भारतीय प्रौद्योगिकी संस्थान तिरुपति

TIRUPATI

Bioheat Equation

for Cancer Treatment

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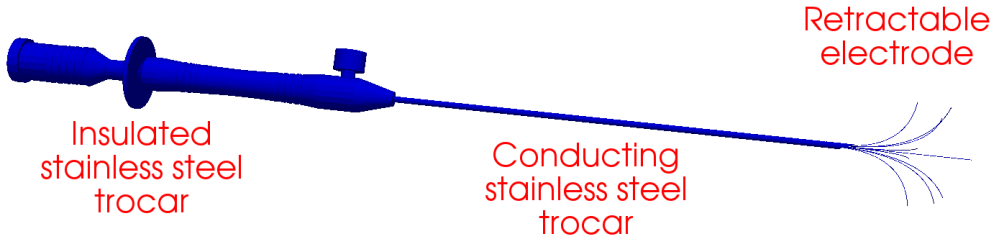
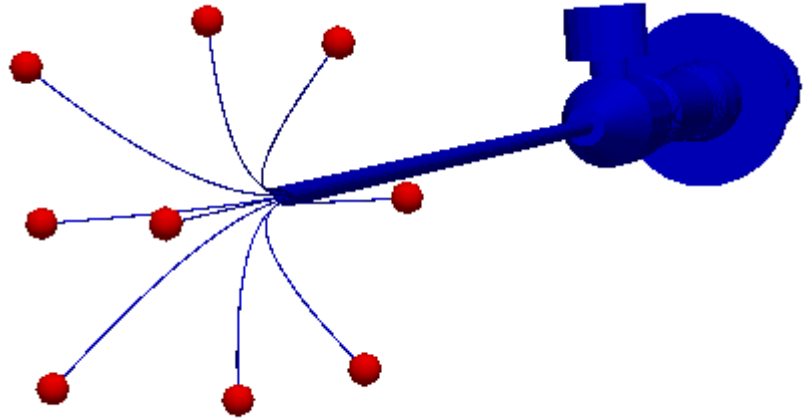
- MICT: Recommended by IRs
  - Minimal Invasive Cancer Treatment
  - Under treatment:
    - Recurrence of Tumour
  - Over Treatment:
    - Kills unnecessary cells → Death
- RFA is most common
  - Requires more experienced IRs
- RFA Procedure
  - Locoregional treatment
  - Use RFA needle (umbrella shaped) to induce heat energy

Needles


Retractable
electrode
Insulated
stainless steel
trocar
Conducting
stainless steel
trocar




Insulated
Stainless steel
Trocar
Conducting
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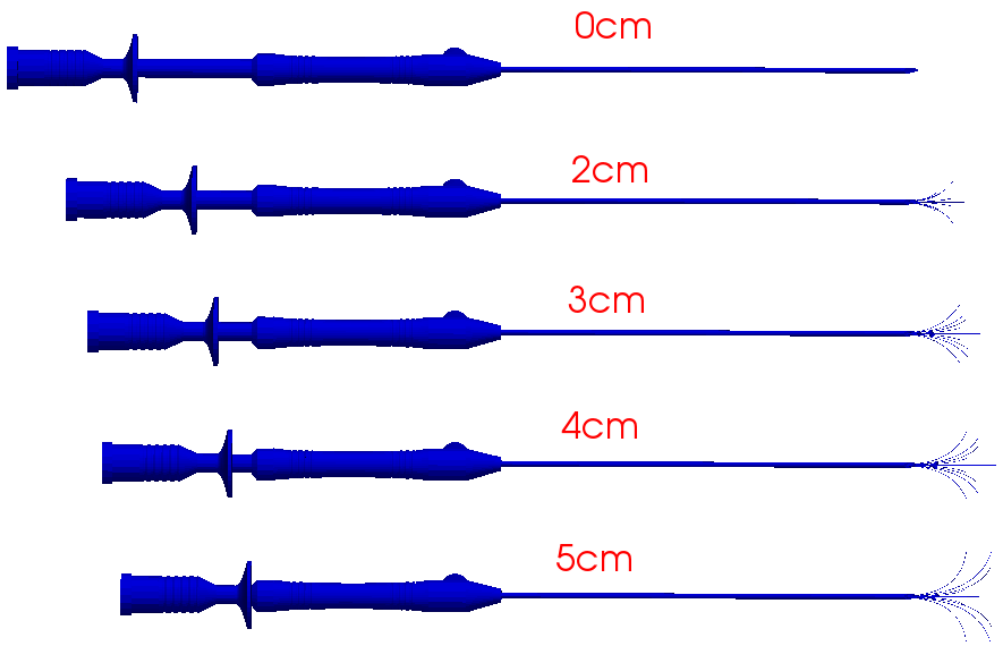
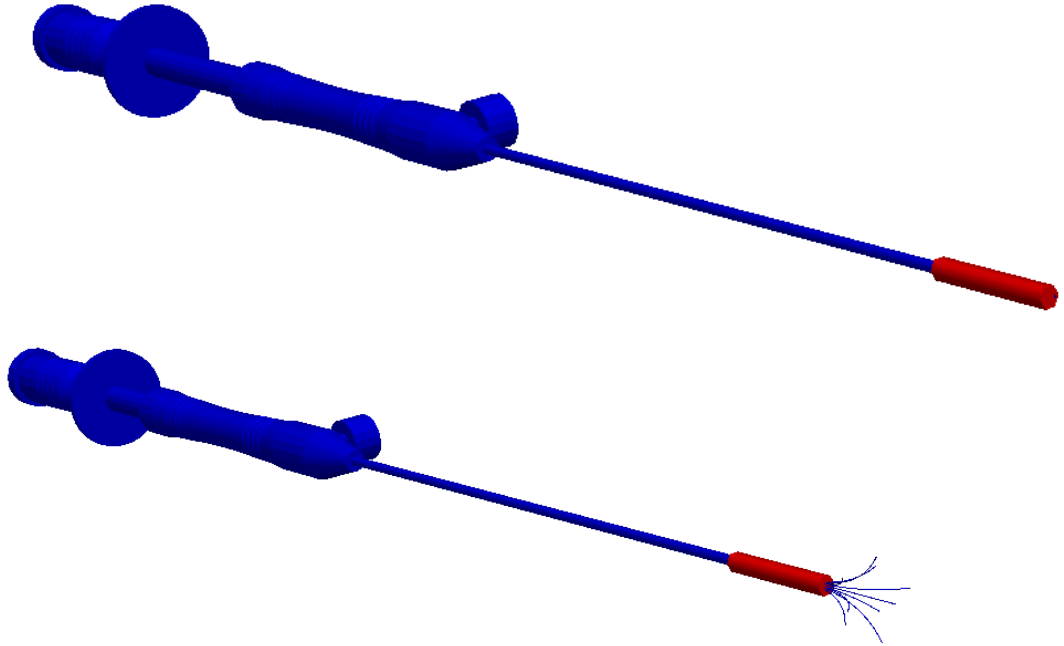
Retractable
Electrode

