MA621L-Mathematical Modeling

Lecture 9 : Limitations, Properties, Errors and Non-uniqueness

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Errors

Mars Climate Orbiter Failure

Example 1

In 1998, NASA launched the Mars Climate Orbiter to study the Martian climate and weather patterns.

Mission: To enter orbit around Mars and send back valuable data about the planet's atmosphere. Precise calculations were required to ensure that the spacecraft would enter the correct orbit around Mars.

Mission Failed.

Link : https://science.nasa.gov/mission/mars-climate-orbiter/

Mars Climate Orbiter Failure

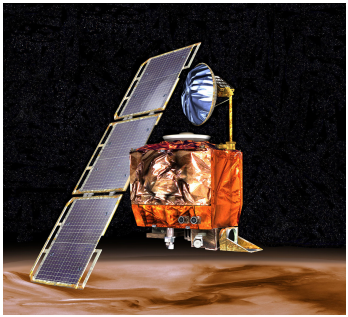

Figure 1: Mars Climate Orbiter, Source: Wikipedia

Mars Climate Orbiter Failure

Numerical Error: Failure due to metric conversion mistakes. The Jet Propulsion Laboratory (JPL) engineers used meters and kilograms in their spacecraft's thrust calculations, whereas Lockheed Martin Astronautics, which designed and built the spacecraft, provided the data in feet. Therefore, the acceleration calculation in pounds-seconds^2 was not translated properly to Newtons-seconds^2. When the probe came too close to Mars and maneuvered into orbit and was destroyed. Later, it was found that it was due to the failed translation of the English units into metric units in a piece of software.

Financial Loss: $327 million

The Patriot Missile Failure

Example 2

The American Patriot Missile battery in Saudi Arabia during the Gulf War failed to attack the Iraqi Scud missile and caused human deaths. The cause was an inaccurate calculation of the time since boot due to computer arithmetic errors. The time in tenths of a second, as measured by the internal clock, was multiplied by 10 to produce the time in seconds. This calculation was performed in 24 bit fixed point register. In particular, the value 1/10, which has a non-terminating binary expansion, was chopped at 24 bits after the radix point. The small chopping error, when multiplied by the large number giving the time in tenths of a second, led to a significant error.

The Patriot Missile Failure

Indeed, the Patriot battery had been up around 100 hours, and an easy calculation shows that the resulting time error due to the magnified chopping error was about 0.34 seconds. (The number 1/10 equals 1/24+1/25+1/28+1/29+1/212+1/213+.... In other words, the binary expansion of 1/10 is 0.0001100110011001100110011001100.... Now the 24-bit register in the Patriot stored instead 0.00011001100110011001100, introducing an error of 0.0000000000000000000000011001100... binary, or about 0.000000095 decimal. Multiplying by the number of tenths of a second in 100 hours gives $0.000000095 \times 100 \times 60 \times 60 \times 10 = 0.34$..

The Patriot Missile Failure

A Scud travels at about 1,676 meters per second, and so travels more than half a kilometer in this time. This was far enough that the incoming Scud was outside the "range gate" that the Patriot tracked. More details can be found here.

For video details: See this video

The Lesson

> **A model is only as trustworthy as its weakest assumption, unit, and datum.**

Sources of Error

Mistakes vs. Errors (Dym, Ch. 4)

Definition 3 (Mistake vs. error)
A **mistake** is a blunder that could in principle be avoided (wrong sign, wrong unit). An **error** is an unavoidable discrepancy inherent to modelling or measurement.

The Mars orbiter was a **mistake**. Rounding $\pi \approx 3.14$ introduces an **error**.

The Four Sources of Error

1. **Modelling error**: reality $\neq$ model (neglected friction, linearization).
2. **Data / measurement error**: inputs and parameters are known only approximately.
3. **Numerical error**: discretization, truncation of series, iteration stopped early.
4. **Round-off error**: finite precision arithmetic (missile failure).

Total error is their *combination*; reducing one may expose another.

