y,u)$ and $\psi(x,y,u)$ are functionally independent integrals and

$$\begin{aligned}\phi(x(s),y(s),u(s)) &= C_1 \\ \psi(x(s),y(s),u(s)) &= C_2\end{aligned} \implies \begin{aligned}\phi_x\frac{dx}{ds}+\phi_y\frac{dy}{ds}+\phi_u\frac{du}{ds} &= 0 \\ \psi_x\frac{dx}{ds}+\psi_y\frac{dy}{ds}+\psi_u\frac{du}{ds} &= 0\end{aligned}$$

$$\implies \begin{aligned}\phi_x a(x,y,u)+\phi_y b(x,y,u)+\phi_u c(x,y,u) &= 0 \\ \psi_x a(x,y,u)+\psi_y b(x,y,u)+\psi_u c(x,y,u) &= 0\end{aligned}$$

Therefore, ϕ and ψ are functionally independent first integrals iff

$$\frac{a(x,y,u)}{\begin{vmatrix}\phi_y & \phi_u \\ \psi_y & \psi_u\end{vmatrix}} = \frac{b(x,y,u)}{\begin{vmatrix}\phi_u & \phi_x \\ \psi_u & \psi_x\end{vmatrix}} = \frac{c(x,y,u)}{\begin{vmatrix}\phi_x & \phi_y \\ \psi_x & \psi_y\end{vmatrix}} \tag{11}$$

General Solution

Theorem 4 (General Solution)

If $\phi(x,y,u) = C_1$ and $\psi(x,y,u) = C_2$ be two independent solutions of the ODEs

$$C : \begin{cases} \frac{dx}{dt} = a(x,y,u) \\ \frac{dy}{dt} = b(x,y,u) \\ \frac{du}{dt} = c(x,y,u) \end{cases}$$

and $\phi_u^2 + \psi_u^2 \neq 0$, then the general solution to (9) is given by

$$f(\phi(x,y,u), \psi(x,y,u)) = 0$$

where f is an arbitrary function.

General Solution

Proof: Let $u = u(x, y)$ be a function for which

$$f(\phi(x, y, u(x, y)), \psi(x, y, u(x, y))) = 0$$

Differentiating it with respect to x, y, we have

$$f_\phi(\phi_x + \phi_u u_x) + f_\psi(\psi_x + \psi_u u_x) = 0$$

$$f_\phi(\phi_y + \phi_u u_y) + f_\psi(\psi_y + \psi_u u_y) = 0$$

If $(f_\phi, f_\psi) \neq (0, 0)$, then

$$\begin{vmatrix} \phi_x + \phi_u u_x & \psi_x + \psi_u u_x \\ \phi_y + \phi_u u_y & \psi_y + \psi_u u_y \end{vmatrix} = 0$$

General Solution

Proof (Contd): On simplification,

$$(\phi_u\psi_y - \phi_y\psi_u)u_x + (\phi_x\psi_u - \phi_u\psi_x)u_y = \phi_y\psi_x - \phi_x\psi_y \tag{12}$$

By comparing (12) and (11) we can obtain that

$$au_x + bu_y = c$$

Conversely, suppose $u = u(x, y)$ is a solution of (9), $\phi(x, y, u)$ and $\psi(x, y, u)$ are functionally independent first integrals of (9). Then, by (11), we obtain that (12). Now, we have function $\Phi = \phi(x, y, u(x, y))$ and $\Psi = \psi(x, y, u(x, y))$.

General Solution

Proof (Contd): If $(f_\phi, f_\psi) \neq (0,0)$, then

$$\begin{aligned}\begin{vmatrix}\Phi_x & \Psi_x \\ \Phi_y & \Psi_y\end{vmatrix} &= \begin{vmatrix}\phi_x + \phi_u u_x & \psi_x + \psi_u u_x \\ \phi_y + \phi_u u_y & \psi_y + \psi_u u_y\end{vmatrix} \\ &= (\phi_u\psi_y - \phi_y\psi_u)u_x + (\phi_x\psi_u - \phi_u\psi_x)u_y - \phi_y\psi_x - \phi_x\psi_y \\ &= \lambda(au_x + bu_y - c) \\ &= 0\end{aligned}$$

