MA621L-Mathematical Modeling

Lecture 15 : Hovering Bird

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Model of the Week: How Big Can a Hovering Bird Be?

The Data

von Kármán (1954) plotted wing loading against weight for flying birds on log–log axes (Dym Fig. 3.9).

- A straight line of slope $1{:}3$ fits: $W \propto (\text{wing loading})^3$.
- No flying bird in the data heavier than about 35–40 lb.

Two questions:

1. Can dimensional reasoning explain the slope?
2. Is there an upper limit on the weight of a flying bird?

The Data

Slope. Under geometric similarity, $W \propto L^3$ and wing area $A \propto L^2$, so

$$\frac{W}{A} \propto \frac{L^3}{L^2} = L \qquad \Longrightarrow \qquad W \propto L^3 \propto \left(\frac{W}{A}\right)^3.$$

Consistent with the data. Dym calls W/A "wing loading per unit area"; aerodynamicists simply call W/A the wing loading.

Dym's Ten Questions for the Hovering Model

Why?	Explain why no large bird hovers, and why the data stop near 40 lb.
Find?	How power needed and power available scale with size L; the existence of a limit.
Given?	Air density ρ; bird size L; weight W; wing area A.
Assume?	Ledger A1–A7 (next slide).
How?	Momentum balance on the air jet; kinetic energy rate; heat, oxygen and muscle limits.
Predict?	$P_{\text{need}} \propto L^{7/2}$, $P_{\text{avail}} \propto L^{2}$, hence a maximum size.
Valid?	Dimensions of every step; exponents consistent with the wing-loading data.
Verified?	Largest true hoverers are hummingbirds (~ 20 g); large birds soar or hover only in a headwind.
Improve?	Lift A6 (Kleiber), A1 (non-similar wings), A5 (profile drag).
Use?	Size limits for flapping drones; why helicopters need engines, not muscles.

Assumption Ledger: Hovering Model

No.	Assumption	Liftable?
A1	Birds are geometrically similar (all lengths $\propto L$)	hummingbird wings are not
A2	Body density is the same for all birds $\Rightarrow W \propto L^3$	hollow bones vary
A3	Lift comes from a downward air jet through area $A \propto L^2$	actuator-disc model
A4	Air density ρ is the same for all birds	altitude dependence
A5	Power needed $=$ rate of kinetic energy given to the jet	add profile drag power
A6	Power available $\propto L^2$	Kleiber: $\propto L^{9/4}$
A7	Muscle efficiency ($\approx 25\%$) independent of size	measured data

Grading reminder

Every exponent below depends on some of A1–A7. When you quote a result, quote the assumptions it used.

Power Needed to Hover

How? The wings push air downward with speed v through an area A (A3).

$$\text{mass flux} = \rho A v, \qquad \text{momentum flux} = \rho A v \cdot v.$$

Newton's second law: the momentum flux equals the lift, which equals the weight,

$$W = \rho A v^2.$$

With $W \propto L^3$ (A1, A2), $A \propto L^2$ (A1), ρ constant (A4):

$$v^2 \propto \frac{L^3}{L^2} = L \quad \Longrightarrow \quad v \propto L^{1/2}.$$

Power is the rate of kinetic energy given to the jet (A5):

$$P_{\text{need}} \propto \tfrac{1}{2}(\rho A v)\, v^2 = \tfrac{1}{2}\rho A v^3 \propto L^2 \cdot L^{3/2} = \boxed{L^{7/2}}$$

Equivalently $P_{\text{need}} = Wv/2$ with $v = \sqrt{W/(\rho A)}$: power per unit weight grows like $L^{1/2}$.

Power Available: Three Independent Arguments



1. **Heat loss.** Muscles are about 25% efficient (A7); the rest is heat leaving through the skin, area $\propto L^2$. To avoid overheating, $P_{\text{avail}} \propto L^2$.
2. **Oxygen supply.** Power needs oxygen; oxygen flow $\propto$ blood flow $\propto$ cross-section of vessels $\propto L^2$.
3. **Muscle stress.** Muscle force $F \leq \sigma_{\max} \times$ area $\propto L^2$; contraction $\propto L$; limb mass $\propto L^3$. Work–energy:

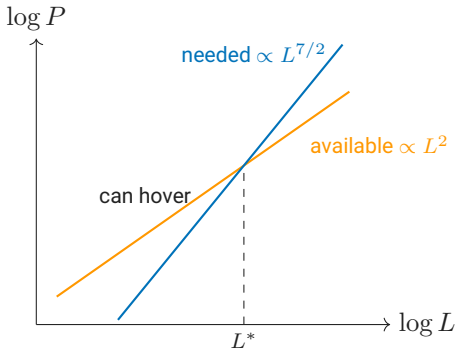
$$F \cdot \Delta\ell = \tfrac{1}{2} m_{\text{limb}} u^2 \;\Rightarrow\; L^2 \cdot L \propto L^3 u^2 \;\Rightarrow\; u \text{ independent of } L.$$

Contraction time $\propto L/u \propto L$, so power $= F\Delta\ell/t \propto L^2 \cdot L/L = L^2$.

Three arguments, one exponent

Agreement of independent arguments is a validity check (Dym's *Valid?*), not a proof. All three rest on A1.

So There Is a Hovering Limit



On log axes the two laws are lines of slope $7/2$ and 2.

$$\frac{P_{\text{need}}}{P_{\text{avail}}} \propto L^{3/2} \propto W^{1/2}.$$

So There Is a Hovering Limit

- Whatever the constants, the lines cross at some L^*.
- Above L^*, hovering is impossible.
- Doubling length needs $2^{3/2} \approx 2.8$ times more power per unit available.

What the model does not give

The *value* of L^*. Scaling fixes exponents, not constants. We need one measured hoverer to calibrate.

Improve? Lifting One Assumption at a Time

Lift A6. Replace $P_{\text{avail}} \propto L^2$ by Kleiber's law, $P_{\text{avail}} \propto M^{3/4} \propto L^{9/4}$:

$$\frac{P_{\text{need}}}{P_{\text{avail}}} \propto L^{7/2-9/4} = L^{5/4}.$$

Still increasing: the limit survives. The conclusion is **robust** to this assumption.

Lift A1. Allow wing area $A \propto L^{2+\epsilon}$ (relatively larger wings in big birds). Then

$$v^2 \propto L^{1-\epsilon}, \qquad P_{\text{need}} \propto L^{(7-\epsilon)/2}.$$

A limit exists whenever $(7-\epsilon)/2 > 2$, i.e. $\epsilon < 3$. Wings would have to grow absurdly fast to escape it.

Lesson

Test a conclusion by perturbing each exponent. A conclusion that survives every reasonable perturbation is worth trusting; one that flips is a warning.

Exercises: Hovering



Exercise 1: Checks (Dym 3.8–3.11)

Confirm $v \propto L^{1/2}$ and $P_{\text{need}} \propto L^{7/2}$. Check dimensionally that power is the rate of kinetic energy, and that u is independent of L.

Exercise 2: Running speed (Dym 3.28)

Using the same work–energy reasoning, show that the top running speed of similar animals is independent of size. Compare a mouse, a dog and a horse. Which assumption fails?

Exercises: Hovering

Exercise 3: Calibrate the limit

Suppose (illustrative numbers) a 4 g hummingbird needs half of its maximum available power to hover. Using $P_{\text{need}} \propto W^{7/6}$ and $P_{\text{avail}} \propto W^{2/3}$, estimate the largest hovering mass. State which ledger items your estimate depends on.



4. Size and Function: Hearing and Speech

Small Animals Hear High



Hearing ranges (Dym): humans 20–$20{,}000$ Hz, dogs up to $50{,}000$ Hz, bats up to $100{,}000$ Hz.

Model: the eardrum is a stretched circular membrane. Below its lowest natural frequency it behaves like a stiff spring and follows the incoming signal faithfully. Fundamental frequency:

$$f_{\text{membrane}} = \frac{2.40}{2\pi r}\sqrt{\frac{F}{\rho h}}$$

r: radius, h: thickness, F: tension per unit length, ρ: density.

> **Dimension check (Dym 3.13)**
> $[F] = \mathrm{M\,T^{-2}}$, $[\rho h] = \mathrm{M\,L^{-2}}$, so $\sqrt{F/\rho h}$ has units $\mathrm{L\,T^{-1}}$ and f has units $\mathrm{T^{-1}}$.

