MA621L-Mathematical Modeling

Lecture 12 : Scaling: Gulliver to Microduck Problem

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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भारतीय प्रौद्योगिकी संस्थान तिरुपति
TIRUPATI



Why Does Size Matter?

A Story: Gulliver's Rations

Swift, *Gulliver's Travels* (1726).



Movie

A Story: Gulliver's Travels, Chapter 1



Jonathan Swift, *Gulliver's Travels* (1726), Part I.

- A ship's surgeon is wrecked and wakes up tied to the ground by people about six inches tall.

A Story: Gulliver's Travels, Chapter 1



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- The Lilliputians cannot weigh him, and they cannot measure all of him. So they **measure a small part and scale up**.

A Story: Gulliver's Travels, Chapter 1

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- The King's mathematicians take his height with a quadrant: twelve of theirs.
- The seamstresses measure only his thumb. Twice round the thumb is once round the wrist, and so on to the neck and the waist. From that they cut a whole suit, and it fits.

A Story: Gulliver's Travels, Chapter 1



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- The seamstresses measure only his thumb. Twice round the thumb is once round the wrist, and so on to the neck and the waist. From that they cut a whole suit, and it fits.
- The ministers then decide that, since his body is shaped like theirs, he needs $12 \times 12 \times 12 = 1728$ Lilliputian rations a day.

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What the Lilliputians actually did

They never counted Gulliver. They measured one unit, assumed a shape rule, and multiplied. That is the method of this whole lecture.

A Story: Gulliver's Travel, Chapter 2

Chapter Two: My Life in Lilliput



A Story: Gulliver's Travel, Chapter 2



Chapter Two: My Life in Lilliput

Lilliput. Freedom. War with Blefuscu. Escape to Blefuscu. I return to England.

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People in Lilliput seemed more friendly now. Boys and girls played in my hair and they called me Man Mountain. The horses weren't scared of me now. People enjoyed coming to see me: I was now a tourist attraction.

A Story: Gulliver's Travel, Chapter 2



Chapter Two: My Life in Lilliput

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One day, the emperor asked me to stand up with my feet apart. He then told his soldiers to march under me. Everyone enjoyed the parade.

A Story: Gulliver's Travel, Chapter 2

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One day, the emperor asked me to stand up with my feet apart. He then told his soldiers to march under me. Everyone enjoyed the parade.

'Now I'll give you your freedom,' the emperor said, 'But there are some rules. So, there are nine rules. If you agree, you'll be free.'

A Story: Gulliver's Travel, Chapter 2

Chapter Two: My Life in Lilliput

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'Now I'll give you your freedom,' the emperor said, 'But there are some rules. So, there are nine rules. If you agree, you'll be free.'

The emperor gave me a piece of paper.

A Story: Gulliver's Travel, Chapter 2

RULES

1. The Man Mountain must not leave Lilliput without permission.
2. He must not come into the city without permission.
3. He must stay on the roads.
4. He must be careful (our people are very small).
5. He must carry urgent messages for the emperor.
6. He must help us in our war with Blefuscu.
7. He must help us build new houses.
8. He must make a map of our country.
9. He can have food and drink = 1,728 Lilliputians.

A Story: Gulliver's Travel, Chapter 2

I had no problem with the rules. I asked a friend about how they decided how much food and drink to give me.

'The best mathematicians in the country decided this,' he said. 'They thought about your height – you're exactly 12 times taller than us. Your volume is therefore 12^3. As you know, 12^3 equals 1,728.'

I checked the mathematics: 12 x 12 x 12 = 1,728. 'You're very clever people,' I answered, 'And very good at economics!'

Is 1728 the Right Answer?

The arithmetic is not in doubt. The **assumption** is. Gulliver needs food for something, and the scaling rule depends on what that something is.

If food is mainly used to …	rations scale like	factor
build and maintain body mass	volume, L^3	$12^3 = 1728$
replace heat lost through the skin	area, L^2	?
power the body (measured animals, Kleiber's Law)	$M^{3/4}$	?

Before the lecture continues

Fill in the two question marks. Which row would you defend, and what single measurement would settle it?

A Story: Gulliver's Travel, Chapter 2



The Lilliputians see that Gulliver is twelve times their height. Since his body has the same shape as theirs, the King's ministers decide he must contain $12^3 = 1728$ Lilliputians, and they feed him 1728 rations a day.

A Story: Gulliver's Travel, Chapter 2



The Lilliputians see that Gulliver is twelve times their height. Since his body has the same shape as theirs, the King's ministers decide he must contain $12^3 = 1728$ Lilliputians, and they feed him 1728 rations a day.

The modelling question

Is 1728 right? The answer depends entirely on what you assume food is *for*.

- Food builds and maintains body mass $\Rightarrow$ rations scale with volume: $12^3 = 1728$.
- Food replaces heat lost through the skin $\Rightarrow$ rations scale with area: $12^2 = 144$.
- Measured metabolic rates of mammals (Kleiber's Law) $\Rightarrow 12^{9/4} \approx 268$.

A Story: Gulliver's Travel, Chapter 2

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Three assumptions, three models, a factor of twelve between them. By the end of tomorrow, you should be able to defend one of them.



