

MA621L-Mathematical Modeling

Lecture 11 : Buckingham's Π Theorem

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A Story: The Dye That Named a Number

Manchester, 1883: Osborne Reynolds' Pipe



- Reynolds injects a thin thread of coloured dye into water flowing through a glass pipe.
- At low speed the thread stays **straight** – smooth, *laminar* flow.
- Increase the speed past a threshold and the thread suddenly **bursts into swirls** – *turbulence*.
- The transition always happens at the same value of one dimensionless combination:

$$Re = \frac{\rho v D}{\mu} \approx 2300.$$

Why One Number Rules Them All



- Big pipe or small, water or oil, fast or slow – the *flavour* of the flow depends only on Re .
- This is why a small wind-tunnel wing predicts a real aircraft: match Re and the flows are **similar**.
- Where do such magic groups come from, and how many are there? The **Buckingham Π theorem** answers both.



The Theorem

Example 1: the simple pendulum



What does the period T of a pendulum depend on?

variable	T	ℓ	m	g
dimension	T	L	M	LT^{-2}

Try $T^a \ell^b m^c g^d$ dimensionless:

$$\text{M} : c = 0, \quad \text{L} : b + d = 0, \quad \text{T} : a - 2d = 0.$$

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The mass drops out without any mechanics. If we add the amplitude θ_0 (already dimensionless), we get $T = \sqrt{\ell/g} \Phi(\theta_0)$.

Example 2: drag on a sphere



Force F on a sphere of diameter D moving at speed U through a fluid of density ρ and viscosity μ .

variable	F	ρ	U	D	μ
dimension	MLT^{-2}	ML^{-3}	LT^{-1}	L	$\text{ML}^{-1}\text{T}^{-1}$

Five variables, three dimensions: two groups,

$$C_D = \frac{F}{\rho U^2 D^2}, \quad \text{Re} = \frac{\rho U D}{\mu}, \quad \Rightarrow \quad \boxed{F = \rho U^2 D^2 \Phi(\text{Re})}$$

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A function of four variables becomes a function of one. Every experiment, any fluid, any sphere, lies on a single curve $C_D(\text{Re})$. This is why wind-tunnel models work.

Example 3: G. I. Taylor and the blast wave



A point explosion releases energy E into air of density ρ . How does the radius R of the fireball grow with time t ?

variable	R	t	E	ρ
dimension	L	T	ML^2T^{-2}	ML^{-3}

One group, $\Pi = \frac{\rho R^5}{Et^2}$, hence

$$R = C \left(\frac{E t^2}{\rho} \right)^{1/5}$$

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Taylor (1950) used published photographs of the 1945 Trinity test: plotting $\log R$ against $\log t$ gives slope $2/5$, and the intercept gives E . His estimate, about 17 kilotons, was close to the classified figure.

What the examples have in common



example	variables n	dimensions k	groups $n - k$
pendulum	4	3	1
sphere drag	5	3	2
blast wave	4	3	1

In each case:

- the physics does not care which units we use;
- the dimensionless combinations are the solutions of a linear system;
- the number of groups was always $n - k$.

Is this always true, and why?

The Buckingham Π Theorem



Theorem 1 (Buckingham, 1914)

A physically meaningful relation among n variables involving k independent fundamental dimensions can be rewritten as a relation among $n - k$ independent **dimensionless** groups $\Pi_1, \dots, \Pi_{n-k}$:

$$\phi(\Pi_1, \Pi_2, \dots, \Pi_{n-k}) = 0.$$

- k = rank of the dimensional matrix (usually the number of independent dimensions present).
- The $n - k$ groups are the true “degrees of freedom” of the problem.

Setting

Fundamental dimensions $L_1, \dots, L_k$ (e.g. mass, length, time).

Physical variables $q_1, \dots, q_n > 0$ with

$$[q_j] = L_1^{a_{1j}} L_2^{a_{2j}} \cdots L_k^{a_{kj}}, \quad a_{ij} \in \mathbb{R}.$$

Dimension matrix $A = (a_{ij}) \in \mathbb{R}^{k \times n}$.

k independent fundamental dimensions $\iff \text{rank } A = k$.



Change of units



Rescaling the unit of L_i by $\lambda_i > 0$ changes the value of q_j to

$$\tilde{q}_j = \left(\prod_{i=1}^k \lambda_i^{a_{ij}} \right) q_j, \quad \text{written } \tilde{q} = T_\lambda q.$$

Definition 2

$f(q_1, \dots, q_n) = 0$ is **physically meaningful** if

$$f(q) = 0 \iff f(T_\lambda q) = 0 \quad \forall q \in (0, \infty)^n, \lambda \in (0, \infty)^k.$$

A law of nature should not depend on whether we measure in metres or feet.