Accuracy and Precision

Errors

- Inaccuracy or bias is defined as systematic deviation from the truth
- Imprecision or uncertainty refers to the magnitude of the scatter
- There should be enough precision for adequate design

Errors

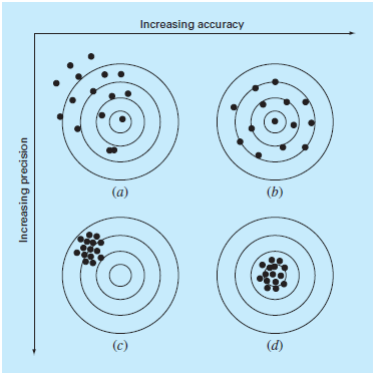

Figure 2: Accuracy and Precision, (a) inaccurate and imprecise, (b) accurate and imprecise, (c) inaccurate and precise, (d) accurate and precise. Source: S. Chapra, R. P. Canale, Numerical Methods for Engineers

Error Definitions

Error Definitions

$$\text{True Value} = \text{Approximation} + \text{True Error} \tag{1}$$

Let E_t denote the true error or exact value of the error or absolute error, then we obtain

$$\text{True Error } (E_t) = \text{True Value} - \text{Approximation} \tag{2}$$

Error Definitions

The disadvantage of this definition is that it does not take into account the order of magnitude of the value. Alternatively, you can normalize it and define True fractional relative error or true percentage relative error

$$\text{True fractional relative error}(\varepsilon_t) = \frac{\text{true value} - \text{approximation}}{\text{true value}} \quad (3)$$

$$\text{true percentage relative error } (\varepsilon_t\%) = \frac{\text{true value} - \text{approximation}}{\text{true value}} 100\% \quad (4)$$

Error Definitions

Let us use the following definitions

Definition 4 (True Error, Absolute Error and Relative Error)
Suppose that $\tilde{a}$ is an approximation to the true value a. Then the true error or actual error is

$$E_t = a - \tilde{a},$$

the absolute error is

$$E_{tabs} = |a - \tilde{a}|$$

and the relative error is

$$\varepsilon_t = \frac{|a - \tilde{a}|}{|a|},$$

provided that $a \neq 0$.

Problem

Suppose that you have the task of measuring the lengths of a bridge and a rivet, and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute (a) E_t and ε_t for each case.

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Answer: $E_t = 1$ cm and $\varepsilon_t = 0.01\%$ for bridge and $E_t = 1$ cm and $\varepsilon_t = 10\%$ for rivet. The relative error for the rivet is much greater.

Error Definitions

In practice, true value is rarely available; for example, in numerical methods, true values are known only when an analytical solution or true answer is known a priori. An alternative to normalize the error using the best available estimate of the true value, that is, the approximation itself,

$$\text{percent relative error } (\varepsilon_a\%) = \frac{\text{present approx.} - \text{previous approx.}}{\text{present approx.}} 100\% \quad (5)$$

Error Definitions

Definition 5 (Relative Error)

Suppose that $\tilde{a}$ is the present approximation to the previous approximation a. The relative error is

$$\varepsilon_a = \frac{|a - \tilde{a}|}{|a|},$$

provided that $a \neq 0$.

Error Definitions

Often, we may not concern the sign of the error, in computations, but are interested in the absolute value of the percent relative error, which is lower than the prescribed tolerance ε_s. We repeat the computation until it satisfies the following stopping criterion.

$$|\text{percent relative error}| < \varepsilon_s \tag{6}$$

If

$$\varepsilon_s = (0.5 \times 10^{2-n})\%$$

it is assured that the result is correct to at least n significant figures.

Significant Figures and Propagation

- Report only digits you can defend.
- First-order propagation: if $y = f(x)$ and x has error Δx, then $\Delta y \approx |f'(x)|\,\Delta x$.
- Relative error of a product adds: $\dfrac{\Delta(ab)}{ab} \approx \dfrac{\Delta a}{a} + \dfrac{\Delta b}{b}$.

Quick check

A pendulum period $T = 2\pi\sqrt{L/g}$. If L is 2% uncertain, how uncertain is T? (1% – the square root halves it.)