Gulliver in Reverse: The Microduck

Microduck Scaling

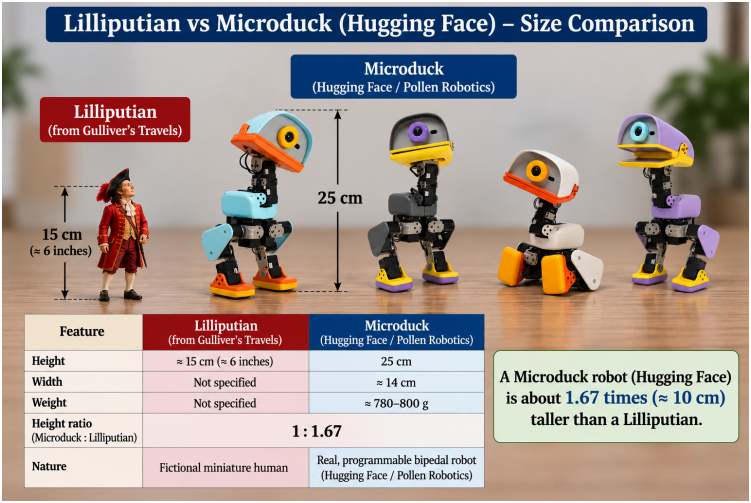


Microduck Video

A Microduck Robot

- **Microduck** (Pollen Robotics, Hugging Face, 2026): a two-legged duck robot, 25 cm tall, 15 motors, about $400.
- Walking skills are trained in simulation and transferred to the real robot. It started as a research platform and is now sold as a companion robot.
- It is a small version of full-size walking robots. So we ask the Lilliputians' question in reverse: *what changes when a walking machine is made smaller?*

Microduck Scaling



Feature	Lilliputian (from Gulliver's Travels)	Microduck (Hugging Face / Pollen Robotics)
Height	≈ 15 cm (≈ 6 inches)	25 cm
Width	Not specified	≈ 14 cm
Weight	Not specified	≈ 780–800 g
Height ratio (Microduck : Lilliputian)	1 : 1.67	
Nature	Fictional miniature human	Real, programmable bipedal robot (Hugging Face / Pollen Robotics)

A Microduck robot (Hugging Face) is about **1.67 times (≈ 10 cm)** taller than a Lilliputian.

Torque Tap

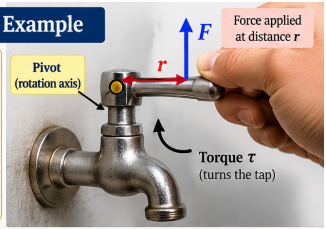
Torque (Moment of a Force) – Water Tap Example

Definition. If a force $\boldsymbol{F}$ acts at a distance $\mathbf{r}$ from a pivot,

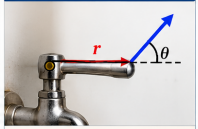
$$\tau = \boldsymbol{r} \times \boldsymbol{F} \quad \text{with magnitude} \quad \tau = r\,F\sin\theta$$

where θ is the angle between $\mathbf{r}$ and $\boldsymbol{F}$. Only the part of the force **perpendicular to the lever arm** turns anything.

Equivalently, $\tau = F \times d$, where d is the perpendicular distance from the pivot to the force's line of action, called the **lever arm**.



1. General case (angle θ)



$\tau = r\,F\sin\theta$

- θ is the angle between $\mathbf{r}$ and $\boldsymbol{F}$.
- Only the perpendicular component $F_\perp = F\sin\theta$ produces torque.

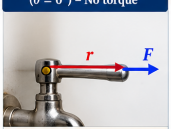
2. Force perpendicular to lever arm ($\theta = 90°$) – Maximum torque



$\tau = r\,F\sin 90° = r\,F$

- Full force contributes to turning the tap.
- This gives the maximum torque.

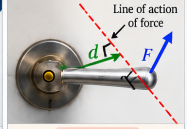
3. Force along the lever arm ($\theta = 0°$) – No torque



$\tau = r\,F\sin 0° = 0$

- The force is along the lever arm.
- No turning effect (the tap does not rotate).

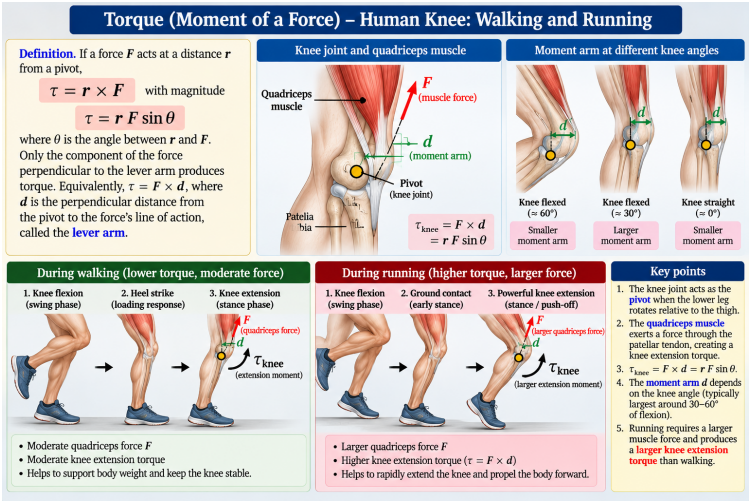
4. Lever arm (perpendicular distance)



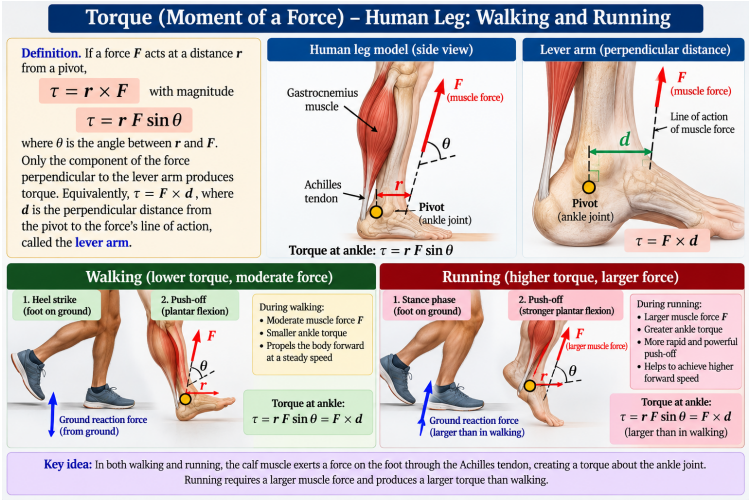
$\tau = F \times d$

- d is the perpendicular distance from the pivot to the force's line of action.
- d is called the **lever arm**.