At similar F, ρ, h: $f \propto 1/r$. A smaller eardrum is stiffer and reaches higher frequencies.

Large Animals Speak Low

Two models for the voice:

- **Vocal cords as strings**:

$$f_{\text{string}} = \frac{1}{2\ell}\sqrt{\frac{F}{\rho A}}$$

  Longer, heavier cords $\Rightarrow$ lower pitch.

- **Larynx as a Helmholtz resonator**, neck area A, neck length ℓ, cavity volume V_0, sound speed c_0:

$$f_{\text{cavity}} = \frac{c_0}{2\pi}\sqrt{\frac{A}{\ell V_0}}$$

  Under geometric similarity, $A/(\ell V_0) \propto L^2/L^4$, so $f_{\text{cavity}} \propto 1/L$.

Common structure
All three frequencies have the form (speed)/(length). The length is set by anatomy, so pitch scales inversely with size. Birds chirp, bears growl.

Exercises: Hearing and Speech



Exercise 4: Dimensions (Dym 3.14–3.15)

Confirm that f_{string} and f_{cavity} have dimensions T^{-1}.

Exercise 5: Elephants and whales (Dym 3.12)

Look up the hearing ranges of an elephant and a blue whale. Does the $1/r$ model predict the right ordering? Elephants detect sound partly through their feet. Which assumption of the eardrum model does that violate?

Exercise 6: Bottle resonator (hostel)

Blow across the mouth of an empty bottle and record the note with a phone tuner app. Add water in steps to reduce V_0. Test $f \propto V_0^{-1/2}$ on a log–log plot and estimate the slope with an interval.



5. Size and Limits: Scale in Equations

When a Model Is No Longer Applicable

A box of sweets flown across the country at 965 km/h $= 268$ m/s:

$$\left(\frac{v}{c}\right)^2 = \left(\frac{268}{3\times 10^8}\right)^2 \approx 7.98\times 10^{-13}, \qquad \sqrt{1-(v/c)^2} \approx 1 - 3.99\times 10^{-13}.$$

- Newtonian mechanics is the leading-order model in the small parameter $\varepsilon = (v/c)^2$.
- The error is $O(\varepsilon)$; here, four parts in 10^{13}.
- The model has not become wrong at high speed. It has left its **range of validity**, which was stated by a dimensionless group from the start.

Definition 1 (Range of validity)
The set of values of a model's dimensionless groups for which its neglected terms are below a stated tolerance.

Writing down the range of validity is part of the model, not an afterthought.

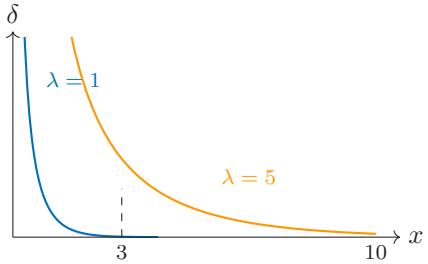
Scale Factors Move Boundaries

Approximate $\sinh x = \frac{1}{2}(e^x - e^{-x})$ by $\frac{1}{2}e^x$. The relative error is

$$\delta(x) = \frac{\frac{1}{2}e^x - \sinh x}{\sinh x} = \frac{1}{e^{2x} - 1}.$$

At $x = 3$: $\delta \approx 0.25\%$. Now introduce a scale factor λ:

$$\sinh \frac{x}{\lambda} \approx \frac{1}{2}e^{x/\lambda} \quad \text{acceptable for} \quad \frac{x}{\lambda} \geq 3 \iff x \geq 3\lambda.$$



- $\lambda = 1$: good for $x \geq 3$.
- $\lambda = 5$: good for $x \geq 15$.
- Changing λ moves the boundary of validity. The physics did not change; the ruler did.

Arguments of Transcendental Functions Are Dimensionless

$$e^{x/\lambda} = 1 + \frac{x}{\lambda} + \frac{1}{2!}\left(\frac{x}{\lambda}\right)^2 + \frac{1}{3!}\left(\frac{x}{\lambda}\right)^3 + \cdots$$

- If x/λ had units, the series would add metres to square metres. Not a rational equation.
- So every $\exp$, $\sin$, $\log$, $\sinh$ in a model hides a **characteristic scale** λ.
- That scale usually has meaning: for a catenary cable $y = c\,(\cosh(x/c) - 1)$, $c = T_0/\gamma$ is the ratio of horizontal tension to weight per length.