Needles





System:

- Entire Body, Clinical Environment of RFA Treatment

Questions:

1. Can we create a simple model for RFA Treatment?
   a) Can we create liver, vessel, tumour model?
   b) Can we produce the same power produced by the needle virtually?
   c) Can we implement the same protocol followed by doctors?
2. Can we predict the temperature profile before the treatment?
3. Can we predict the cell deaths?

Mathematical Statements for:

1. Heat transfer from RFA needle to the liver
2. Power generation or Heat Source from needle
3. Temperature controlled power supply
4. Cell death due to temperature

Input-Output System

Input

- CT Scan Images
- Blood Perfusion
- Body Temperature
- Protocol
- Needle location on image
- Power output from machine

Output

- Liver Geometry, Tumour Geometry
- Vessel Geometry, Needle Geometry
- Temperature around tumour cells
- Cell States around tumour cells

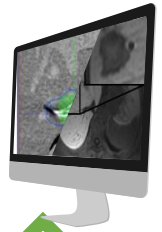
Cancer Treatments

Common Liver Cancers

HCCs

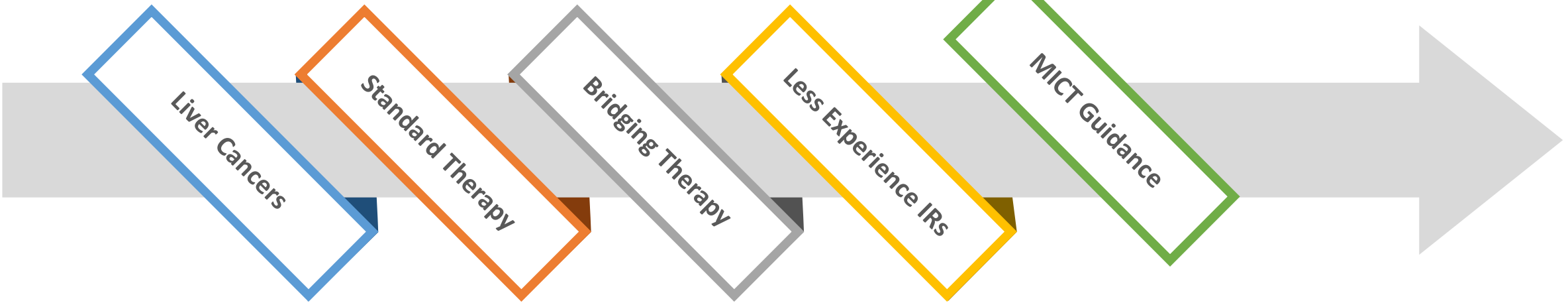
Bridging Therapy

RFA, MWA
IRE, Cryo
FUS, Laser



MICT Guidance

Help IRs
Change of parameters



Standard Therapy

- Surgical Treatments: 75-90% unsuitable[1]
- Transplantation

Less Experienced IRs

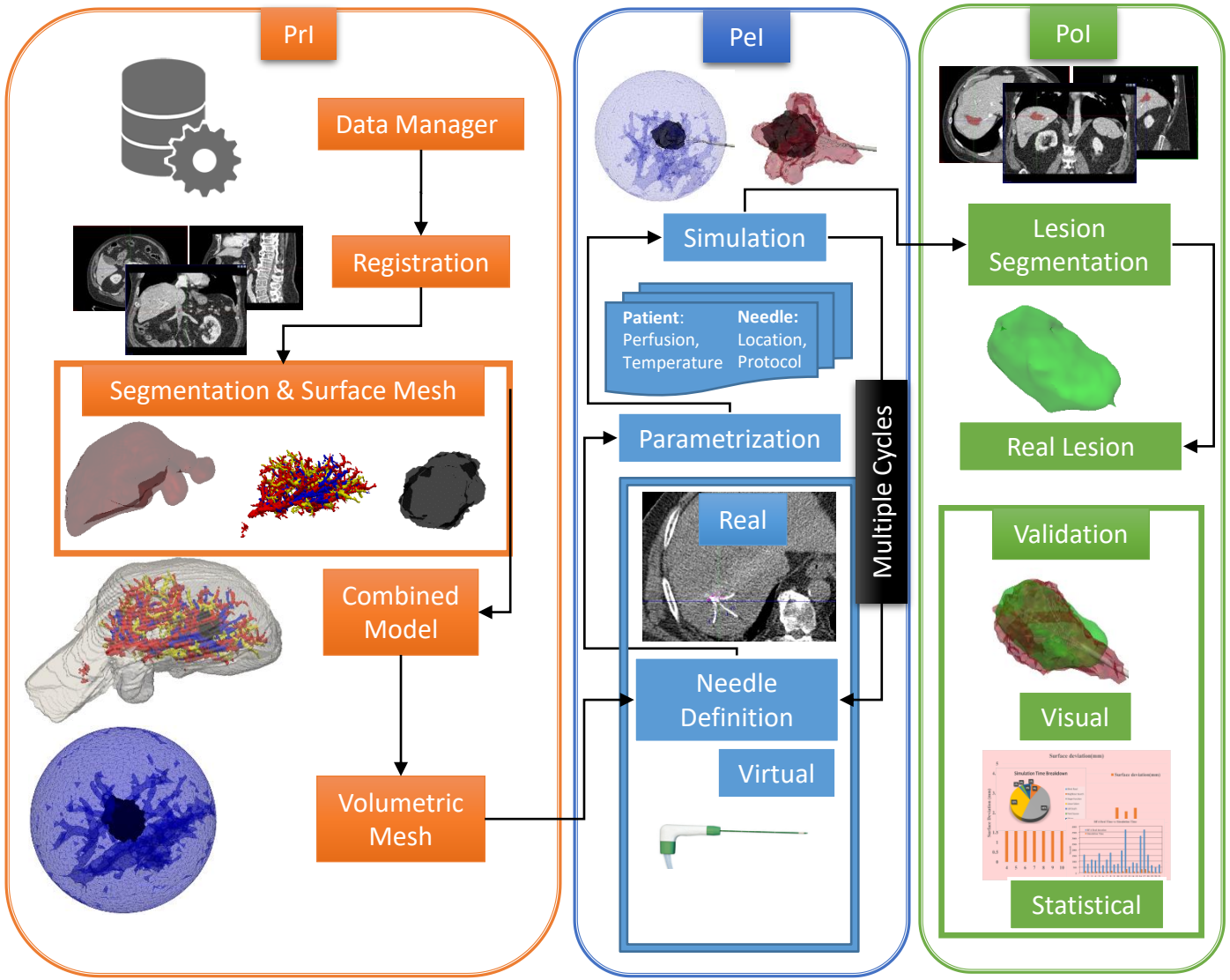
Under treatment: Recurrence
Over Treatment: Death