Torque: Knee


Torque (Moment of a Force) – Human Knee: Walking and Running
Definition. If a force F acts at a distance r from a pivot,
τ = r × F with magnitude
τ = r F sin θ
where θ is the angle between r and F. Only the component of the force perpendicular to the lever arm produces torque. Equivalently, τ = F × d, where d is the perpendicular distance from the pivot to the force's line of action, called the lever arm.
Knee joint and quadriceps muscle
Quadriceps muscle
F (muscle force)
d (moment arm)
Pivot (knee joint)
Patella
tibia
τknee = F × d = r F sin θ
Moment arm at different knee angles
Knee flexed (≈ 60°)
Smaller moment arm
Knee flexed (≈ 30°)
Larger moment arm
Knee straight (≈ 0°)
Smaller moment arm
During walking (lower torque, moderate force)
1. Knee flexion (swing phase)
2. Heel strike (loading response)
3. Knee extension (stance phase)
F (quadriceps force)
d
τknee (extension moment)
• Moderate quadriceps force F
• Moderate knee extension torque
• Helps to support body weight and keep the knee stable.
During running (higher torque, larger force)
1. Knee flexion (swing phase)
2. Ground contact (early stance)
3. Powerful knee extension (stance / push-off)
F (larger quadriceps force)
d
τknee (larger extension moment)
• Larger quadriceps force F
• Higher knee extension torque (τ = F × d)
• Helps to rapidly extend the knee and propel the body forward.
Key points
1. The knee joint acts as the pivot when the lower leg rotates relative to the thigh.
2. The quadriceps muscle exerts a force through the patellar tendon, creating a knee extension torque.
3. τKnee = F × d = r F sin θ.
4. The moment arm d depends on the knee angle (typically largest around 30–60° of flexion).
5. Running requires a larger muscle force and produces a larger knee extension torque than walking.

Torque: Leg


Torque (Moment of a Force) – Human Leg: Walking and Running
Definition. If a force F acts at a distance r from a pivot,
τ = r × F with magnitude
τ = r F sin θ
where θ is the angle between r and F. Only the component of the force perpendicular to the lever arm produces torque. Equivalently, τ = F × d, where d is the perpendicular distance from the pivot to the force's line of action, called the lever arm.
Human leg model (side view)
Gastrocnemius muscle
F (muscle force)
θ
Achilles tendon
r
Pivot (ankle joint)
Torque at ankle: τ = r F sin θ
Lever arm (perpendicular distance)
F (muscle force)
Line of action of muscle force
d
Pivot (ankle joint)
τ = F × d
Walking (lower torque, moderate force)
1. Heel strike (foot on ground)
2. Push-off (plantar flexion)
F
θ
r
Ground reaction force (from ground)
During walking:
• Moderate muscle force F
• Smaller ankle torque
• Propels the body forward at a steady speed
Torque at ankle:
τ = r F sin θ = F × d
Running (higher torque, larger force)
1. Stance phase (foot on ground)
2. Push-off (stronger plantar flexion)
F (larger muscle force)
r
θ
Ground reaction force (larger than in walking)
During running:
• Larger muscle force F
• Greater ankle torque
• More rapid and powerful push-off
• Helps to achieve higher forward speed
Torque at ankle:
τ = r F sin θ = F × d
(larger than in walking)
Key idea: In both walking and running, the calf muscle exerts a force on the foot through the Achilles tendon, creating a torque about the ankle joint. Running requires a larger muscle force and produces a larger torque than walking.

Torque: Hand

Torque (Moment of a Force) – Human Hand and Arm

Definition. If a force F acts at a distance r from a pivot, the torque (moment of the force) is

$$\tau = r \times F$$

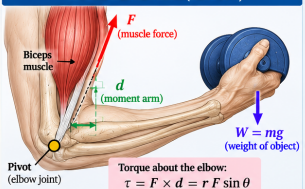
$$\tau = r\,F \sin\theta$$

where θ is the angle between r and F. Only the component of the force perpendicular to the lever arm produces torque. Equivalently,

$$\tau = F \times d$$

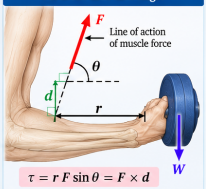
where d is the perpendicular distance from the pivot to the force's line of action, called the **moment arm** (or **lever arm**).

Human arm and hand (side view)



Torque about the elbow: $\tau = F \times d = r\,F \sin\theta$

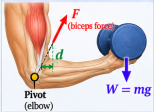
Moment arm and angle



$\tau = r\,F \sin\theta = F \times d$

Examples of torque in daily activities

1. Lifting a weight (bicep curl)



- Biceps muscle exerts a force F through a small moment arm d.
- Produces a flexion torque about the elbow.
- Larger weight or larger moment arm requires larger muscle force.

2. Holding a bag in the hand



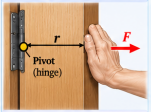
- The weight of the bag produces a torque about the wrist.
- $\tau = W \times r$.
- A heavier bag or a longer distance r requires a larger muscle torque to keep the hand level.