Practical rule

Whenever you write $e^{(\cdot)}$, find the scale inside the bracket and say what it is physically. If you cannot, the model is not finished.

Characteristic Times

A capacitor discharging through a resistor:

$$\frac{dV}{dt} = -\lambda V \quad \Longleftrightarrow \quad \frac{dV}{V} = -\lambda\, dt.$$

The left side is dimensionless, so $[\lambda] = \mathrm{T}^{-1}$ (Dym 3.16). Solution $V = V_0 e^{-\lambda t}$.

Definition 2 (Characteristic time)
A time scale built from the parameters of a model, such that the solution depends on t only through t/t_c.

- e-folding time: $t_c = 1/\lambda = RC$.
- Decay to $1/10$: $\lambda t_{\text{decay}} = \ln 10$, so $\lambda \approx 2.303/t_{\text{decay}}$ and $V \approx V_0 e^{-2.303\, t/t_{\text{decay}}}$.
- Half-life: $t_{1/2} = \ln 2/\lambda \approx 0.693/\lambda$.

All three are the same scale multiplied by a pure number. Choose whichever is easiest to measure.

Scaling an Equation: A Recipe



1. Write $u = u_c\,\tilde{u}$, $t = t_c\,\tilde{t}$ with unknown scales u_c, t_c.
2. Substitute and divide by the coefficient of one term.
3. Choose u_c, t_c so that as many coefficients as possible become 1.
4. What remains are dimensionless groups. Small ones are candidates for approximation.

RC circuit charged by a constant voltage V_0

$$R\frac{dq}{dt} + \frac{q}{C} = V_0. \qquad q = V_0 C\,\tilde{q}, \quad t = RC\,\tilde{t} \quad \Longrightarrow \quad \frac{d\tilde{q}}{d\tilde{t}} + \tilde{q} = 1.$$

No parameters left: every RC circuit is the same circuit, $\tilde{q} = 1 - e^{-\tilde{t}}$.

This is Chapter 2 applied to equations instead of variable lists. The number of groups that survive equals what Buckingham predicts.

Exercises: Size and Limits

Exercise 7: cosh (Dym 3.33)

Find the range of x for which $\cosh(x/\lambda) \approx \frac{1}{2}e^{x/\lambda}$ has relative error below 1%, for $\lambda = 1$ and $\lambda = 6$. Plot both.

Exercise 8: Decay time (Dym 3.17–3.18)

Verify $V = V_0 e^{-\lambda t}$ solves the ODE, and that $\lambda \approx 2.303/t_{\text{decay}}$.

Exercise 9: Scale the pendulum

Scale $m\ell\ddot{\theta} + mg\sin\theta = 0$ with $t = t_c\tilde{t}$. Which t_c removes all parameters? Which dimensionless quantity decides whether $\sin\theta \approx \theta$ is valid, and what is its range of validity at 1% error?



6. Consequences of Choosing a Scale

Scale and Data Acquisition: The Spinning Wheel



A falling weight drives a wheel of radius R through a string. Measure rim speed v_f at time t_f, starting from rest:

$$I_{\text{rot}} = \frac{\tau}{\alpha}, \qquad \alpha_{\text{exp}} = \frac{v_f}{R\,t_f}.$$

t_f (s)	revs	v_f (m/s)	α_{exp}
2	1/5	0.55	0.55
6	2.4	2.47	0.82
10	7	4.48	0.90
20	30	9.50	0.95
100	790	49.68	0.99

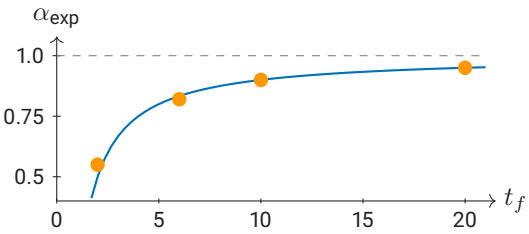
Dym Table 3.1 (α in rad/s^2).

- α_{exp} creeps up towards a constant.
- Static friction holds the wheel back at the start. In a short run this start-up occupies a large fraction of the run.
- The experiment's time scale t_f decides how large that bias is.