1. Bruix J, Sherman M (2011) Management of hepatocellular carcinoma: an update. Hepatology 53(3):1020–1022

Image Sources: https://angiodynamics.com
https://http://www.mermaidmedical.dk/

Clinical Model

Clinical Workflow



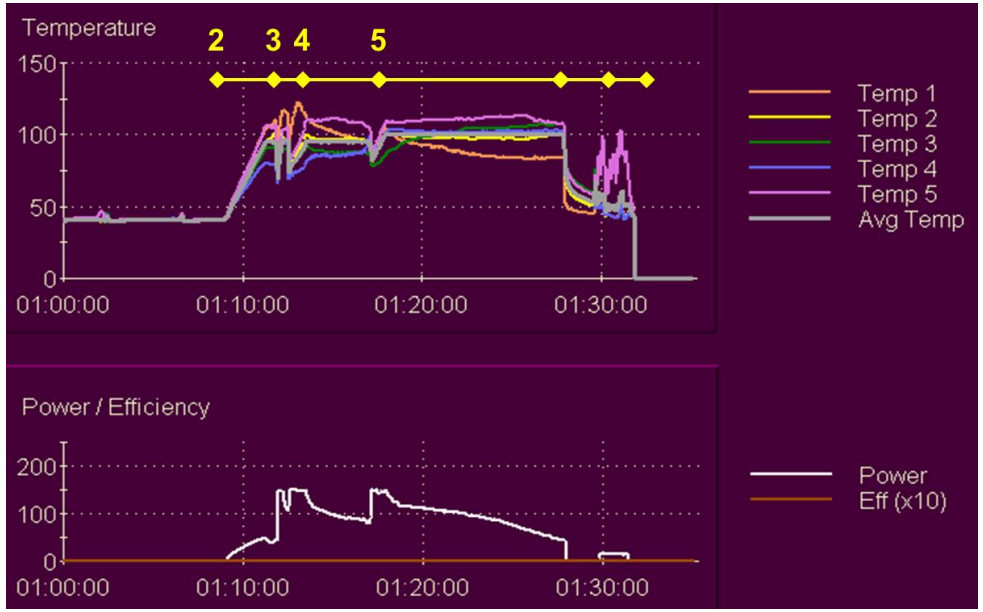
Device-Specific Parameters

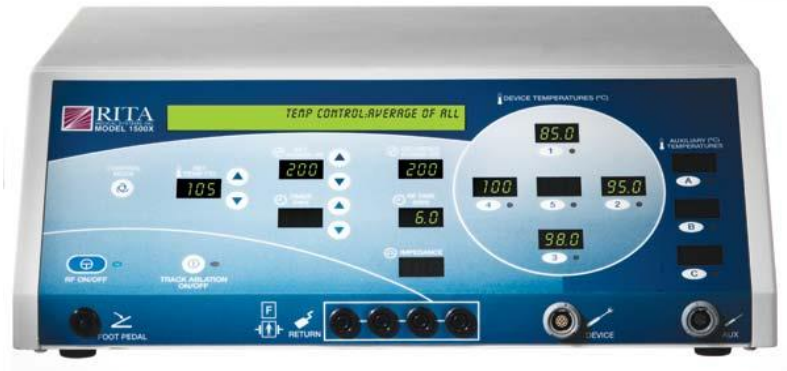
- Power Input
- Needle Geometry
- Temperature Controlled Algorithm
- Treatment Protocol
- Cooling Procedure