Limitations and Validity

Every Model Has a Domain of Validity

- Hooke's law $F = kx$ – only below the elastic limit.
- Ideal gas $PV = nRT$ – fails at high pressure / low temperature.
- Newton cooling – fails when radiation ($\propto T^4$) dominates.
- Malthus $\dot{N} = rN$ – fails as resources saturate (L18).

Verification vs. Validation
Verification: "are we solving the equations right?" (maths/code correct). **Validation**: "are we solving the right equations?" (model matches reality).

Well-Posedness

A model is **well-posed** (Hadamard) if a solution

1. **exists**,
2. is **unique**,
3. depends **continuously** on the data.

Property (3) is exactly what Lorenz's weather *lacked* in practice, and what the backward heat equation lacks entirely

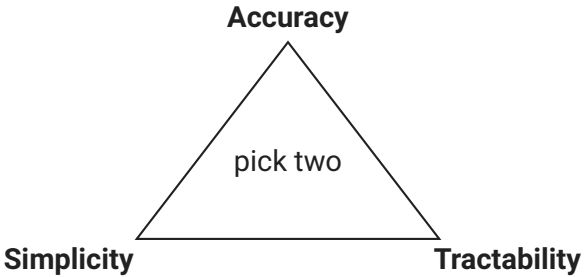
Non-uniqueness

Non-uniqueness: Many Models, One Reality

- The Earth: sphere (Eratosthenes) $\subset$ ellipsoid $\subset$ geoid – all valid at different fidelities.
- Drag: Stokes ($\propto v$) at low speed, quadratic ($\propto v^2$) at high speed – different models, same falling body.
- Data fitting: infinitely many curves pass through finitely many points.

Occam's razor
Among models that fit adequately, prefer the **simplest**. Complexity must be *earned* by the purpose.

The Modeller's Trade-off Triangle



More detail buys accuracy but costs simplicity and (often) tractability. Good modelling is choosing *where* on this triangle to sit – for the purpose.

Exercises

1. Classify each as a mistake or an error: (a) using $g = 9.8$ vs 9.81; (b) mixing metric and imperial; (c) linearizing $\sin\theta \approx \theta$; (d) a typo in a formula.
2. Density $\rho = m/V$ with m to $\pm 1\%$, V to $\pm 2\%$. Estimate the relative error in ρ.
3. State the domain of validity of the small-angle pendulum. For what θ is the error in $\sin\theta \approx \theta$ below 1%?
4. Give two genuinely different models of the same system you know, and argue which Occam prefers.
5. Explain why “the code ran without errors” guarantees neither verification nor validation.

Dimensional Analysis

How Long is a Pendulum's Swing?

- Suppose you had *never* solved the pendulum, and knew only what the period T **could depend on**: length L, mass m, gravity g.
- Their dimensions: $[L] = \mathsf{L}$, $[m] = \mathsf{M}$, $[g] = \mathsf{L}\,\mathsf{T}^{-2}$, $[T] = \mathsf{T}$.
- There is **only one** way to combine L, m, g into a time:

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- There is **only one** way to combine L, m, g into a time: $\sqrt{L/g}$. Mass *cannot* appear (nothing cancels M).
- Therefore $T = C\sqrt{L/g}$ – **without solving a single differential equation**.

The Power of Dimensions

- We deduced the *form* of the law, that period is mass-independent, and that quadrupling L doubles T – all from bookkeeping of units.
- The pure number C ($= 2\pi$) needs one experiment or one calculation.

Unit II's promise
Dimensional analysis extracts the maximum physics from the minimum assumptions – and it is the theory behind *scale models* (Previous wind tunnel).

