MA621L-Mathematical Modeling

Lecture 14 : Scaling: 4 Questions

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Recap

Recall: Buckingham Π Theorem



- Every model equation must be **dimensionally homogeneous**.
- Buckingham Pi: n variables in m primary dimensions reduce to $n - m$ dimensionless groups, $\Pi_1 = \Phi(\Pi_2, \ldots, \Pi_{n-m})$.
- Dimensional analysis tells us *which* groups matter, not the function Φ.

Recall: Buckingham Π Theorem



- Every model equation must be **dimensionally homogeneous**.
- Buckingham Pi: n variables in m primary dimensions reduce to $n - m$ dimensionless groups, $\Pi_1 = \Phi(\Pi_2, \ldots, \Pi_{n-m})$.
- Dimensional analysis tells us *which* groups matter, not the function Φ.

Today: dimensions told us how quantities are *measured*. Scale tells us how they *change* when the object gets bigger or smaller, and when a model stops being valid.

Dym's four questions about scale



Dym's four questions about scale

1. Abstraction: what level of detail does the question need?
2. Size and shape: how do areas, volumes and masses change with length?
3. Size and function: how does size limit what a system can do?
4. Choice of scale: how does the scale we choose affect data, experiments and their interpretation?

Roadmap

#	Topic	Model of the lecture
1	Abstraction and scale	lumped spring; Newton vs. relativity
2	Size and shape	Gulliver; flight muscles; cathedrals
3	Size and function I	the hovering limit of birds
4	Size and function II	hearing and speech
5	Size and limits	scale factors, characteristic times
6	Consequences of a scale	data, experiments, similitude, graphs

Model of the Week: the hovering limit (Section 3), developed with both Dym's ten questions and Velten's (S, Q, M).

Model Workshop: cooling time of hot water in vessels of different sizes (hostel experiment, thermometer only).



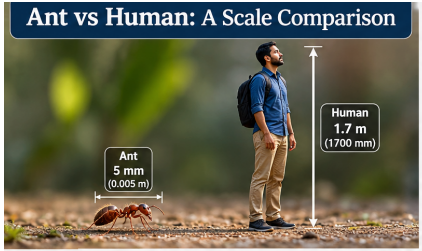
1. Abstraction and Scale

One Equation, Three Scales



Definition 1 (Abstraction)
Abstraction is the choice of the level of detail a model must carry to answer its question. Choosing the level of detail is the same act as choosing the scale of the model.

One Equation, Three Scales



Key Numbers

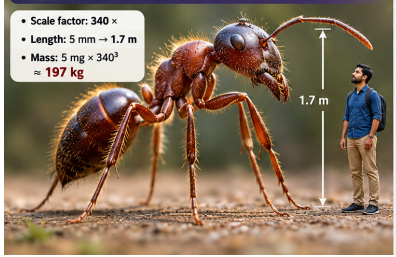
Property	Ant (typical)	Human (typical)	Ratio (Human / Ant)
Length / Height	5 mm	1.7 m (1,700 mm)	**340**
Mass	≈ 5 mg (0.005 g)	≈ 70 kg (70,000 g)	≈ **14 million**
Volume ($\propto L^3$)	1.25×10^{-7} m³	0.10 m³	≈ **39.3 million**
Density	≈ 1,000 kg/m³ (≈ water)	≈ 1,000 kg/m³	≈ **1**

Length ratio: 1.7 m / 5 mm = **340**

Volume (and mass) ratio: $340^3 = 39{,}304{,}000 \approx 3.93 \times 10^7$

If an Ant Is Enlarged to Human Size

- **Scale factor: 340 ×**
- **Length:** 5 mm → **1.7 m**
- **Mass:** 5 mg × 340^3 ≈ **197 kg**



Why a Giant Ant Would Not Be as Strong

Square–Cube Law

- Muscle force ∝ cross-sectional area ∝ L^2
- Body weight ∝ volume ∝ L^3
- As size increases, weight grows faster than muscle force.



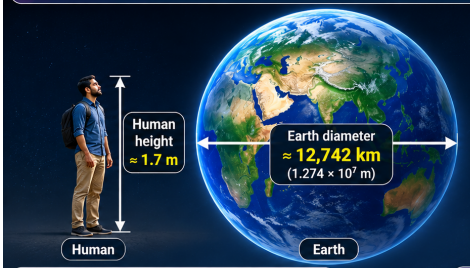
Relative strength per unit weight scales as **1/L**.
So, a human-sized ant would be ≈ 1/340 as strong (relative to its weight) compared to a real ant.