A Model for the Start-Up Bias

Assume: the wheel is effectively stuck for a time t_s, then accelerates at the true constant α_∞. Then $v_f = \alpha_\infty R(t_f - t_s)$ and

$$\alpha_{\text{exp}} = \alpha_\infty \left(1 - \frac{t_s}{t_f}\right).$$



With $\alpha_\infty = 1$ rad/s^2 and $t_s = 1$ s the curve matches the table to within 0.05.

- Relative bias $= t_s/t_f$: a ratio of two time scales.
- For 1% accuracy, run for $t_f \geq 100\, t_s$.
- Better: fit v_f against t_f and take the slope. The intercept absorbs t_s.

Scale Hides or Reveals: The Oscilloscope



A square wave goes into a suspected faulty amplifier.

- At 0.5 s/division the output looks like a clean square wave.
- At 0.5 ms/division and 0.5 μs/division, the edges ring and distort.

Rule

A measurement can only show features whose time scale lies between the resolution of the instrument and the length of the record. Before trusting “no effect”, ask at which scale you looked.

The same holds for spatial data: mesh size in simulation, pixel size in imaging, sampling interval in a survey.

Scale Models and Similitude

A beam of length L, bending stiffness EI, central load P, midpoint deflection w_{mp}:

$$w_{mp} = f(P, EI, L) \quad \xrightarrow{\text{Pi theorem}} \quad \Pi_1 = \frac{w_{mp}}{L}, \quad \Pi_2 = \frac{PL^2}{EI}, \quad \frac{w_{mp}}{L} = \Phi\Big(\frac{PL^2}{EI}\Big).$$

Definition 3 (Complete similarity)
Model (m) and prototype (p) are completely similar if every dimensionless group of the problem has the same value for both: $(\Pi_i)_m = (\Pi_i)_p$.

Define scaling factors (prototype/model):

$$n_w = \frac{(w_{mp})_p}{(w_{mp})_m}, \quad n_P = \frac{P_p}{P_m}, \quad n_E = \frac{E_p}{E_m}, \quad n_I = \frac{I_p}{I_m}, \quad n_L = \frac{L_p}{L_m}.$$

Similarity conditions:

$$\frac{n_w}{n_L} = 1, \qquad \frac{n_P n_L^2}{n_E n_I} = 1.$$

Five factors, two constraints: we may choose **three** freely.

Worked Example: Steel Beam, Balsa Model

Laboratory choices: $n_L = 20$, $n_I = 1000$, $n_E = 50$. Then

$$n_w = n_L = 20, \qquad n_P = \frac{n_E n_I}{n_L^2} = \frac{50 \times 1000}{400} = 125.$$

- Load the model with $1/125$ of the prototype load.
- The measured model deflection is $1/20$ of the prototype deflection.
- No knowledge of Φ was needed. That is the whole point of a scale model.

Dym's scaling factors n vs. graph scale factor λ

n_L is a ratio of two physical objects. λ in $\sinh(x/\lambda)$ is a reference length inside one model. Do not confuse them.

When Complete Similarity Is Impossible

A ship model in a towing tank. Wave drag depends on the Froude number, viscous drag on the Reynolds number:

$$\mathrm{Fr} = \frac{V}{\sqrt{gL}}, \qquad \mathrm{Re} = \frac{VL}{\nu}.$$

- Match Fr (same g): $V_m = V_p\sqrt{L_m/L_p}$.
- Match Re: $V_m L_m/\nu_m = V_p L_p/\nu_p$.
- Both together require $\nu_m/\nu_p = (L_m/L_p)^{3/2}$. For a $1{:}25$ model, a fluid 125 times less viscous than water. None is practical.

Partial similarity

Match Fr exactly, accept the Re mismatch, and correct the viscous part separately with a friction-line formula. Every towing tank works this way. The choice of which group to sacrifice is an assumption and belongs in the ledger.