3. Using a hammer (pulling a nail)



- A downward force F at a long handle distance r produces a large torque.
- A longer handle makes it easier to produce torque.
- $\tau = F \times r$ (when force is perpendicular).

4. Pushing a door (about hinge)



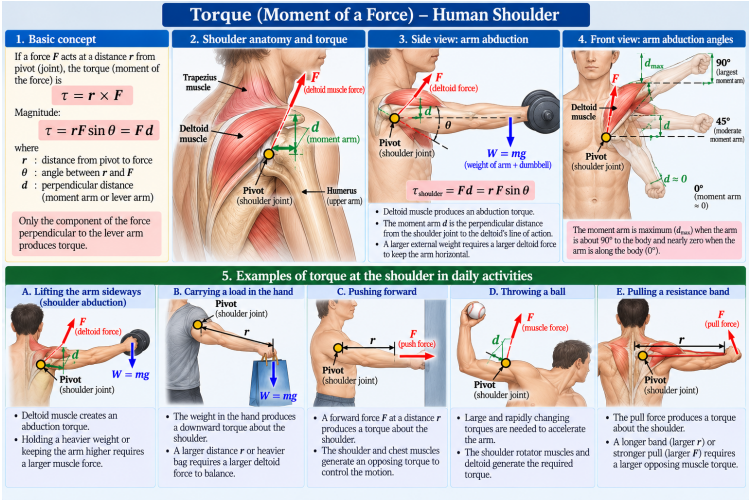
- A push at a distance r from the hinge produces a torque.
- A larger distance (near the edge) makes it easier to open the door.
- $\tau = F \times r$.

5. Turning a doorknob



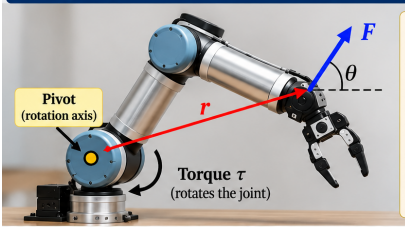
- A tangential force F produces a rotational torque about the knob axis.
- $\tau = F \times r$.
- A larger knob radius r or a larger force makes it easier to turn.

Torque: Shoulder



Torque: Robot Arms

Torque (Moment of a Force) – Robot Arm Example



Definition. If a force F acts at a distance r from a pivot,

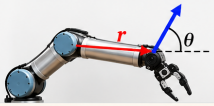
$\tau = \boldsymbol{r} \times \boldsymbol{F}$ with magnitude $\tau = rF\sin\theta$

where θ is the angle between $\boldsymbol{r}$ and F.

Only the component of the force **perpendicular to the lever arm** produces torque.

Equivalently, $\tau = F \times d$, where d is the perpendicular distance from the pivot to the force's line of action, called the **lever arm**.

1. General case (angle θ)



$\tau = rF\sin\theta$

- θ is the angle between $\boldsymbol{r}$ and F.
- Only the perpendicular component $F_\perp = F\sin\theta$ produces torque.

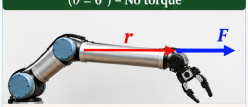
2. Force perpendicular to lever arm ($\theta = 90°$) – Maximum torque



$\tau = rF\sin 90° = rF$

- Full force contributes to rotating the joint.
- This gives the maximum torque.

3. Force along the lever arm ($\theta = 0°$) – No torque



$\tau = rF\sin 0° = 0$

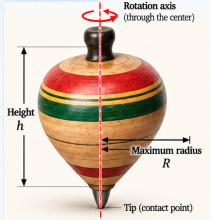
- The force is along the lever arm.
- No turning effect (the joint does not rotate).

Moment of Inertia: Earth

Moment of Inertia – Examples: Spinning Top and Earth

A. Spinning Top (Toy)

1. Physical model



2. Moment of inertia (model)

Model the spinning top as a solid body with rotational symmetry. A simple approximation is a solid disk or a solid sphere (both give similar order of magnitude).

Solid disk approximation
(radius R, thickness h, mass M)

$$I \approx \frac{1}{2} M R^2$$

Solid sphere approximation
(radius R, mass M)

$$I \approx \frac{2}{5} M R^2$$

3. Example calculation (typical spinning top)

Assumed values:

- Mass (wood): $M = 50\ g = 0.05\ kg$
- Maximum radius: $R = 2.5\ cm = 0.025\ m$
- Height: $h = 4\ cm = 0.04\ m$

Using solid disk approximation:

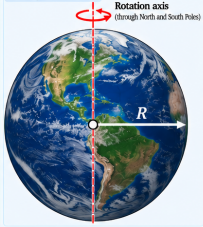
$$I \approx \frac{1}{2} M R^2 = \frac{1}{2}(0.05)(0.025)^2 = 1.56 \times 10^{-5}\ kg\,m^2$$

Using solid sphere approximation (for comparison):

$$I \approx \frac{2}{5} M R^2 = \frac{2}{5}(0.05)(0.025)^2 = 1.25 \times 10^{-5}\ kg\,m^2$$

B. Earth – The Rotating Planet

1. Physical model



2. Moment of inertia (uniform sphere)

Approximate the Earth as a uniform solid sphere of mass M and radius R rotating about its central axis.

$$I_{Earth} = \frac{2}{5} M R^2$$

3. Values for the Earth

- **Mass:** $M = 5.97 \times 10^{24}\ kg$
- **Radius:** $R = 6.37 \times 10^{6}\ m$