Key Takeaways

- A human is ≈ **340 times taller** than an ant.
- Volume and mass are ≈ **39 million** times larger.
- An ant scaled to human height would weigh ≈ **197 kg** (assuming the same density).
- It would not retain the ant's extraordinary strength because of the square-cube law.
- In short: ants are incredibly strong for their size, but simply making them human-sized would not make them super-strong.

One Equation, Three Scales

Human vs Earth: Size Comparison



Property	Human (typical)	Earth	Ratio (Earth / Human)
Height / Diameter	1.7 m	12,742 km (1.274×10^7 m)	$\approx 7.5 \times 10^6$
Volume	≈ 0.07 m^3 (≈ 70 L)	1.08×10^{12} km^3 (1.08×10^{21} m^3)	$\approx 1.5 \times 10^{22}$
Mass	≈ 70 kg	5.97×10^{24} kg	$\approx 8.5 \times 10^{22}$

- Earth's diameter is ≈ **7.5 million** (7,495,000) times a human's height.
- Earth's volume is ≈ **1.5×10^{22} times larger.**
- Earth's mass is ≈ **8.5×10^{22} times larger.**

If a Human Were Scaled Up to Earth's Size...

- Scale factor: 7.5×10^6
- **Height:** 1.7 m → 12,742 km
- **Mass:** 70 kg × $(7.5 \times 10^6)^3$ ≈ 2.95×10^{22} kg (about 0.5% of Earth's mass)

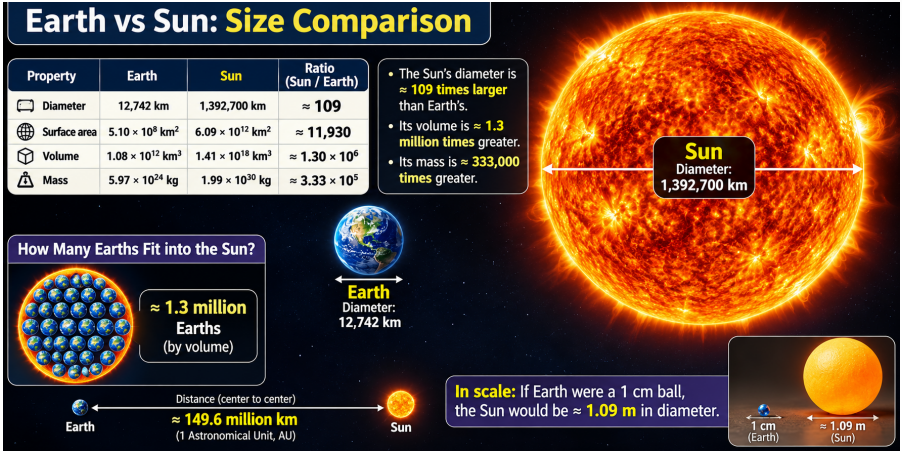


Perspective: How Many Humans Fit on Earth?



- Earth's surface area: 5.10×10^8 km^2 = **5.10×10^{14} m^2**
- If each person needs 1 m^2, the surface could hold ≈ **5.1×10^{14} people** (≈ **510 trillion**) in a single layer.

One Equation, Three Scales



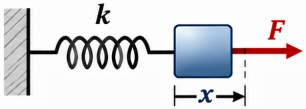
One Equation, Three Scales

One Equation, Three Scales: Linear Spring

$$F = kx$$

F : restoring force (N)
k : effective stiffness (N m^{-1})
x : displacement (m)

The same equation, $F = kx$, describes elastic behavior at very different scales.



System	What k lumps together	Scale
Diving board	material, length, cross-section of the board	metres (≈ 1 – 5 m)



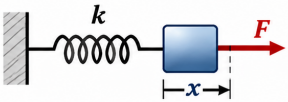
- The diving board behaves like a spring.
- Its effective stiffness k depends on the board's material (e.g., steel, wood, composite), length, and cross-sectional geometry.
- A larger k (stiffer board) gives a smaller deflection x for the same load F.

One Equation, Three Scales

One Equation, Three Scales: Linear Spring

$$F = kx$$

The linear spring law $F = kx$, captures elastic behavior at very different scales.