The theorem



Theorem 3 (Buckingham, 1914)

Let $\text{rank } A = k$ and let $f(q) = 0$ be physically meaningful. Then there are $n - k$ dimensionless products

$$\Pi_\ell = \prod_{j=1}^n q_j^{x_j^{(\ell)}}, \quad \ell = 1, \dots, n - k,$$

with linearly independent exponent vectors, and a function ϕ such that

$$f(q_1, \dots, q_n) = 0 \iff \phi(\Pi_1, \dots, \Pi_{n-k}) = 0.$$

Step 1: dimensionless products = $\ker A$

For $x \in \mathbb{R}^n$ let $q^x = \prod_j q_j^{x_j}$. Then

$$[q^x] = \prod_{j=1}^n \prod_{i=1}^k L_i^{a_{ij}x_j} = \prod_{i=1}^k L_i^{(Ax)_i}.$$

So

$$q^x \text{ is dimensionless} \iff Ax = 0, \quad (T_\lambda q)^x = \left(\prod_i \lambda_i^{(Ax)_i} \right) q^x.$$



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Rank-nullity: $[\ker]A = n - \text{rank } A = n - k$.

Still to show: f depends on q *only* through these products.

Step 2: a convenient basis



Since $\text{rank } A = k$, renumber so that the first k columns are independent:

$$A = [B \mid C], \quad B \in \mathbb{R}^{k \times k} \text{ invertible.}$$

$q_1, \dots, q_k$ are the **repeating variables**. Let $D = B^{-1}C = (d_{i\ell})$ and

$$x^{(\ell)} = \begin{pmatrix} -De_{\ell} \\ e_{\ell} \end{pmatrix}, \quad Ax^{(\ell)} = -BB^{-1}Ce_{\ell} + Ce_{\ell} = 0.$$

The last $n - k$ entries form the identity, so the $x^{(\ell)}$ are independent: a basis of $\ker A$.

$$\Pi_{\ell} = q_{k+\ell} \prod_{i=1}^k q_i^{-d_{i\ell}}, \quad \ell = 1, \dots, n - k$$

Step 3: change of variables



Define $\Psi : (0, \infty)^n \rightarrow (0, \infty)^n$,

$$\Psi(q) = (q_1, \dots, q_k, \Pi_1, \dots, \Pi_{n-k}).$$

Ψ is a bijection (indeed a diffeomorphism), with inverse

$$q_{k+\ell} = \Pi_\ell \prod_{i=1}^k q_i^{d_{i\ell}}, \quad q_1, \dots, q_k \text{ unchanged.}$$

With $F = f \circ \Psi^{-1}$:

$$f(q) = 0 \iff F(q_1, \dots, q_k, \Pi_1, \dots, \Pi_{n-k}) = 0.$$

Step 4a: choosing units – take logarithms



Goal. For a fixed point q , find units $\lambda = (\lambda_1, \dots, \lambda_k)$ such that

$$\tilde{q}_1 = \dots = \tilde{q}_k = 1.$$

Difficulty. The unknowns sit in exponents:

$$\tilde{q}_j = \left(\prod_{i=1}^k \lambda_i^{a_{ij}} \right) q_j \quad (\text{nonlinear in } \lambda).$$

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Remedy. All quantities are positive, so take logarithms and put $\mu_i = \log \lambda_i$ (equivalently $\lambda_i = e^{\mu_i}$; no $\lambda_i > 0$ is lost):

$$\log \tilde{q}_j = \log q_j + \sum_{i=1}^k a_{ij} \mu_i = \log q_j + (A^T \mu)_j,$$

since the (j, i) entry of A^T is a_{ij} . **The problem is now linear in μ .**

Step 4b: solving for the units



Only $j = 1, \dots, k$ matter. Their columns of A form B , so the rows $j \leq k$ of A^T form B^T :

$$\begin{pmatrix} \log \tilde{q}_1 \\ \vdots \\ \log \tilde{q}_k \end{pmatrix} = b + B^T \mu, \quad b := \begin{pmatrix} \log q_1 \\ \vdots \\ \log q_k \end{pmatrix}.$$

We want the left side to be 0, i.e. $B^T \mu = -b$.

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$\det B^T = \det B \neq 0$, so B^T is invertible, with $(B^T)^{-1} = (B^{-1})^T =: B^{-T}$. Hence

$$\mu = -B^{-T} b, \quad \lambda_i = e^{\mu_i}$$

μ depends on q . That is allowed: homogeneity holds for every λ , so for each q we may pick the λ that suits it.

Example: $\lambda_T = 1/t$, $\lambda_L = 1/(gt^2)$.