Dimensions and Homogeneity

Fundamental and Derived Dimensions

- Mechanics uses three **fundamental dimensions**: mass M, length L, time T (add temperature Θ, charge Q as needed).
- Every physical quantity has dimensions $\mathsf{M}^a\mathsf{L}^b\mathsf{T}^c$:

Quantity	Symbol	Dimensions
velocity	v	$\mathsf{L}\,\mathsf{T}^{-1}$
acceleration	a	$\mathsf{L}\,\mathsf{T}^{-2}$
force	F	$\mathsf{M}\,\mathsf{L}\,\mathsf{T}^{-2}$
energy	E	$\mathsf{M}\,\mathsf{L}^2\,\mathsf{T}^{-2}$
pressure	p	$\mathsf{M}\,\mathsf{L}^{-1}\,\mathsf{T}^{-2}$

The Principle of Dimensional Homogeneity

Fourier's principle (1822)
Every term in a physically meaningful equation must have the **same dimensions**. You cannot add a length to a time.

- Check: $s = ut + \frac{1}{2}at^2$: $[ut] = \mathsf{L}\,\mathsf{T}^{-1} \cdot \mathsf{T} = \mathsf{L}$; $[at^2] = \mathsf{L}\,\mathsf{T}^{-2} \cdot \mathsf{T}^2 = \mathsf{L}$. ✓
- Arguments of $\sin, \exp, \ln$ must be **dimensionless**. So in $e^{-t/\tau}$, τ must be a time.
- A wrong-dimension equation is *certainly* wrong – a free error check (recall Mars).

The Method

The Method of Dimensional Analysis

To find how quantity Q depends on variables $x_1, \ldots, x_n$:

1. List all relevant variables and their dimensions (the crucial modelling step!).
2. Assume a power-law monomial $Q = C\, x_1^{a_1} \cdots x_n^{a_n}$.
3. Impose homogeneity: match exponents of $\mathsf{M}, \mathsf{L}, \mathsf{T} \Rightarrow$ linear equations for a_i.
4. Solve; group into dimensionless combinations.

Worked Example: Drag on a Sphere (slow flow)

Assume drag F depends on radius r, speed v, viscosity μ ($[\mu] = \mathrm{M\,L^{-1}T^{-1}}$):

$$F = C\, r^a v^b \mu^c$$

Match $\mathrm{M\,L\,T^{-2}} = \mathrm{L}^a(\mathrm{L\,T^{-1}})^b(\mathrm{M\,L^{-1}T^{-1}})^c$:

$$\mathrm{M}:\ c = 1; \quad \mathrm{T}:\ -b - c = -2 \Rightarrow b = 1; \quad \mathrm{L}:\ a + b - c = 1 \Rightarrow a = 1.$$

$$\boxed{F = C\,\mu\, r\, v} \qquad (\text{Stokes: } C = 6\pi)$$

Linear in radius and speed – obtained *without* solving the flow.

What Dimensional Analysis Cannot Do

- It cannot find the pure constant C (needs theory or one experiment).
- It fails if you **omit** a relevant variable or **include** an irrelevant one – the physics is in the variable list.
- When several dimensionless groups exist, it gives a *relation among groups*, not a formula (next lecture: Buckingham Π).

Exercises

1. Give the dimensions of: power, momentum, angular momentum, surface tension, Young's modulus.
2. By dimensional analysis, find how the speed of a deep-water wave depends on gravity g and wavelength λ.
3. The period of a mass m on a spring of stiffness k ($[k] = \mathrm{M\,T}^{-2}$): show $T = C\sqrt{m/k}$ and that g cannot enter.
4. Check the dimensional homogeneity of Bernoulli's equation $p + \frac{1}{2}\rho v^2 + \rho g h = \text{const}$.
5. Find, up to a constant, the energy radiated per second by a blackbody in terms of σ and T given $[\sigma T^4]$ is a flux.

A Story: The Dye That Named a Number

Manchester, 1883: Osborne Reynolds' Pipe

- Reynolds injects a thin thread of coloured dye into water flowing through a glass pipe.
- At low speed the thread stays **straight** – smooth, *laminar* flow.
- Increase the speed past a threshold and the thread suddenly **bursts into swirls** – *turbulence*.
- The transition always happens at the same value of one dimensionless combination:

$$Re = \frac{\rho v D}{\mu} \approx 2300.$$

Why One Number Rules Them All

- Big pipe or small, water or oil, fast or slow – the *flavour* of the flow depends only on Re.
- This is why a small wind-tunnel wing predicts a real aircraft: match Re and the flows are **similar**.
- Where do such magic groups come from, and how many are there? The **Buckingham** Π **theorem** answers both.