System	What k lumps together	Scale
Tall building	framing, geometry, materials of the whole frame	10^2 metres (≈ 100 m)



- A tall building can be modeled as an effective spring.
- The effective stiffness k lumps together the entire structural system: framing, geometry, and material properties (e.g., steel, concrete).
- Under a lateral load F (e.g., wind or earthquake), the building deflects by x.

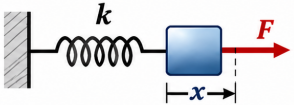
One Equation, Three Scales

One Equation, Three Scales: Linear Spring

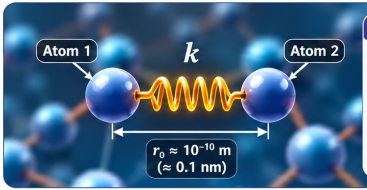
$$F = kx$$

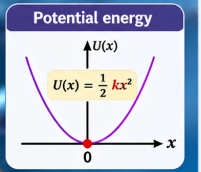
F : restoring force (N)
k : effective stiffness (N m^{-1})
x : displacement (m)

The same linear spring law
$F = kx$
also describes the elasticity of an atomic bond, at a nanometre scale.



System	What k lumps together	Scale
Atomic bond	interatomic forces and distances	10^{-10} metres (≈ 0.1 nm)





- Near its equilibrium distance r_0, the interatomic potential is approximately quadratic: $U(x) \approx \frac{1}{2} kx^2$.
- Hence, the bond force is $F = -\frac{dU}{dx} = kx$ (Hooke's law), with k set by the strength of the atomic interaction and the bond length.

One Equation, Three Scales: Linear Spring

The linear spring $F = kx$ serves at three very different scales:

System	What k lumps together	Scale
diving board	material, length, cross-section of the board	metres
tall building	framing, geometry, materials of the whole frame	10^{2} metres
atomic bond	interatomic forces and distances	10^{-10} metres

- The equation does not change. What k *means* does.
- Validity check for the board: load it with divers of different weight and plot tip deflection. A straight line supports the spring model.

Lumped Element Models



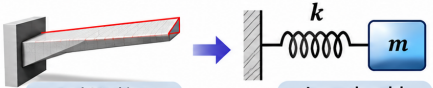
Definition 2 (Lumped element model)
A model in which the distributed properties of a real three-dimensional object are aggregated into a few parameters, such as a single mass, stiffness or resistance.

An aeroplane, three questions, three models:

Lumped Element Models

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A model in which the distributed properties of a real three-dimensional object are aggregated into a few parameters, such as a single mass, stiffness or resistance.



Real 3D object (distributed properties)

Lumped model (few parameters)

Lumped Element Models

An aeroplane, three questions, three models:

1 ***Flight plan:***

a point mass moving over a sphere, with a drag force depending on speed.



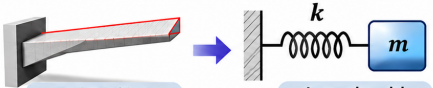
Lumped model (point mass)

- **State:** position $r(t)$ on a sphere ($r = R_{\text{Earth}}$).
- **Forces:** drag $F_D = C_D v^2$ (opposes motion).
- **Model:** $m\ddot{r} = -C_D v v +$ (gravity/constraint).

Lumped Element Models

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Real 3D object (distributed properties)

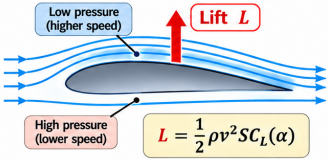
Lumped model (few parameters)

Lumped Element Models

An aeroplane, three questions, three models:

2 *Lift on the wing:*

wing surface area and aerodynamics enter; the point mass is no longer enough.



Aerodynamic model (wing)

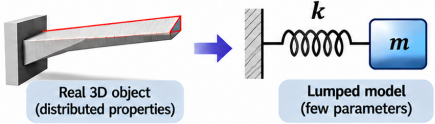
- **Model:** wing represented by its area S and aerodynamic coefficients, e.g., lift coefficient $C_L(\alpha)$ (depends on angle of attack α).
- **Lift:** $L = \frac{1}{2}\rho v^2 S C_L(\alpha)$.
- **State:** at least the aircraft's position, velocity and attitude (e.g., α, pitch).