Step 4c: what the new units do

The Π 's do not change. Write $\tilde{\Pi}_\ell$ for Π_ℓ evaluated at $T_\lambda q$. By Step 1,

$$\tilde{\Pi}_\ell = (T_\lambda q)^{x^{(\ell)}} = \left(\prod_{i=1}^k \lambda_i^{(Ax^{(\ell)})_i} \right) q^{x^{(\ell)}} = \Pi_\ell, \quad \text{since } Ax^{(\ell)} = 0.$$



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So the rescaled point has simple coordinates:

$$\Psi(T_\lambda q) = (\tilde{q}_1, \dots, \tilde{q}_k, \tilde{\Pi}_1, \dots, \tilde{\Pi}_{n-k}) = (1, \dots, 1, \Pi_1, \dots, \Pi_{n-k}).$$

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Conclusion of Step 4.

$$f(q) = 0 \xLeftrightarrow{(a)} f(T_\lambda q) = 0 \xLeftrightarrow{(b)} F(1, \dots, 1, \Pi_1, \dots, \Pi_{n-k}) = 0.$$

(a) homogeneity with this λ ; (b) Step 3 applied at the point $T_\lambda q$.

Step 5: conclusion

Define

$$\phi(\Pi_1, \dots, \Pi_{n-k}) := F(1, \dots, 1, \Pi_1, \dots, \Pi_{n-k}).$$

Then

$$f(q_1, \dots, q_n) = 0 \iff \phi(\Pi_1, \dots, \Pi_{n-k}) = 0.$$

- each Π_ℓ is dimensionless (Step 1);
- the Π_ℓ are independent: no one is a product of powers of the others (Step 2).



Remarks



- **Explicit form.** If $f = 0$ can be solved for q_{k+1} , then $\Pi_1 = \Phi(\Pi_2, \dots, \Pi_{n-k})$.
- **Rank deficiency.** If $\text{rank } A = r < k$, repeat the proof with r in place of k ; one gets $n - r$ groups.
- **Non-uniqueness.** Any basis of $\ker A$ works; two choices are related by $\Pi'_m = \prod_{\ell} \Pi_{\ell}^{m_{\ell m}}$, $(m_{\ell m})$ invertible. The number $n - k$ is fixed.
- **Signs.** Positivity is only needed for real powers; otherwise work with $|q_j|$.

Recipe for the Π Groups



1. List n variables and their dimensions; determine k .
2. Choose k **repeating variables** that (i) together contain all k dimensions and (ii) cannot form a dimensionless group by themselves.
3. Form each Π_i by multiplying one *non-repeating* variable by powers of the repeating ones; solve for powers to make it dimensionless.
4. Interpret and, if possible, recognise the named groups.



Worked Examples

Example: Pressure Drop in a Pipe

Variables: $\Delta p, \rho, v, D, \mu, \ell$ ($n = 6$), dimensions M, L, T ($k = 3$) $\Rightarrow$ 3 groups.

Repeating: ρ, v, D . Then:

$$\Pi_1 = \frac{\Delta p}{\rho v^2}, \quad \Pi_2 = \frac{\rho v D}{\mu} = Re, \quad \Pi_3 = \frac{\ell}{D}.$$

Hence $\frac{\Delta p}{\rho v^2} = \phi\left(Re, \frac{\ell}{D}\right)$ – the basis of the Darcy friction-factor chart.



Example: Rederiving the Pendulum



Variables T, L, g, m ($n = 4$), dimensions M, L, T. But M appears *only* in m , so m drops; effectively $n = 3, k = 2 \Rightarrow 1$ group:

$$\Pi_1 = T\sqrt{g/L} = \text{const} \Rightarrow T = C\sqrt{L/g}.$$

One group $\Rightarrow$ the relation is just " $\Pi_1 = \text{constant}$ " – the strongest possible conclusion.

Named Dimensionless Groups (a modeller's toolkit)



Group	Definition	Balances
Reynolds Re	$\rho v L / \mu$	inertia / viscous
Froude Fr	$v / \sqrt{gL}$	inertia / gravity
Mach Ma	v / c	speed / sound speed
Péclet Pe	$v L / \alpha$	advection / diffusion
Nusselt Nu	$h L / k$	convective / conductive heat

These reappear throughout Unit VI (heat, traffic, Navier–Stokes).



Summary

Exercises



1. Using Π , find the groups for drag F on a sphere from ρ, v, D, μ . Show $F/(\rho v^2 D^2) = \phi(Re)$.
2. For a vibrating drop held by surface tension σ , density ρ , radius r : find the single group controlling its frequency.
3. Determine the number of dimensionless groups for heat convection $h = f(\rho, v, L, \mu, c_p, k)$.
4. Build the dimensional matrix for $\{\Delta p, \rho, v, D, \mu\}$ and compute its rank.
5. Explain, via similarity, why a 1:50 ship model is towed at a speed set by matching Froude number, not Reynolds.

Thanks

Doubts and Suggestions

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