Scale and Perception of Data

- **Compressed axis.** $\sinh(x/\lambda)$ and $\frac{1}{2}e^{x/\lambda}$ plotted on $0 \le x \le 60$ look identical, though they differ greatly near $x = 0$ (Dym Fig. 3.15b).
- **Stretched axis.** On $0 \le x \le 0.25$ the same two curves look very different (Fig. 3.15c).
- **Truncated time window.** Connecticut traffic deaths, 1955 vs. 1956, show a sharp drop after stricter speed enforcement. Over 1951–59, and normalised per $100{,}000$ people against neighbouring states, the drop is smaller and follows an unusually high year (Tufte, 1983; Fig. 3.16).
- **Truncated vertical axis.** A three-day Dow Jones chart with the axis from $10{,}060$ to $10{,}360$ makes a fall of about 2% look like a crash (Fig. 3.17).

Rules for presenting model results

Show the full range, label the origin if it is cut, normalise counts by the population at risk, and give a comparison group. The credibility of a model depends on how its output is shown.

Exercises: Consequences of Scale



Exercise 10: Beam (Dym 3.19 and 3.25)

Derive the two groups in $w_{mp} = f(P, EI, L)$ and rewrite the similarity conditions in terms of n_w, n_P, n_E, n_I, n_L.

Exercise 11: Timber prototype and steel model (Dym 3.36)

A 20 cm steel beam models a 3.6 m timber beam carrying 9000 N at 1.5 m from the left end. Using the group $P/(EL^2)$, find the model load and the model load position. Take $E_{\text{steel}} = 200$ GPa, $E_{\text{timber}} = 10$ GPa.

Exercises: Consequences of Scale (contd.)



Exercise 12: Beam frequencies (Dym 3.35)

Natural frequency ω depends on ρ, E, ℓ, h, A. Model and prototype use the same material with lengths in ratio $1{:}5$. How do their frequencies relate?

Exercise 13: Wheel (Dym 3.20–3.24)

Why is the torque constant? Express it in terms of the falling mass and the axle radius. Check the revolution counts in Table 3.1.



Model Workshop: Cooling Time and Size

Workshop: Does a Bigger Vessel Cool Slower?

Given: hot water, steel tumblers or vessels of several sizes, a thermometer, a phone stopwatch.

Lumped model (Newton cooling, uniform water temperature):

$$\rho c V \frac{dT}{dt} = -hA_s(T - T_a) \quad \Longrightarrow \quad T - T_a = (T_0 - T_a)e^{-t/t_c}, \qquad t_c = \frac{\rho c}{h}\frac{V}{A_s}.$$

- Geometrically similar vessels: $V/A_s \propto L$, so $t_c \propto L \propto V^{1/3}$.
- Doubling the volume should raise t_c by $2^{1/3} \approx 1.26$.

The trap
Filling *one* tumbler to different heights is not a geometrically similar family. Derive V/A_s for a cylinder of radius r filled to height H (open top), and decide what exponent you expect instead.

Workshop Deliverable

1. Dym's ten questions, in order, one or two lines each.
2. (S, Q, M) and the SQM classification.
3. Assumption ledger A1–An. Suggested start: uniform water temperature; constant h; no evaporation; vessel wall has no heat capacity; constant room temperature T_a; same h for all sizes.
4. Data: $T(t)$ for at least three sizes, every minute for 20 minutes.
5. Fit $\ln(T - T_a)$ against t for each size to get t_c. Plot $\log t_c$ against $\log(V/A_s)$ and report the slope with an interval.
6. Valid? Verified? Which ledger item explains the largest misfit?

Grading

Ledger 30 marks, mathematical execution 15. An honest misfit with the right assumption blamed scores higher than a perfect fit with no ledger.

Summary

Idea	Key relation	Example
geometric scaling	$A \propto L^2$, $V \propto L^3$, $A/V \propto L^{-1}$	Gulliver, cubes
allometry	$Y = aX^b$, slope on log–log	flight muscles, naves
size limits function	exponents of need vs. supply	hovering, bones
frequency scaling	$f \sim$ speed/length	hearing, speech
range of validity	dimensionless group $\ll 1$	v/c, x/λ
characteristic time	solution depends on t/t_c	RC circuit
similitude	$(\Pi_i)_m = (\Pi_i)_p$	beam, ship model
presentation	axis range changes the message	Tufte

Take-home

Scaling fixes exponents, not constants. It tells you what must change with size, where a model stops working, and how to test a big thing with a small one.

Thanks

Doubts and Suggestions

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MA621L-Mathematical Modeling

Lecture 15 : Hovering Bird

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