An aeroplane, three questions, three models:

Lumped Element Models

Definition (Lumped element model)

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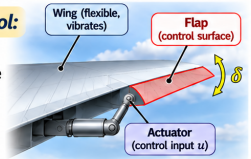


Lumped Element Models

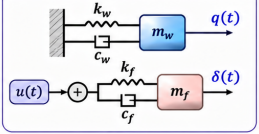
An aeroplane, three questions, three models:

3 *Flap control:*

add a control system and the vibration of the wing.



Lumped model (wing–flap–control)



- **States:** wing vibration $q(t)$ (e.g., tip deflection) and flap angle $\delta(t)$.
- **Lumped dynamics:**

$$\begin{cases} m_w\ddot{q} + c_w\dot{q} + k_w q = a\,\delta, \\ m_f\ddot{\delta} + c_f\dot{\delta} + k_f\delta = bu(t). \end{cases}$$

- **Purpose:** actively move the flap (δ) to reduce wing vibration (q) and improve stability.

Lumped Element Models



An aeroplane, three questions, three models:

1. *Flight plan:* a point mass moving over a sphere, with a drag force depending on speed.
2. *Lift on the wing:* wing surface area and aerodynamics enter; the point mass is no longer enough.
3. *Flap control:* add a control system and the vibration of the wing.

Scale Relative to a Standard

We call a quantity large or small only by comparing it with a reference of the ***same dimensions***.

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-(v/c)^2}} \quad \text{(relativistic mass)}$$

- The only thing that matters is the dimensionless ratio v/c.

Car: $v = 30$ m/s



$\frac{v}{c} \approx 10^{-7}$

Airplane: $v = 250$ m/s



$\frac{v}{c} \approx 8 \times 10^{-7}$

Rocket: $v = 0.8c$



$\frac{v}{c} = 0.8$

$\Longrightarrow$ The ratio $\frac{v}{c}$ tells us how large or small the quantity is, **relative to the standard.**

Scale Relative to a Standard

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$$\frac{m}{m_0} = \frac{1}{\sqrt{1-(v/c)^2}} \qquad \text{(relativistic mass)}$$

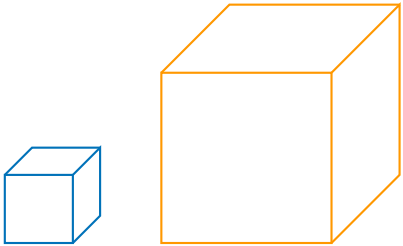
- The only thing that matters is the dimensionless ratio v/c.
- A rocket at escape speed, $45{,}000$ km/h, has $v/c \approx 4 \times 10^{-5}$: it "stands still" compared to light.
- Newtonian mechanics is the model obtained by setting $(v/c)^2 \approx 0$.

Never say "z is small"
Say "$z \ll R_e$" or "$v \ll c$". A smallness statement without a reference has no meaning.



2. Size and Shape: Geometric Scaling

Two Cubes



	unit cube	side L
length	1	L
surface area	6	$6L^2$
volume	1	L^3
area/volume	6	$6/L$

Geometric similarity



Definition 3 (Geometric similarity)
Two objects are geometrically similar if every length of one is the same multiple L of the corresponding length of the other. Then areas scale as L^2 and volumes (and masses, at fixed density) as L^3.

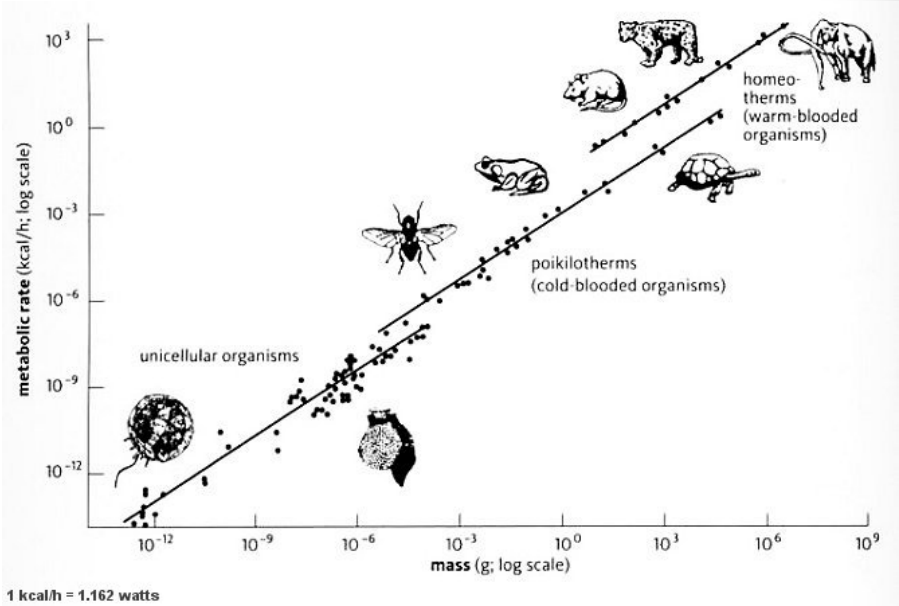
Doubling the side multiplies area by 4 and volume by 8. The area-to-volume ratio *falls* as $1/L$: this single fact drives most of today's examples.