Device-Specific Parameters

RITA Starburst Needle and RF Generator

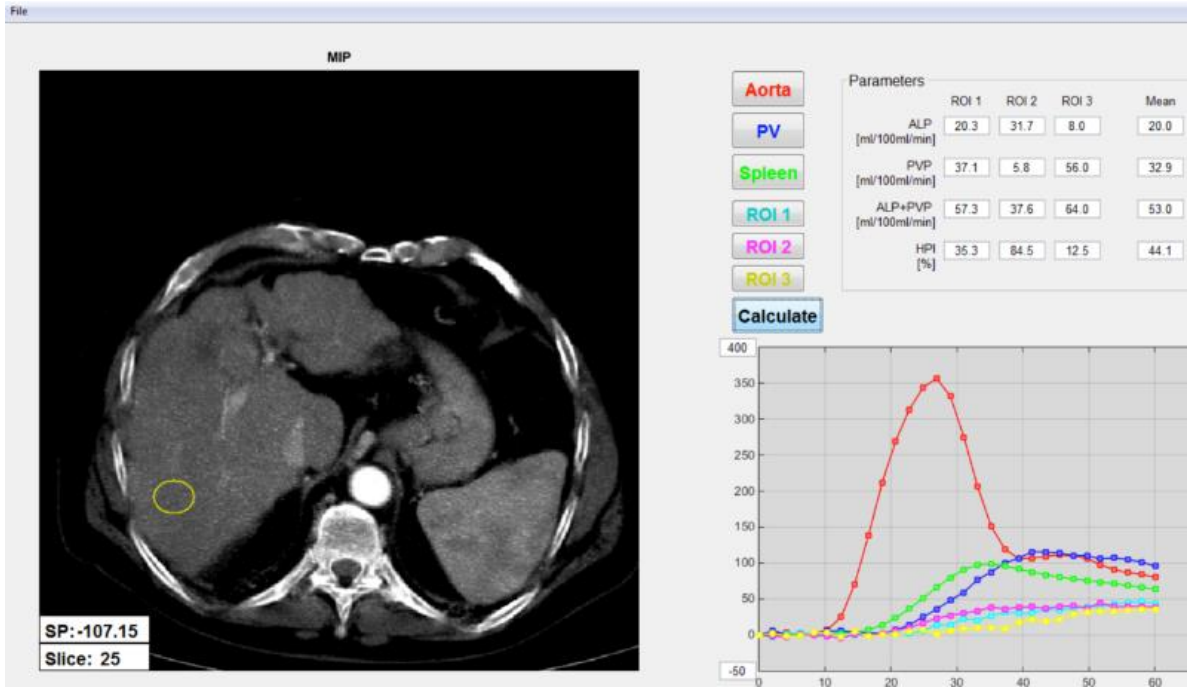
Needle length	Target temp	Power	Time	Wait
2 cm	95° C	150 W	15.0 min	untill target temp.
3 cm	95° C	150 W	14.5 min	untill target temp.
4 cm	95° C	150 W	14.0 min	4 min untill target emp.
5 cm	100° C	150 W	10.0 min	untill
Automatic "Cool Down" - mode 30 sec to 70° C				
If nec. repeat				
Withdraw needle in "Track ablation"- mode				





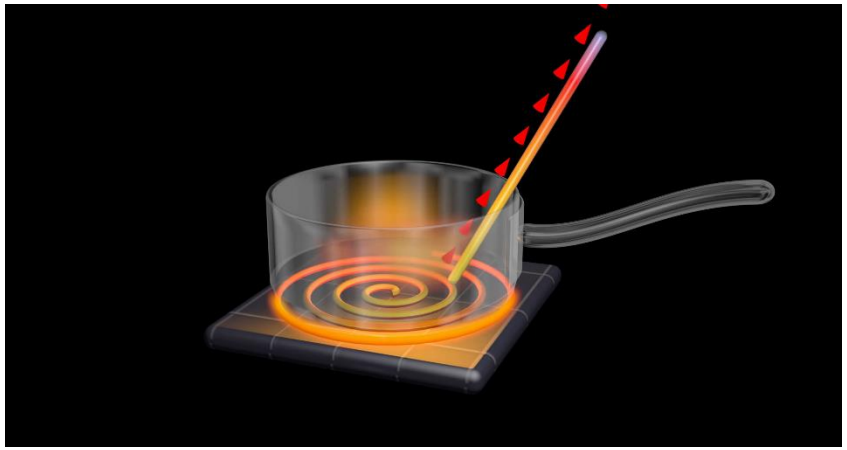
Patient-Specific Parameters

- Blood Perfusion
  - Tissue
  - Tumour
  - TACE
- Specific heat capacity
- Thermal conductivity
- Many more…



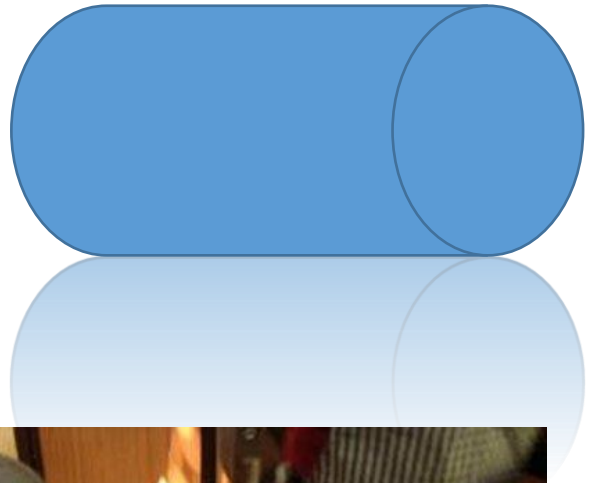
Mathematical Model

- A discipline of thermal engineering
- Conversion, exchange of heat between physical systems