From the rank theorem of Calculus, it follows that one of the functions Φ and Ψ can be expressed as a function of the other. That is, there exists a function g such that

$$\psi(x, y, u(x,y)) = g(\phi(x, y, u(x,y)))$$

$$\implies f(\phi(x,y,u), \psi(x,y,u)) = 0$$

Examples

MA612L PARTIAL DIFFERENTIAL EQUATIONS

Example 5

Show that

$$(y + 2ux)u_x - (x + 2uy)u_y = \frac{1}{2}(x^2 - y^2)$$

with

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = s \\ u = u_0(s) = 0 \end{cases}$$

has exactly one solution.

Solution: The characteristic equations are

$$C : \begin{cases} \frac{dx}{dt} = y + 2ux \\ \frac{dy}{dt} = -(x + 2uy) \\ \frac{du}{dt} = 0.5(x^2 - y^2) \end{cases}$$

Examples

Solution (Contd): One First integral we can obtain from

$$\frac{xdx + ydy}{2u(x^2 - y^2)} = \frac{2du}{x^2 - y^2}$$

$$\implies \phi(x, y, u) = x^2 + y^2 - 4u^2 = C_1$$

We can obtain another independent first integral from

$$\frac{ydx + xdy}{y^2 - x^2} = \frac{2du}{x^2 - y^2}$$

$$\implies \psi(x, y, u) = xy + 2u = C_2$$

The general integral solution is given by

$$x^2 + y^2 - 4u^2 = g(xy + 2u)$$

Examples

Solution (Contd): For the given Cauchy data, we have

$$2s^2 = C_1, s^2 = C_2 \implies C_1 = 2C_2$$

$$\implies f(\phi, \psi) = \phi - 2\psi$$

$$\implies x^2 + y^2 - 4u^2 = 2(xy + 2u)$$

$$\implies x^2 + y^2 - 2xy = 4u^2 + 4u$$

$$\implies u = \frac{1}{2}\left[\sqrt{(x-y)^2 + 1} - 1\right]$$

It is the only solution that satisfies all conditions.

Examples

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Example 6

Find the general solution of the equation

$$(u-y)u_x + yu_y = x+y$$

with

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = 1 \\ u = u_0(s) = 2+s \end{cases}$$

has exactly one solution.

Solution: The characteristic equations are

$$\frac{dx}{u-y} = \frac{dy}{y} = \frac{du}{x+y}$$

Examples

Solution (Contd): One First integral we can obtain from

$$\frac{dx + du}{u + x} = \frac{dy}{y}$$

$$\implies \phi(x, y, u) = \frac{u + x}{y} = C_1$$

We can obtain another independent first integral from

$$\frac{dx + dy}{u} = \frac{du}{x + y}$$

$$\implies \psi(x, y, u) = (x + y)^2 - u^2 = C_2$$

The general integral solution is given by

$$(x + y)^2 - u^2 = g\left(\frac{u + x}{y}\right)$$

Examples

Solution (Contd): For the given Cauchy data, we have

$$\frac{2s+2}{1} = C_1, (s+1)^2 - (s+2)^2 = C_2$$

$$2s+2 = C_1, -2s-3 = C_2 \implies C_1 + C_2 + 1 = 0$$

$$\implies f(\phi, \psi) = \phi + \psi + 1$$

$$(x+y)^2 - u^2 + 1 + \frac{u+x}{y} = 0, y \neq 0$$

It is the only solution which satisfies all conditions.

Exercise

Exercise 2: General Solution

Find the general solution of the following equations

1. $(x-y)y^2u_x - (x-y)x^2u_y - (x^2+y^2)u = 0$
2. $(y-u)u_x + (u-x)u_y = x - y$
3. $x(y-u)u_x + y(u-x)u_y = (x-y)u$
4. $uu_x + (u^2 - x^2)u_y + x = 0$
5. $u_y - \left(\frac{y}{x}u\right)_x = 0$

Let us wrap the first-order linear and quasilinear PDEs for the moment and solve the big three PDEs. Let us begin with the Heat Equation and the separation of variables first.

Thanks

Doubts and Suggestions

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