4. Numerical calculation

$$I_{Earth} = \frac{2}{5}(5.97 \times 10^{24})(6.37 \times 10^{6})^2 \approx 9.7 \times 10^{37}\ kg\,m^2$$

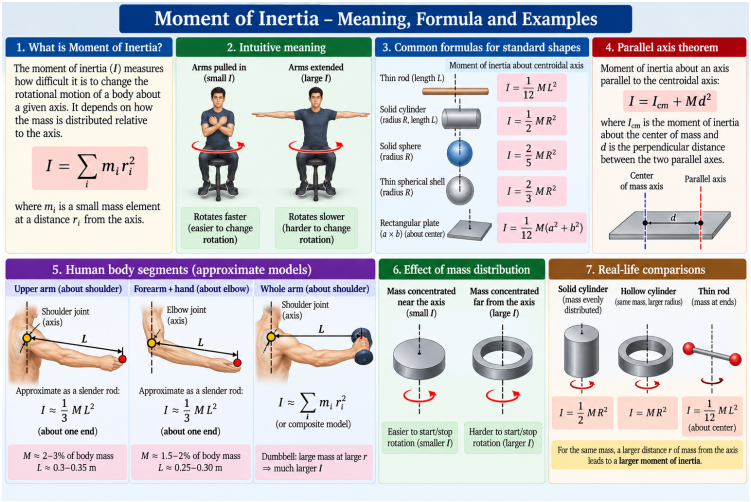
C. Comparison

Object	Mass (kg)	Typical radius (m)	Moment of inertia (kg m^2)
Spinning top	~ 0.05	~ 0.025	~ 10^{-5} (typical)
Earth	5.97×10^{24}	6.37×10^{6}	9.7×10^{37}

D. Key takeaways

- The moment of inertia depends on both the amount of mass and how far the mass is from the rotation axis.
- The Earth's moment of inertia is about 10^{42} **times larger** than that of a **spinning top**.
- A larger moment of inertia means it is much harder to change the rotational motion (start, stop, or change the rotation rate).
- A spinning top can speed up or slow down quickly because it has a small mass and radius, so it has a small moment of inertia.
- The Earth rotates much more slowly and is very difficult to speed up or slow down because of its enormous moment of inertia.

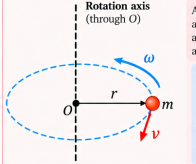
Moment of Inertia: Human



Moment of Inertia: Derivation

Derivation of $I = \sum m_i r_i^2$ for a Rigid Body

1 A particle rotating about a fixed axis



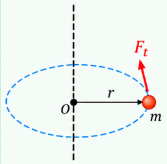
A particle of mass m at a distance r from a fixed axis rotates in a circle with angular velocity ω and angular acceleration α.

Tangential acceleration:

$$a_t = r\alpha$$

(because the tangential linear acceleration is the distance r times the angular acceleration α.).

2 Apply Newton's second law

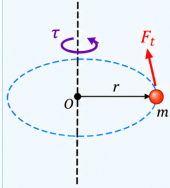


The tangential force required to produce the tangential acceleration a_t is given by Newton's second law:

$$F_t = ma_t = mr\alpha$$

Here, F_t is the tangential component of the force, which is perpendicular to the radius.

3 Calculate the torque



The torque about the axis due to this tangential force is

$$\tau = rF_t$$

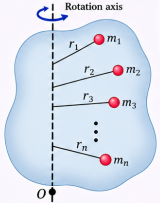
Substitute $F_t = mr\alpha$:

$$\tau = r(mr\alpha) = mr^2\alpha$$

Thus, for a single particle,

$$\tau = mr^2\alpha$$

4 Extend to many particles (rigid body)



For a rigid body consisting of many particles, each particle contributes a torque

$$\tau_i = m_i r_i^2 \alpha$$

The net torque is the sum of all torques:

$$\tau_{\text{net}} = \sum_{i=1}^{n} m_i r_i^2 \alpha$$

Since α is the same for all particles, we can write

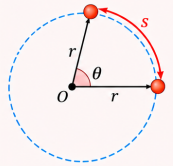
$$\tau_{\text{net}} = \left(\sum_{i=1}^{n} m_i r_i^2 \right) \alpha = I\alpha$$

where $I = \sum_{i=1}^{n} m_i r_i^2$

Moment of Inertia: Circular Motion

Derivation of $a_t = r\alpha$ for a Particle in Circular Motion

1 Geometry of the circular motion

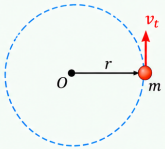


A particle moves in a circle of radius r. If it rotates through an angle θ (in radians), the distance travelled along the circular path (arc length) is

$$s = r\theta$$

Here, r is constant because the particle moves on a circle of fixed radius.

2 Differentiate to get the tangential speed



Differentiate $s = r\theta$ with respect to time t:

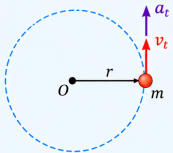
$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

The tangential (linear) speed is

$$v_t = \frac{ds}{dt} = r\omega$$

where $\omega = \frac{d\theta}{dt}$ is the angular velocity.

3 Differentiate again to get the tangential acceleration



Differentiate $v_t = r\omega$ with respect to time t:

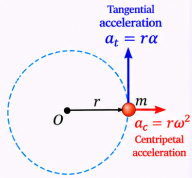
$$\frac{dv_t}{dt} = r\frac{d\omega}{dt}$$

The tangential acceleration is

$$a_t = \frac{dv_t}{dt} = r\alpha$$

where $\alpha = \frac{d\omega}{dt}$ is the angular acceleration.