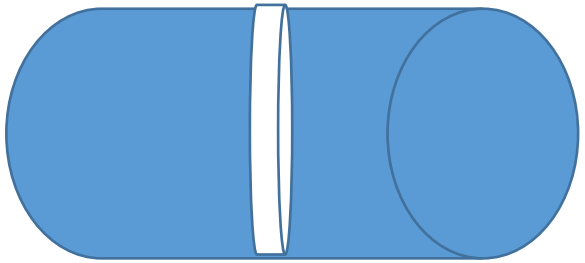


Heat Equation

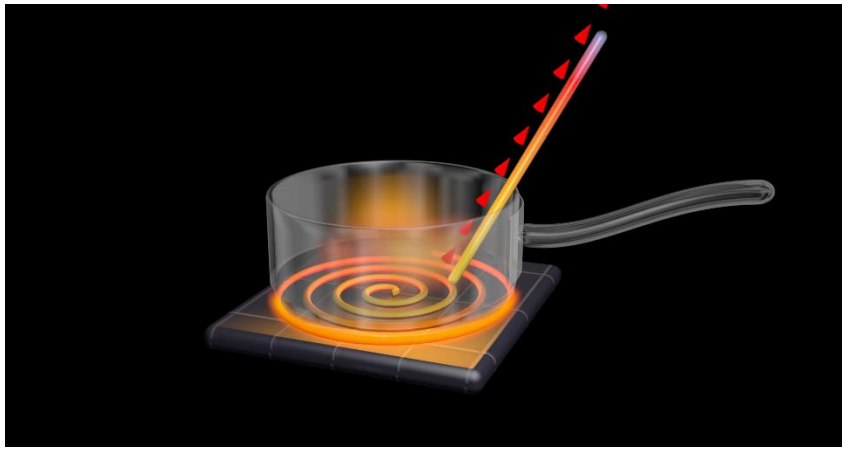
- Heat flow in a solid rod with circular cross section A
- Perfectly insulated rod. No heat escapes radially
- Heat flow only along the axis of symmetry of the rod
- Initial temperature = room temperature



Heat Equation

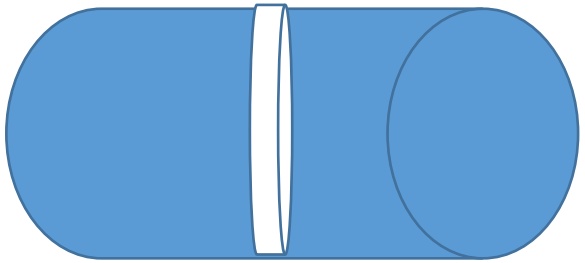


- At one end temperature raised suddenly
- Heat flow from hot to cold
- Let δx denote thickness of a section through the rod located at the point x
- Some of the heat absorbed by the rod
- Heat flow at $x \neq$ Heat flow at $x + \delta x$
- Conservation Energy

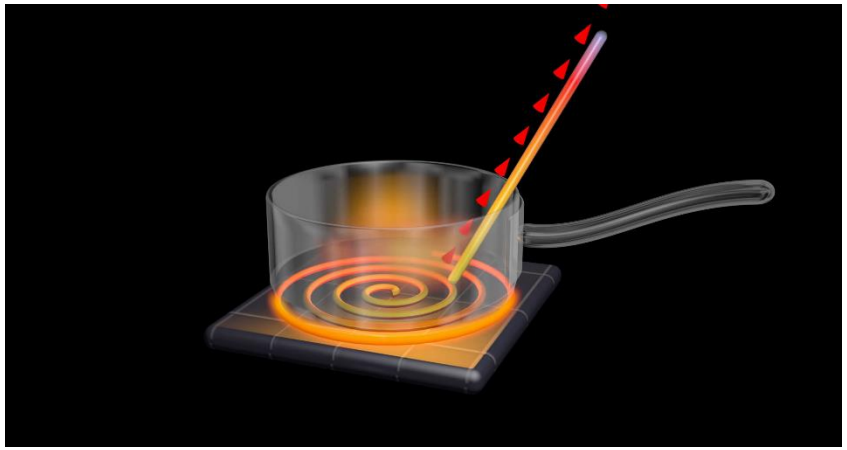


Rate of change of heat content = net rate of heat conducted in and out of section

Heat Equation



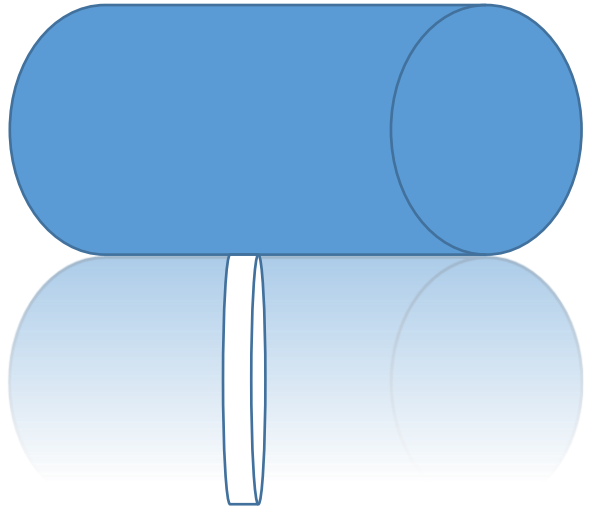
- Assume no heat production inside the rod or heat loss from the surface of the rod.



Rate of change of heat content = net rate of heat conducted in and out of section

Derivation: Heat Equation

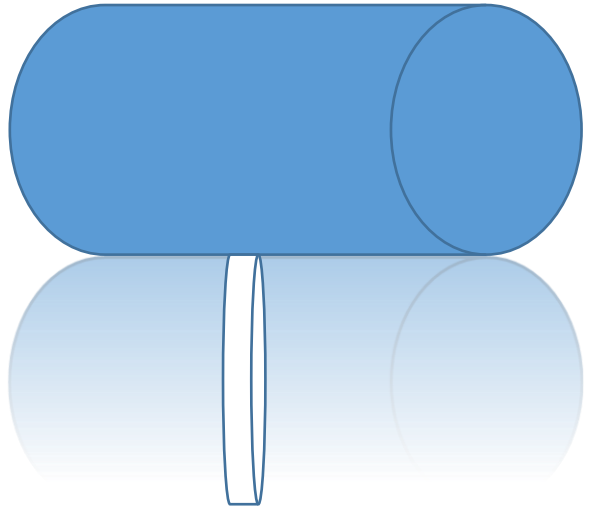
Rate of change of heat content = net rate of heat conducted in and out of section



- $T(x,t)$ denotes the temperature of the rod at position x, at time t.
- Some of the heat energy is absorbed by the rod.
- Changes in temperature at time $t + \delta t$
- Temperature difference: $T(x, t+\delta t) - T(x,t)$

Derivation: Heat Equation

- Temperature difference: $T(x, t + \delta t) - T(x, t)$
- Amount of heat required to change the temperature of the entire mass of the cross section??
- Proportional to mass of section??
- Mass = density x volume
- Volume = $A \times \delta x$