The Theorem

Example 1: the simple pendulum

What does the period T of a pendulum depend on?

variable	T	ℓ	m	g
dimension	T	L	M	LT^{-2}

Try $T^a \ell^b m^c g^d$ dimensionless:

$$\mathsf{M}: \ c = 0, \qquad \mathsf{L}: \ b + d = 0, \qquad \mathsf{T}: \ a - 2d = 0.$$

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The mass drops out without any mechanics. If we add the amplitude θ_0 (already dimensionless), we get $T = \sqrt{\ell/g}\ \Phi(\theta_0)$.

Example 2: drag on a sphere

Force F on a sphere of diameter D moving at speed U through a fluid of density ρ and viscosity μ.

variable	F	ρ	U	D	μ
dimension	MLT^{-2}	ML^{-3}	LT^{-1}	L	$\mathsf{ML}^{-1}\mathsf{T}^{-1}$

Five variables, three dimensions: two groups,

$$C_D = \frac{F}{\rho U^2 D^2}, \qquad \mathrm{Re} = \frac{\rho U D}{\mu}, \qquad \Longrightarrow \qquad \boxed{F = \rho U^2 D^2 \, \Phi(\mathrm{Re})}$$

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A function of four variables becomes a function of one. Every experiment, any fluid, any sphere, lies on a single curve $C_D(\mathrm{Re})$. This is why wind-tunnel models work.

Example 3: G. I. Taylor and the blast wave

A point explosion releases energy E into air of density ρ. How does the radius R of the fireball grow with time t?

variable	R	t	E	ρ
dimension	L	T	$\mathrm{ML^2T^{-2}}$	$\mathrm{ML^{-3}}$

One group, $\Pi = \dfrac{\rho R^5}{E t^2}$, hence

$$\boxed{R = C \left(\frac{E\, t^2}{\rho} \right)^{1/5}}$$

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Taylor (1950) used published photographs of the 1945 Trinity test: plotting $\log R$ against $\log t$ gives slope $2/5$, and the intercept gives E. His estimate, about 17 kilotons, was close to the classified figure.

What the examples have in common

example	variables n	dimensions k	groups $n-k$
pendulum	4	3	1
sphere drag	5	3	2
blast wave	4	3	1

In each case:

- the physics does not care which units we use;
- the dimensionless combinations are the solutions of a linear system;
- the number of groups was always $n-k$.

Is this always true, and why?

The Buckingham Π Theorem

Theorem 6 (Buckingham, 1914)
A physically meaningful relation among n variables involving k independent fundamental dimensions can be rewritten as a relation among $n-k$ independent **dimensionless** groups $\Pi_1, \ldots, \Pi_{n-k}$:

$$\phi(\Pi_1, \Pi_2, \ldots, \Pi_{n-k}) = 0.$$

- k = rank of the dimensional matrix (usually the number of independent dimensions present).
- The $n-k$ groups are the true “degrees of freedom” of the problem.

Setting

Fundamental dimensions $L_1, \ldots, L_k$ (e.g. mass, length, time).

Physical variables $q_1, \ldots, q_n > 0$ with

$$[\, q_j \,] = L_1^{a_{1j}} L_2^{a_{2j}} \cdots L_k^{a_{kj}}, \qquad a_{ij} \in \mathbb{R}.$$

Dimension matrix $A = (a_{ij}) \in \mathbb{R}^{k \times n}$.

k independent fundamental dimensions $\iff \operatorname{rank} A = k$.

Change of units

Rescaling the unit of L_i by $\lambda_i > 0$ changes the value of q_j to

$$\tilde{q}_j = \Big(\prod_{i=1}^{k} \lambda_i^{a_{ij}}\Big) q_j, \qquad \text{written } \tilde{q} = T_\lambda q.$$

Definition 7

$f(q_1, \ldots, q_n) = 0$ is **physically meaningful** if

$$f(q) = 0 \iff f(T_\lambda q) = 0 \qquad \forall\, q \in (0,\infty)^n,\ \lambda \in (0,\infty)^k.$$

A law of nature should not depend on whether we measure in metres or feet.