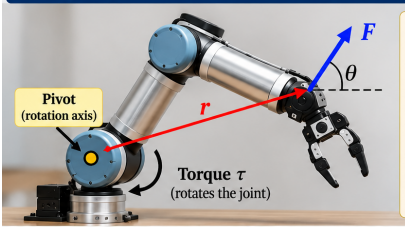
4 Meaning



- $a_t = r\alpha$ is the tangential acceleration, which changes the speed of the particle.
- There is also a centripetal acceleration $a_c = r\omega^2$, which changes the direction of the velocity.
- The two accelerations are perpendicular to each other.

Torque: Robot Arms

Torque (Moment of a Force) – Robot Arm Example



Definition. If a force F acts at a distance r from a pivot,

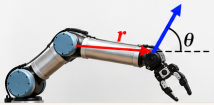
$\tau = \boldsymbol{r} \times \boldsymbol{F}$ with magnitude $\tau = rF\sin\theta$

where θ is the angle between $\boldsymbol{r}$ and F.

Only the component of the force **perpendicular to the lever arm** produces torque.

Equivalently, $\tau = F \times d$, where d is the perpendicular distance from the pivot to the force's line of action, called the **lever arm**.

1. General case (angle θ)



$\tau = rF\sin\theta$

- θ is the angle between $\boldsymbol{r}$ and F.
- Only the perpendicular component $F_\perp = F\sin\theta$ produces torque.

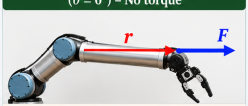
2. Force perpendicular to lever arm ($\theta = 90°$) – Maximum torque



$\tau = rF\sin 90° = rF$

- Full force contributes to rotating the joint.
- This gives the maximum torque.

3. Force along the lever arm ($\theta = 0°$) – No torque



$\tau = rF\sin 0° = 0$

- The force is along the lever arm.
- No turning effect (the joint does not rotate).

Torque

Translational Motion	**Rotational Motion**
Mass: m (kg)	Moment of inertia: I (kg m^2)
Force: F (N)	Torque: τ (N m)
Acceleration: a (m/s^2)	Angular acceleration: α (rad/s^2)
Momentum: $p = mv$ (kg m/s)	Angular momentum: $L = I\omega$ (kg m^2/s)
Newton's law: $F = ma$	Rotational law: $\tau = I\alpha$

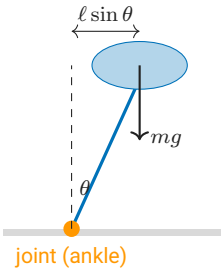
Table 1: Analogy between translational and rotational motion, including SI units

A Microduck Robot

Quantity	Scales as	Derived on slide
torque needed at a joint	L^4	Torque
torque a motor can give	$L^\alpha, 3 \leq \alpha \leq 4$	Torque
natural step frequency	$L^{-1/2}$	Frequency
maximum walking speed	$L^{1/2}$	Walking speed
fall energy per unit volume	L	Falls

L: a characteristic length (height or leg length). Assume geometric similarity and the same materials.

Torque Needed to Hold a Leg Pose



Moment balance about the ankle joint, body of mass m at lever arm $\ell \sin\theta$:

$$\tau_{\text{req}} = m\, g\, \ell \sin\theta.$$

Geometric similarity, density ρ, fixed angle θ:

$$m = c_m \rho L^3, \qquad \ell = c_\ell L$$

$$\Longrightarrow \quad \boxed{\tau_{\text{req}} = c_m c_\ell\, \rho g \sin\theta\; L^4 \;\propto\; L^4}$$

where c_m and c_l are mass and lever-arm constants (shape constants) resp.

Torque Needed to Hold a Leg Pose



Dimension check

$[\tau] = [m][g][\ell] = \mathrm{M}\cdot\mathrm{L}\,\mathrm{T}^{-2}\cdot\mathrm{L} = \mathrm{M}\,\mathrm{L}^2\,\mathrm{T}^{-2}$ (N m). $\quad [\rho g L^4] = \mathrm{M}\,\mathrm{L}^{-3}\cdot\mathrm{L}\,\mathrm{T}^{-2}\cdot\mathrm{L}^4 = \mathrm{M}\,\mathrm{L}^2\,\mathrm{T}^{-2}$. ✓

Walking adds inertial torques $I\ddot{\theta}$ with $I \propto mL^2 \propto L^5$ and $\ddot{\theta} \propto g/L$ (next slides): again $\propto L^4$.

Torque a Motor Can Supply



Definition 1 (Scaling exponent α)
If every length of a motor is multiplied by k, its maximum torque is multiplied by k^{α}. Equivalently, $\tau_{\text{motor}} \propto L^{\alpha}$.

Where torque comes from. A wire carrying current in a magnetic field B feels the Lorentz force. Per unit volume of winding with current density J (A/m^2), force $= BJ$. Over the active volume V at radius r:

$$\tau_{\text{motor}} \approx B\,J\,V\,r, \qquad V \propto L^3, \quad r \propto L \quad \Longrightarrow \quad \tau_{\text{motor}} \propto B\,J\,L^4.$$

Torque a Motor Can Supply

B is set by the magnet material (about 1 T for any size). So α is decided entirely by **how J changes with size**:

1. J **the same at all sizes:** $\tau \propto L^4$, so $\alpha = 4$.
2. J **limited by heating.** Copper loss (heat made) $\propto J^2 V \propto J^2 L^3$. Heat removed through the surface $\propto L^2$ (same allowed temperature rise). Setting them equal: $J^2 L^3 \propto L^2 \Rightarrow J \propto L^{-1/2}$, so $\tau \propto L^{-1/2} L^4 = L^{7/2}$: $\alpha = 7/2$.
3. **Catalogue rule:** for real gearmotors, torque per kilogram is roughly constant, so $\tau \propto$ mass $\propto L^3$: $\alpha = 3$.