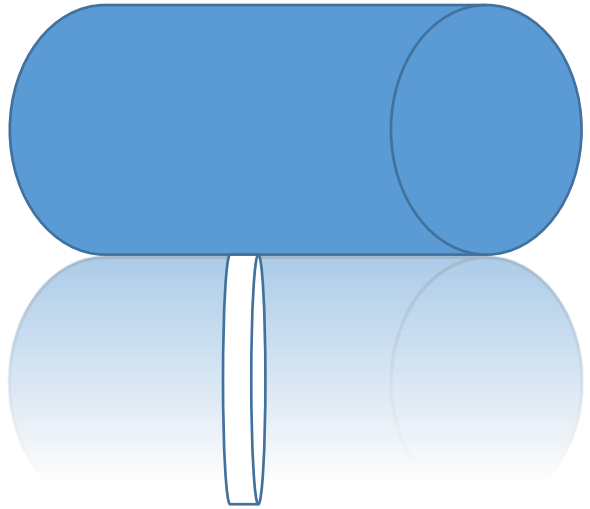


Rate of change of heat content = net rate of heat conducted in and out of section

Derivation: Heat Equation

Rate of change of heat content = net rate of heat conducted in and out of section

- Let $Q(t)$ be the heat content of the section between x and $x + \delta x$



HEAT vs. TEMPERATURE RISE

SETUP

- A fixed pot of water (mass is constant)
- Steady flame delivers heat at a constant rate
- Therefore, time on the stove ∝ heat supplied (Q)

Time on the stove
∝
Heat supplied (Q)

OBSERVATION

10° → 20°

Takes some time

ΔT = 10°C

10° → 30°

Takes about twice that time

ΔT = 20°C

10° → 40°

Takes about three times that time

ΔT = 30°C

SUMMARY (mass of water is fixed)

Temperature Change (ΔT)	Heat Supplied (Q)	Time on Stove
10° → 20° (ΔT = 10°C)	Q	t
10° → 30° (ΔT = 20°C)	2Q	2t
10° → 40° (ΔT = 30°C)	3Q	3t

CONCLUSION

Q ∝ ΔT

(For fixed mass and constant heating rate)

The heat required is directly proportional to the temperature rise.

The more you want to raise the temperature (higher ΔT), the more heat (Q) you must supply, and it takes proportionally more time on a steady flame.

Remember: Time ∝ Q ∝ ΔT

Experiment 2

HEAT vs. MASS

SETUP (same flame, same temperature rise)

- Same steady flame
- Same temperature rise: 10° → 20° (ΔT = 10°C)
- Change the mass of water
- Time on the stove ∝ heat supplied (Q)

Intuition

Warming 2 kg of water is the same job as warming two separate 1 kg pots by the same amount. Heat is extensive—it adds up when you combine bodies.

OBSERVATION

1 kg of water

Takes some time

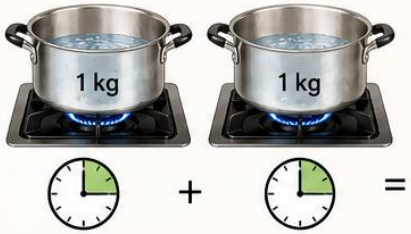
10° → 20° (ΔT = 10°C)

2 kg of water

Takes about twice as long

10° → 20° (ΔT = 10°C)

Same job, two pots of 1 kg



Together takes about twice as long

10° → 20° (ΔT = 10°C)

SUMMARY (ΔT fixed at 10°C)

Mass of Water (m)	Temperature Change (ΔT)	Heat Supplied (Q)	Time on Stove
1 kg	10° → 20° (10°C)	Q	t
2 kg	10° → 20° (10°C)	2Q	2t
3 kg	10° → 20° (10°C)	3Q	3t

CONCLUSION

$Q \propto m$ (ΔT fixed)

For a fixed temperature rise, the heat required is directly proportional to the mass of the body.

General relation

$$Q = c\,m\,\Delta T$$

c: specific heat; m: mass; ΔT: temperature rise

For the same temperature rise, more mass requires proportionally more heat, so it takes proportionally more time on the same flame.

Remember:
At the same ΔT on the same flame → Time ∝ Q ∝ m

Observation

FROM WHAT WE OBSERVED TO THE GENERAL FORMULA

From the two experiments we learned: (i) For fixed mass: $Q \propto \Delta T$ | (ii) For fixed ΔT: $Q \propto m$

1) Dependence on temperature rise (mass fixed)

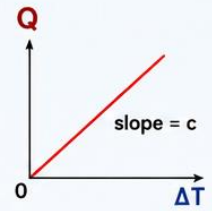
Process	ΔT	Heat Supplied Q	Time on stove (same flame)
10° → 20°	10°C	Q	t
10° → 30°	20°C	2Q	2t
10° → 40°	30°C	3Q	3t

Since Q is proportional to ΔT (mass fixed), we can write

$Q = f(\Delta T)$

For a constant mass, f is a linear function:

$f(\Delta T) = c\,\Delta T$



2) Dependence on mass (ΔT fixed)

ΔT = 10°C
Heat = Q
Time = t

ΔT = 10°C
Heat = 2Q
Time = 2t

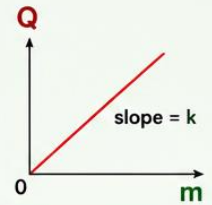
ΔT = 10°C
Heat = Q + Q = 2Q
Time = t + t = 2t

Since Q is proportional to mass m (ΔT fixed), we can write

$Q = g(m)$

For a fixed ΔT, g is a linear function:

$g(m) = k\,m$



3) Combining both dependences

From (1): $Q = f(\Delta T) = c\,\Delta T$

From (2): $Q \propto m$ → multiply by m

→ $Q = c\,m\,\Delta T$

where:
- m = mass of the body
- ΔT = temperature rise
- c = constant (depends on the material and units)

c is whatever constant makes the equation true for that material and unit system.