The theorem

Theorem 8 (Buckingham, 1914)
Let $\operatorname{rank} A = k$ and let $f(q) = 0$ be physically meaningful. Then there are $n - k$ dimensionless products

$$\Pi_\ell = \prod_{j=1}^{n} q_j^{x_j^{(\ell)}}, \qquad \ell = 1, \ldots, n-k,$$

with linearly independent exponent vectors, and a function ϕ such that

$$f(q_1, \ldots, q_n) = 0 \iff \phi(\Pi_1, \ldots, \Pi_{n-k}) = 0.$$

Step 1: dimensionless products $= \ker A$

For $x \in \mathbb{R}^n$ let $q^x = \prod_j q_j^{x_j}$. Then

$$[\, q^x \,] = \prod_{j=1}^{n} \prod_{i=1}^{k} L_i^{a_{ij} x_j} = \prod_{i=1}^{k} L_i^{(Ax)_i}.$$

So

$$q^x \text{ is dimensionless } \iff Ax = 0, \qquad (T_\lambda q)^x = \Big(\prod_i \lambda_i^{(Ax)_i} \Big) q^x.$$

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Rank–nullity: $[\ker] A = n - \operatorname{rank} A = n - k$.

Still to show: f depends on q *only* through these products.

Step 2: a convenient basis

Since $\operatorname{rank} A = k$, renumber so that the first k columns are independent:

$$A = \left[\, B \mid C \,\right], \qquad B \in \mathbb{R}^{k\times k} \text{ invertible.}$$

$q_1, \ldots, q_k$ are the **repeating variables**. Let $D = B^{-1}C = (d_{i\ell})$ and

$$x^{(\ell)} = \begin{pmatrix} -De_\ell \\ e_\ell \end{pmatrix}, \qquad Ax^{(\ell)} = -BB^{-1}Ce_\ell + Ce_\ell = 0.$$

The last $n-k$ entries form the identity, so the $x^{(\ell)}$ are independent: a basis of $\ker A$.

$$\boxed{\Pi_\ell = q_{k+\ell} \prod_{i=1}^{k} q_i^{-d_{i\ell}}, \qquad \ell = 1, \ldots, n-k}$$

Step 3: change of variables

Define $\Psi : (0,\infty)^n \to (0,\infty)^n$,

$$\Psi(q) = (q_1, \ldots, q_k,\ \Pi_1, \ldots, \Pi_{n-k}).$$

Ψ is a bijection (indeed a diffeomorphism), with inverse

$$q_{k+\ell} = \Pi_\ell \prod_{i=1}^{k} q_i^{d_{i\ell}}, \qquad q_1, \ldots, q_k \text{ unchanged.}$$

With $F = f \circ \Psi^{-1}$:

$$f(q) = 0 \iff F(q_1, \ldots, q_k, \Pi_1, \ldots, \Pi_{n-k}) = 0.$$

Step 4: eliminate the repeating variables

Choose units so that $\tilde{q}_1 = \cdots = \tilde{q}_k = 1$. With $\lambda_i = e^{\mu_i}$,

$$\log \tilde{q}_j = \log q_j + (A^{\mathsf{T}}\mu)_j, \qquad \big((A^{\mathsf{T}}\mu)_j\big)_{j \le k} = B^{\mathsf{T}}\mu.$$

B^{T} is invertible, so take

$$\mu = -\,B^{-\mathsf{T}}\big(\log q_1, \ldots, \log q_k\big)^{\mathsf{T}}.$$

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$$\mu = -\,B^{-\mathsf{T}}\big(\log q_1, \ldots, \log q_k\big)^{\mathsf{T}}.$$

The Π_ℓ are dimensionless, so $\tilde{\Pi}_\ell = \Pi_\ell$. Hence

$$f(q) = 0 \iff f(T_\lambda q) = 0 \iff F(1, \ldots, 1, \Pi_1, \ldots, \Pi_{n-k}) = 0.$$

Step 5: conclusion

Define

$$\phi(\Pi_1, \ldots, \Pi_{n-k}) := F(1, \ldots, 1, \Pi_1, \ldots, \Pi_{n-k}).$$

Then

$$f(q_1, \ldots, q_n) = 0 \iff \phi(\Pi_1, \ldots, \Pi_{n-k}) = 0.$$

- each Π_ℓ is dimensionless (Step 1);
- the Π_ℓ are independent: no one is a product of powers of the others (Step 2).