Torque Margin: Who Wins, Small or Large?

Compare with the torque needed, $\tau_{\text{req}} \propto L^4$:

$$\frac{\tau_{\text{motor}}}{\tau_{\text{req}}} \propto \frac{L^\alpha}{L^4} = L^{\alpha-4}.$$

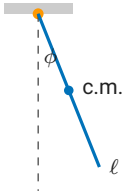
Case	α	margin $\propto$	halve the robot ($k = \frac{1}{2}$): margin $\times$
J fixed	4	L^0	1 (no change)
heating limit	$7/2$	$L^{-1/2}$	$\sqrt{2} \approx 1.41$
torque per kg constant	3	L^{-1}	2

Torque Margin: Who Wins, Small or Large?

Reading the result

Whenever $\alpha < 4$, a smaller robot has more torque to spare relative to its weight. Only an ideal motor with no heating limit ($\alpha = 4$) shows no size effect. Which case applies is an **assumption** for the ledger: test it by plotting $\log(\text{rated torque})$ against $\log(\text{motor mass})$ from a datasheet. A slope of 1 means $\alpha = 3$.

Natural Step Frequency: The Swinging Leg



During swing, the leg hangs from the hip: a **physical pendulum**. Uniform leg, length ℓ, mass m_ℓ, centre of mass at $\ell/2$, moment of inertia $I = \frac{1}{3} m_\ell \ell^2$:

$$I\ddot{\phi} = -m_\ell g \, \frac{\ell}{2} \sin\phi.$$

Small angles, $\sin\phi \approx \phi$:

$$\ddot{\phi} + \frac{3g}{2\ell}\phi = 0 \implies \omega = \sqrt{\frac{3g}{2\ell}}, \qquad \boxed{f = \frac{1}{2\pi}\sqrt{\frac{3g}{2\ell}} \propto L^{-1/2}}$$

Natural Step Frequency: The Swinging Leg

Same answer from dimensional analysis alone

$f = F(g, \ell, m_\ell)$. Mass appears in no other variable, so it drops out. $[f] = \mathrm{T}^{-1}$, $[g] = \mathrm{L}\,\mathrm{T}^{-2}$, $[\ell] = \mathrm{L} \Rightarrow f\sqrt{\ell/g} =$ const. The pendulum model only supplies the constant $\sqrt{3/2}/2\pi$.

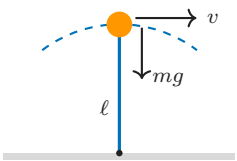
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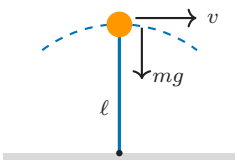
Walking near this natural frequency costs the least energy: the leg swings itself.

Maximum Walking Speed: The Inverted Pendulum



In stance, the body vaults over a straight leg: the hip moves on a circular arc of radius ℓ about the foot.

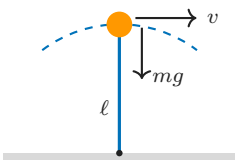
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Moving on a circle needs a centripetal force mv^2/ℓ towards the foot. Only gravity can supply it (feet cannot pull on the ground):

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$$\frac{mv^2}{\ell} \le mg \quad \Longrightarrow \quad \boxed{v_{\max} = \sqrt{g\,\ell} \;\propto\; L^{1/2}}$$

Maximum Walking Speed: The Inverted Pendulum

Check with the frequency result: speed $=$ step length $\times$ step frequency $\propto \ell \cdot \sqrt{g/\ell} = \sqrt{g\ell}$. ✓

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Meaning

Above $\sqrt{g\ell}$, the body would lift off the ground at the top of the arc. A walker *must* switch to running. Small walkers are slow walkers.

The Froude Number: Derivation

Buckingham Pi. Walking speed depends on $v = F(g, \ell, m)$: $n = 4$ variables, $m = 3$ dimensions (M, L, T) $\Rightarrow$ one group.

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$$\mathrm{L\,T^{-1}\,(L\,T^{-2})^a\,L^b\,M^c = L^0T^0M^0}: \qquad \begin{cases} \mathrm{M}: & c = 0 \\ \mathrm{T}: & -1 - 2a = 0 \Rightarrow a = -\frac{1}{2} \\ \mathrm{L}: & 1 + a + b = 0 \Rightarrow b = -\frac{1}{2} \end{cases}$$

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$$\Pi = \frac{v}{\sqrt{g\ell}}, \qquad \boxed{\mathrm{Fr} = \frac{v^2}{g\,\ell}}$$

Physical meaning: $\mathrm{Fr} = \dfrac{mv^2/\ell}{mg} = \dfrac{\text{centripetal (inertial) force}}{\text{weight}}$.

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- Inverted pendulum: walking is possible only for $\mathrm{Fr} < 1$.

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- Inverted pendulum: walking is possible only for $\mathrm{Fr} < 1$.
- Humans and many animals switch from walking to running near $\mathrm{Fr} \approx 0.5$ (Alexander, 1989).
- **Dynamic similarity:** two walkers move alike when their Froude numbers are equal. Then $\dfrac{v_p}{v_m} = \sqrt{\dfrac{\ell_p}{\ell_m}}$ and times scale as $\dfrac{t_p}{t_m} = \sqrt{\dfrac{\ell_p}{\ell_m}}$.

Numbers: Microduck vs. a Human

Assumption: Microduck leg length $\ell \approx 0.10$ m (our estimate from its 25 cm height); human $\ell \approx 0.90$ m. $g = 9.81$ m/s^2.