THE FINAL FORMULA

$$Q = c\ m\ \Delta T$$

(Heat required = specific constant × mass × temperature rise)

WHAT IT MEANS

The heat Q needed to raise the temperature of a body by ΔT is directly proportional to both the mass of the body (m) and the temperature rise (ΔT).

SPECIFIC HEAT CAPACITY

The constant c is called the specific heat capacity of the material.
Units: J kg^{-1} °C^{-1} (or J kg^{-1} K^{-1})

Remember: More mass or more temperature rise → more heat required (same flame).

At the same conditions: $Q \propto m$ and $Q \propto \Delta T \Rightarrow Q \propto m\,\Delta T$

WHAT IS *c*? — THE PRICE TAG OF A MATERIAL

THE CORE EQUATION

$$Q = c\ m\ \Delta T$$

- Q — heat supplied (J)
- c — specific heat capacity (J/(kg·K))
- m — mass (kg)
- ΔT — temperature rise (K or °C)

What is *c*?

c is not derived — it is defined by this equation.

c is the price tag of the material: the heat (in joules) you must pay to raise the temperature of 1 kg of that material by 1 degree.

Units: J/(kg·K) (or J/(kg·°C))

Different materials, different price tags (an experimental fact)

Material	Specific heat capacity *c* (J/(kg·K))	Meaning
Water	$c \approx 4186$	Expensive to heat (temperature rises slowly)
Iron	$c \approx 450$	Cheaper than water
Copper	$c \approx 385$	Cheaper than iron

A DEMONSTRATION STUDENTS REMEMBER

Equal masses, same flames, different materials

Same *m*, Same flame (same *Q* in the same time) but different ΔT → different *c*

Takes longer to heat up. Smaller ΔT in the same time.

Heats up much faster. Larger ΔT in the same time.

Everyday version: nature's demonstration

Same sun (same energy) but different ΔT → different *c*

Small *c* (sand) ← → Large *c* (water)

THE TAKEAWAY

- *c* tells you how hard (or easy) it is to change the temperature of a material.
- Bigger *c* → needs more heat for the same temperature rise.
- Smaller *c* → needs less heat for the same temperature rise.
- *c* is measured in the lab → it is a property of the material.

MEMORY LINE

$$Q = c\ m\ \Delta T$$

The heat you pay equals the price (c) times the amount (m) times the temperature rise (ΔT).

c = price tag of the material

Observation

WHAT IS c?

From the law of heating: $Q = c\,m\,\Delta T$

The crucial framing:

c is not derived — it is defined by this equation.

It's the price tag of the material: joules per kilogram per degree.

Units of c:

$J / (kg \cdot K)$

(joules per kilogram per kelvin or per degree Celsius)

Different materials have different price tags.
This is an experimental fact, not a mathematical one.

Material	Specific heat capacity, c	What it means
Water	c ≈ 4186 J/(kg·K)	Famously expensive to heat
Iron	c ≈ 450 J/(kg·K)	Heats up (and cools down) much faster than water
Copper	c ≈ 385 J/(kg·K)	Heats up even faster than iron

Demonstrations students remember

Equal masses on identical flames:

- Same m
- Same Q (same flame, same time)
- Oil gets hot much faster → larger ΔT
- Therefore different c

Everyday version:

Same sun, same energy. Sand is scorching while the sea is cool. Different ΔT, therefore different c.

CALIBRATION — HOW c IS ACTUALLY MEASURED (METHOD OF MIXTURES)

Drop a hot metal block into cool water. Wait for a common final temperature T_f.

1. Hot metal block
   Mass m_1
   Specific heat c_1
   Initial temp. T_1
   (c_1 is unknown)
2. Cool water in insulated container
   Mass m_2
   Specific heat c_2 (known)
   Initial temp. T_2
3. Put the hot block into the water
4. Wait for thermal equilibrium
   Common final temperature T_f

Conservation of energy: heat lost = heat gained

Metal (loses heat): $Q_{lost} = m_1 c_1 (T_1 - T_f)$

Water (gains heat): $Q_{gain} = m_2 c_2 (T_f - T_2)$

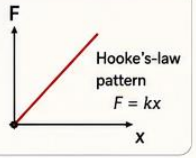
$$m_1 c_1 (T_1 - T_f) = m_2 c_2 (T_f - T_2)$$

Solving for the unknown c_1:

$$c_1 = \frac{m_2 c_2 (T_f - T_2)}{m_1 (T_1 - T_f)}$$

Everything on the right is measurable ($m_1, m_2, c_2, T_1, T_2, T_f$), so c_1 can be determined.

Note for the class:
This experiment presupposes the law from Step 3 ($Q = c\,m\,\Delta T$). Calibration always sits on top of the law it calibrates — just like Hooke's-law experiments.



Observation

CALIBRATION — HOW C IS ACTUALLY MEASURED

METHOD OF MIXTURES

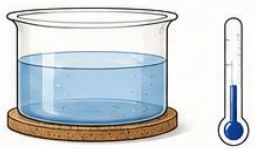
Drop a hot metal block into cool water. Wait for a common final temperature T_f.

1 Hot metal block

Mass m_1
Specific heat c_1 (unknown)
Initial temperature T_1

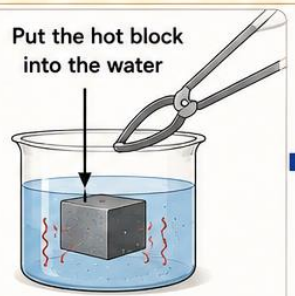
(c_1 is the only unknown)

2 Cool water in insulated container

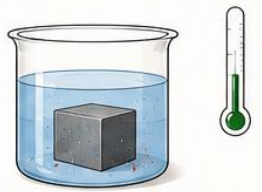


Mass m_2
Specific heat c_2 (known)
Initial temperature T_2

3 Put the hot block into the water



4 Wait for thermal equilibrium



Common final temperature T_f

What we can measure

- ✓ m_1: mass of metal block
- ✓ T_1: initial temperature of block
- ✓ m_2: mass of water
- ✓ T_2: initial temperature of water
- ✓ T_f: final (common) temperature
- ✓ c_2: specific heat of water (known: $c_2 \approx 4186$ J/(kg·K))
- ✗ c_1: specific heat of the metal (unknown)