Gulliver Revisited: Three Models

Let $k = 12$ be the length ratio and assume geometric similarity.

Assumption about food	Scaling law	Exponent	Rations
maintains body mass	$\propto L^3$	3	1728
replaces heat lost through skin	$\propto L^2$	2	144
empirical metabolic rate, $B \propto M^{3/4}$	$\propto L^{9/4}$	2.25	≈ 268

Kleiber's Law



1 kcal/h = 1.162 watts

Kleiber's Law (1932): Metabolic Scaling in Mammals



Max Kleiber (1893–1976)

Kleiber's law (1932): the basal metabolic rate (BMR) of mammals scales with body mass M as

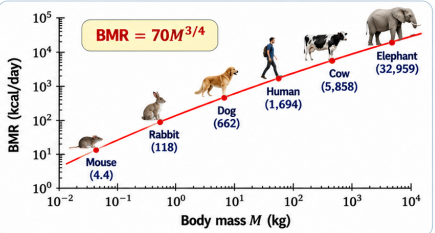
$$\text{BMR} = a\,M^{3/4}$$

where M is body mass (kg) and $a \approx 70\ \text{kcal day}^{-1}\,\text{kg}^{-3/4}$ for mammals.

- It is between linear scaling (M^1, proportional to mass) and surface-area scaling ($M^{2/3}$).
- Larger mammals use more total energy, but less energy per unit mass: $\text{BMR}/M \propto M^{-1/4}$.

Examples (Basal Metabolic Rate, BMR)

Mammal	Body mass M (kg)	BMR $= 70M^{3/4}$ (kcal/day)
Mouse	0.025	≈ 4.4
Rabbit	2	≈ 118
Dog	20	≈ 662
Human	70	$\approx 1{,}694$
Cow	500	$\approx 5{,}858$
Elephant	5,000	$\approx 32{,}959$



Takeaway: BMR increases with size as $M^{3/4}$, an exponent between $2/3$ (surface area) and 1 (mass).

Kleiber's Law (1932): Metabolic Scaling in Mammals



Max Kleiber
(1893–1976)

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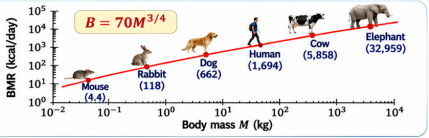
$$B = aM^{3/4}$$

where B is energy per unit time (e.g., kcal/day), M is body mass (kg), and $a \approx 70$ kcal day^{-1} kg$^{-3/4}$.

- It is between linear scaling (M^1, proportional to mass) and surface–area scaling ($M^{2/3}$).
- Larger mammals use more total energy, but less energy per unit mass: $B/M \propto M^{-1/4}$.

Examples: Basal Metabolic Rate (BMR) in Calories

Mammal	Body mass M (kg)	BMR B (kcal/day)	B/M (kcal/kg/day)
Mouse	0.025	≈ 4.4	≈ 176.0
Rabbit	2	≈ 118	≈ 59.0
Dog	20	≈ 662	≈ 33.1
Human	70	≈ 1,694	≈ 24.2
Cow	500	≈ 5,858	≈ 11.7
Elephant	5,000	≈ 32,959	≈ 6.6



1 What does this mean in terms of calories?

If an animal's body mass increases by a factor r, its basal energy requirement increases approximately by

$$\frac{B_2}{B_1} = \left(\frac{M_2}{M_1}\right)^{3/4}.$$

Example: If an animal is 16 times heavier,

$$16^{3/4} = 8.$$

Thus, it requires about **8** times as much energy, rather than **16** times.