■

Remarks

- **Explicit form.** If $f = 0$ can be solved for q_{k+1}, then $\Pi_1 = \Phi(\Pi_2, \dots, \Pi_{n-k})$.
- **Rank deficiency.** If $\operatorname{rank} A = r < k$, repeat the proof with r in place of k; one gets $n - r$ groups.
- **Non-uniqueness.** Any basis of $\ker A$ works; two choices are related by $\Pi'_m = \prod_\ell \Pi_\ell^{m_{\ell m}}$, $(m_{\ell m})$ invertible. The number $n - k$ is fixed.
- **Signs.** Positivity is only needed for real powers; otherwise work with $|q_j|$.

Recipe for the Π Groups

1. List n variables and their dimensions; determine k.
2. Choose k **repeating variables** that (i) together contain all k dimensions and (ii) cannot form a dimensionless group by themselves.
3. Form each Π_i by multiplying one *non-repeating* variable by powers of the repeating ones; solve for powers to make it dimensionless.
4. Interpret and, if possible, recognise the named groups.

Worked Examples

Example: Pressure Drop in a Pipe

Variables: $\Delta p, \rho, v, D, \mu, \ell$ ($n = 6$), dimensions M, L, T ($k = 3$) $\Rightarrow$ 3 groups.
Repeating: ρ, v, D. Then:

$$\Pi_1 = \frac{\Delta p}{\rho v^2}, \qquad \Pi_2 = \frac{\rho v D}{\mu} = Re, \qquad \Pi_3 = \frac{\ell}{D}.$$

Hence $\dfrac{\Delta p}{\rho v^2} = \phi\left(Re,\ \dfrac{\ell}{D}\right)$ – the basis of the Darcy friction-factor chart.

Example: Rederiving the Pendulum

Variables T, L, g, m ($n = 4$), dimensions M, L, T. But M appears *only* in m, so m drops; effectively $n = 3$, $k = 2 \Rightarrow 1$ group:

$$\Pi_1 = T\sqrt{g/L} = \text{const} \;\Rightarrow\; T = C\sqrt{L/g}.$$

One group $\Rightarrow$ the relation is just "$\Pi_1 = \text{constant}$" – the strongest possible conclusion.

Named Dimensionless Groups (a modeller's toolkit)

Group	Definition	Balances
Reynolds Re	$\rho v L/\mu$	inertia / viscous
Froude Fr	$v/\sqrt{gL}$	inertia / gravity
Mach Ma	v/c	speed / sound speed
Péclet Pe	vL/α	advection / diffusion
Nusselt Nu	hL/k	convective / conductive heat

These reappear throughout Unit VI (heat, traffic, Navier–Stokes).

Summary

Exercises

1. Using Π, find the groups for drag F on a sphere from ρ, v, D, μ. Show $F/(\rho v^2 D^2) = \phi(Re)$.
2. For a vibrating drop held by surface tension σ, density ρ, radius r: find the single group controlling its frequency.
3. Determine the number of dimensionless groups for heat convection $h = f(\rho, v, L, \mu, c_p, k)$.
4. Build the dimensional matrix for $\{\Delta p, \rho, v, D, \mu\}$ and compute its rank.
5. Explain, via similarity, why a $1{:}50$ ship model is towed at a speed set by matching Froude number, not Reynolds.

Thanks

Doubts and Suggestions

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MA621L-Mathematical Modeling

Lecture 9 : Limitations, Properties, Errors and Non-uniqueness

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