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	Microduck	Human	Ratio
leg swing frequency $f = \frac{1}{2\pi}\sqrt{3g/2\ell}$	1.93 Hz	0.64 Hz	3
$v_{\max} = \sqrt{g\ell}$ (Fr $= 1$)	0.99 m/s	2.97 m/s	$1/3$
walk–run speed (Fr $= 0.5$)	0.70 m/s	2.10 m/s	$1/3$
$\tau_{\text{motor}}/\tau_{\text{req}}$ (if $\alpha = 3$)	$9\times$	$1\times$	9

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- Length ratio 9, so frequencies differ by $\sqrt{9} = 3$.
- Human walk–run transition at about 2 m/s matches what is measured: a check on the model.
- The duck's control loop must be about three times faster than a human-sized robot's for the same gait.

Falls, and Where Similarity Breaks



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Where the Microduck is *not* a shrunk humanoid

- Off-the-shelf servos: not scaled with the body, so relatively heavy.
- Gear backlash, joint friction and sensor noise do not scale down with L; they matter more at small size.
- This is one cause of the gap between simulation and reality. It is the Lilliputians' error: assuming everything follows one scaling rule.

Exercises: Microduck



Exercise 1: Scale it up

Enlarge the Microduck by a factor of 7 (1.75 m). By what factors do mass, torque needed, torque available, step frequency, and fall energy per volume change? Do the torque part for $\alpha = 3$ and $\alpha = 7/2$. Which factor most explains why a $400 duck is possible but a $400 humanoid is not?

Exercise 2: Point-mass leg

Redo the frequency derivation with all leg mass at the foot ($I = m\ell^2$). How does the constant change? Does the exponent change? What does that say about dimensional analysis?

Exercises: Microduck



Exercise 3: Measure Fr (hostel)

Walk across the corridor at increasing speeds until you must run. Time it with a phone and measure your leg length. Compute your Fr at transition and compare with 0.5. Collect data from the whole class and plot.

From One Classroom to the Institute

You counted switches, fans, and data ports in classrooms. Now estimate the totals for IIT Tirupati, as the Lilliputians did from one thumb.

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- **Scaling question:** fans and switches should grow with floor area ($\propto L^2$); wall data ports with perimeter ($\propto L$).

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$$N_{\text{total}} = \sum_{k} (\text{number of rooms of type } k) \times (\text{mean count per room of type } k).$$

- **Scaling question:** fans and switches should grow with floor area ($\propto L^2$); wall data ports with perimeter ($\propto L$).
- Plot $\log(\text{count})$ against $\log(\text{floor area})$ for each item. Is the slope near 1 or near $1/2$?

From One Classroom to the Institute

You counted switches, fans, and data ports in classrooms. Now estimate the totals for IIT Tirupati, as the Lilliputians did from one thumb.

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Ledger items you must state

Is a board of six one switch or six? Do broken fans count? Are sampled rooms typical?

How Many Leaves on a Tree?

Here a census is impossible. Like the seamstresses, you must measure a part.



How Many Leaves on a Tree?

Here a census is impossible. Like the seamstresses, you must measure a part.

- **Chain of units:**

  $$N_{\text{leaves}} = (\text{leaves per twig}) \times (\text{twigs per branch}) \times (\text{branches per tree}).$$

  Relative errors combine (roughly in quadrature): one careless factor spoils the estimate.

- **Shape rule (pipe model, Shinozaki et al., 1964):** the leaf area a trunk can supply is proportional to its sapwood cross-section, so $N_{\text{leaves}} \propto (\text{girth})^2$. Count small trees or single branches completely, measure the girth of many trees with a tape, and fit a log–log line.

Ledger items you must state
Neem and tamarind have compound leaves: leaf or leaflet? (A factor of ten or more.) Sunny vs. shaded side. Date of counting.

Report a range, not a single number, and say which factor dominates the uncertainty.

Recall: Buckingham Π Theorem



- Every model equation must be **dimensionally homogeneous**.
- Buckingham Pi: n variables in m primary dimensions reduce to $n - m$ dimensionless groups, $\Pi_1 = \Phi(\Pi_2, \ldots, \Pi_{n-m})$.
- Dimensional analysis tells us *which* groups matter, not the function Φ.

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- Dimensional analysis tells us *which* groups matter, not the function Φ.

Today: dimensions told us how quantities are *measured*. Scale tells us how they *change* when the object gets bigger or smaller, and when a model stops being valid.

Dym's four questions about scale



Dym's four questions about scale

1. Abstraction: what level of detail does the question need?
2. Size and shape: how do areas, volumes and masses change with length?
3. Size and function: how does size limit what a system can do?
4. Choice of scale: how does the scale we choose affect data, experiments and their interpretation?

Roadmap

#	Topic	Model of the lecture
1	Abstraction and scale	lumped spring; Newton vs. relativity
2	Size and shape	Gulliver; flight muscles; cathedrals
3	Size and function I	the hovering limit of birds
4	Size and function II	hearing and speech
5	Size and limits	scale factors, characteristic times
6	Consequences of a scale	data, experiments, similitude, graphs

Model of the Week: the hovering limit (Section 3), developed with both Dym's ten questions and Velten's (S, Q, M).

Model Workshop: cooling time of hot water in vessels of different sizes (hostel experiment, thermometer only).

Thanks

Doubts and Suggestions

panch.m@iittp.ac.in





MA621L-Mathematical Modeling

Lecture 12 : Scaling: Gulliver to Microduck Problem

Panchatcharam Mariappan[1]

[1]Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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