2 Calories per kilogram decrease with body size

Divide by M:

$$\frac{B}{M} = aM^{-1/4}.$$



Human (70 kg)

$B \approx 1{,}694$ kcal/day
$B/M \approx 24.2$ kcal/kg/day



Elephant (5,000 kg)

$B \approx 32{,}959$ kcal/day
$B/M \approx 6.6$ kcal/kg/day

Note: These are illustrative estimates from a single scaling relation, not precise dietary recommendations. Actual metabolism depends on species, age, temperature, and physiological conditions.

Gulliver Revisited: Three Models



- The Lilliputians used a **volume** argument
- A warm-blooded body loses heat through its surface the surface law.
- Kleiber's law (1932): basal metabolic rate of mammals scales as $M^{3/4}$, between the two.

Unstated assumptions cost more than arithmetic errors
All three calculations are arithmetically correct. They differ only in what was assumed. Write the assumption first.

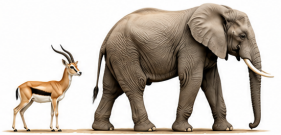
Galileo's Bones (1638)



Galileo Galilei
(1564–1642)

Why? **Why does an elephant not look like a large gazelle?**

- Weight on a leg bone: $W \propto L^3$.
- Load-bearing area of the bone: $A \propto d^2$, where d is its diameter.
- Stress in the bone: $\sigma = \dfrac{W}{A}$.



Scaling the bone

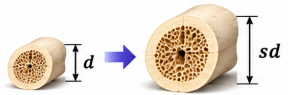
If all linear dimensions are scaled by a factor $s > 1$ (a "bigger copy"), then

$$\begin{cases} W \propto L^3 & \Rightarrow \quad W' = s^3 W, \\ A \propto d^2 & \Rightarrow \quad A' = s^2 A, \\ \sigma = \dfrac{W}{A} & \Rightarrow \quad \sigma' = \dfrac{W'}{A'} = \dfrac{s^3}{s^2}\sigma = s\sigma. \end{cases}$$

Implication

- Stress increases linearly with size: $\sigma' = s\sigma$.
- A large copy of a small animal would have bones that are proportionally too weak.

Visual: Why bones must be thicker



Takeaway: As size increases, bones must become disproportionately thicker ($d \propto L^{3/2}$) to keep the same stress. Hence, an elephant cannot be a geometrically similar, large gazelle.

Galileo's Bones (1638)



Why? Why does an elephant not look like a large gazelle?

- Weight on a leg bone: $W \propto L^3$.
- Load-bearing area of the bone: $A \propto d^2$, where d is its diameter.
- Stress in the bone: $\sigma = W/A$.

Geometrically similar animals ($d \propto L$): $\sigma \propto L^3/L^2 = L$. Stress grows with size until the bone fails.

Constant stress ($\sigma = \sigma_{\max}$ for all sizes): $d^2 \propto L^3$, so

$$d \propto L^{3/2}, \qquad \frac{d}{L} \propto L^{1/2}.$$

Galileo's Bones (1638)

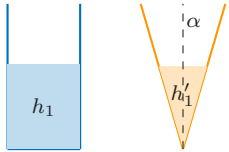


Interpretation
Large animals cannot stay geometrically similar. Their bones become relatively thicker. Shape is forced to change with size.

Galileo, *Two New Sciences*, Day Two. The same argument limits trees and sandcastles.

Linearity Is About Which Variable

Two pairs of glasses: cylinders of radius r, and cones with half-angle α.



Cylinders:

$$V_{cy} = \pi r^2 h_1 + \pi r^2 h_2 = \pi r^2 (h_1 + h_2)$$

Linear in h: superposition works. Not linear in r.

Cones:

$$V_{co} = \frac{\pi}{3} \tan^2\alpha \; h^3$$

$$\frac{\pi}{3} \tan^2\alpha \, (h_1'^3 + h_2'^3) \neq \frac{\pi}{3} \tan^2\alpha \, (h_1' + h_2')^3$$

Linearity Is About Which Variable



Note

The volume of a cone of height h and half-angle α is $\frac{\pi}{3}h^3 \tan^2 \alpha$ (radius $h \tan \alpha$). Check printed formulas dimensionally and geometrically before using them.

Power Laws and Log–Log Plots



Definition 4 (Allometric (power-law) relation)
$Y = aX^b$, with a carrying units and b a dimensionless exponent. $b = 1$: isometric (proportional). $b \neq 1$: allometric.

Taking logarithms,

$$\log Y = \log a + b \log X,$$

a straight line of slope b on log–log axes

Power Laws and Log–Log Plots



- Data spread over several orders of magnitude becomes readable.
- The exponent is read off as a slope and compared with a scaling prediction ($b = 1, 2/3, 3/4, 3, \ldots$).
- a depends on the units of X and Y; b does not. Report b first.

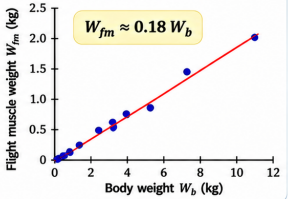
Fitting in log space is a modelling choice

Least squares on $\log Y$ assumes *multiplicative* errors of constant relative size. If errors are additive, the fitted exponent can be biased. Fitting $\neq$ truth $\neq$ usefulness.

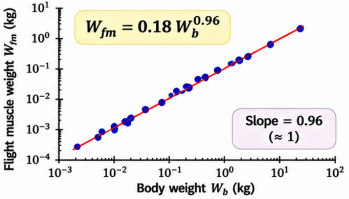
Flight Muscles of Birds

- 14 birds on linear axes: $W_{fm} \approx 0.18\, W_b$.
- All 29 birds, hummingbird to great bustard, need log axes. Regression gives $W_{fm} = 0.18\, W_b^{0.96}$.
- Over four decades, 0.96 is close enough to 1: flight muscle is about 18% of body weight.
- Extending the range exposed a small nonlinearity that the narrow range hid.

(a) 14 Birds: Linear Axes



(b) All 29 Birds: Log–Log Axes





Hummingbird
$W_b \approx 3 \times 10^{-3}$ kg
$W_{fm} \approx 5 \times 10^{-4}$ kg
(≈ 18%)

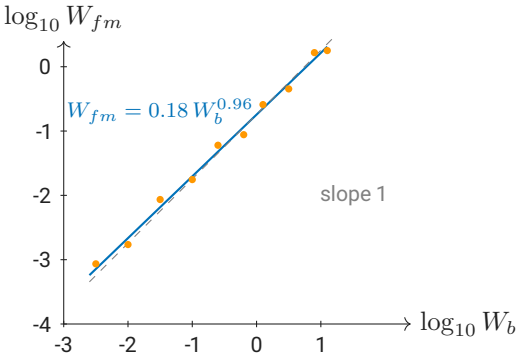
Four decades
(≈ 10^4 in body weight)



Great bustard
$W_b \approx 21$ kg
$W_{fm} \approx 3.8$ kg
(≈ 18%)

Takeaway: Across a very wide range of bird sizes (≈ 10^4), the flight muscle mass scales almost proportionally with body weight, $W_{fm} \approx 0.18\, W_b$, i.e., about 18% of the body weight.

Flight Muscles of Birds



Schematic of Dym Fig. 3.4 (Alexander, 1971); points illustrative.

- 14 birds on linear axes: $W_{fm} \approx 0.18\, W_b$.
- All 29 birds, from hummingbird to great bustard, need log axes. Regression gives

$$W_{fm} = 0.18\, W_b^{0.96}.$$

- Over four decades, 0.96 is close enough to 1: flight muscle is about 18% of body weight.
- Extending the range exposed a small nonlinearity that the narrow range hid.

Exercises: Size and Shape



Exercise 1: Spheres and balloons (Dym 3.3–3.4)

How do the surface area and volume of a sphere scale with radius R? What happens to the angle between two lines drawn on a balloon as it inflates from radius 1 to R? What does your answer say about geometric similarity?

Exercise 2: Stilt and flamingo (Dym 3.30–3.31)

A stilt weighs 4.5 oz with 8 in legs. A flamingo of similar shape weighs 4 lb. Show its legs should be about 20 in. Can you scale a robin (2 oz) from the stilt? Which assumption fails?

Exercises: Size and Shape



Exercise 3: Allometric fit

Show that $Y = aX^b$ fitted by least squares on $(\log X, \log Y)$ minimises the sum of squared *relative* errors to first order. When is that the right error model?

Thanks

Doubts and Suggestions

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MA621L-Mathematical Modeling

Lecture 14 : Scaling: 4 Questions

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