CONSERVATION OF ENERGY: HEAT LOST = HEAT GAINED

Metal (loses heat) $Q_{\text{lost}} = m_1\, c_1\, (T_1 - T_f)$ = Water (gains heat) $Q_{\text{gain}} = m_2\, c_2\, (T_f - T_2)$

$$m_1\, c_1\, (T_1 - T_f) = m_2\, c_2\, (T_f - T_2)$$

Solving for the unknown c_1:

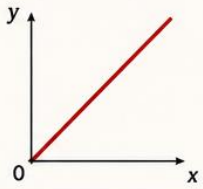
$$c_1 = \frac{m_2\, c_2\, (T_f - T_2)}{m_1\, (T_1 - T_f)}$$

All quantities on the right are measurable or known, so c_1 can be determined.

NOTE FOR THE CLASS

This experiment presupposes the law from Step 3: $Q = c\, m\, \Delta T$.
We use that law to write the heat lost and heat gained. Calibration always sits on top of the law it calibrates — just like Hooke's-law experiments ($F = kx$).

The pattern (again):



- Find a linear law: $y = kx$
- Calibrate the constant k with an experiment
- Then use the law to make predictions

Why this works (ideal conditions)

- The container is insulated → negligible heat exchange with surroundings.
- No heat loss to the air.
- Good stirring → uniform temperature.
- Metal block quickly reaches the same temperature as the water (thermal equilibrium).

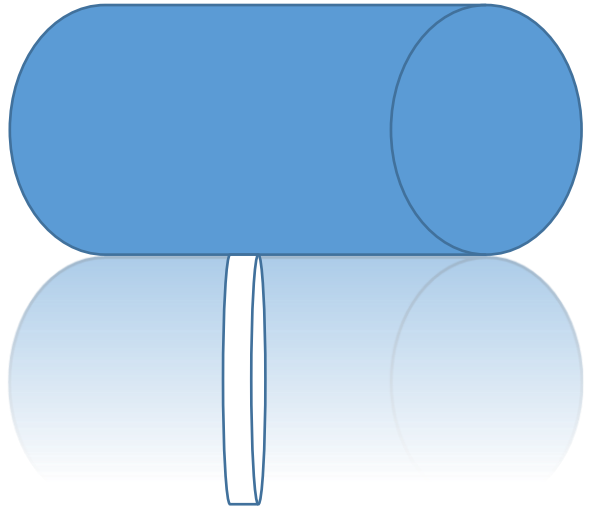
Summary: By mixing a hot object with cooler water and measuring temperatures and masses, we use energy conservation and $Q = c\, m\, \Delta T$ to determine the unknown specific heat c_1.

Typical result (examples)

Iron: $c \approx 450$ J/(kg·K) Copper: $c \approx 385$ J/(kg·K) Water: $c \approx 4186$ J/(kg·K)

Derivation: Heat Equation

Rate of change of heat content = net rate of heat conducted in and out of section



- Let $Q(t)$ be the heat content of the section between x and $x + \delta x$
- How much heat does it take to warm a cup of water?

$$Q = mc\Delta T$$

Derivation: Heat Equation

HEAT CONTENT: How a Body "Stores" Heat Energy

The idea

If raising a mass m from 0 to temperature T costs mcT, then we can think of a body at temperature T as "storing heat energy".

Heat content

$$Q = m\,c\,T$$

- Q — heat content (J)
- m — mass (kg)
- c — specific heat [J/(kg·K)]
- T — temperature (K or °C)

Reference: $T = 0$

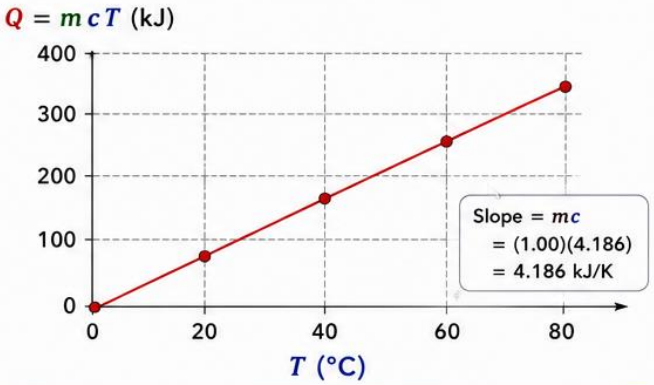
We choose $T = 0$ as the reference (zero of heat content).

Then the heat content at any temperature T is measured relative to this reference.

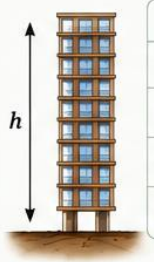
Examples (same material, different temperatures)

	Ice cold $T = 0$ °C	Cool $T = 20$ °C	Warm $T = 40$ °C	Hot $T = 60$ °C	Very hot $T = 80$ °C
Mass m	1.00 kg	1.00 kg	1.00 kg	1.00 kg	1.00 kg
Specific heat c	4.186 kJ/(kg·K)	4.186 kJ/(kg·K)	4.186 kJ/(kg·K)	4.186 kJ/(kg·K)	4.186 kJ/(kg·K)
Temperature T	0 °C	20 °C	40 °C	60 °C	80 °C
Heat content $Q = mcT$ (stored heat energy)	0 kJ	83.7 kJ	167.4 kJ	251.2 kJ	334.9 kJ

Heat content increases linearly with temperature (for fixed m and c)



Analogy: Gravitational potential energy



	Gravitational	Thermal (heat)
Reference level	$h = 0$ (ground)	$T = 0$
Stored energy	$U = m\,g\,h$	$Q = m\,c\,T$
Depends on	height h	temperature T
Proportional to	mass m	mass m

$U = m\,g\,h$ — Zero reference (chosen)

$Q = m\,c\,T$ — Zero reference (chosen)

Key takeaways

- $Q = mcT$ tells us how much heat is required to raise a mass m from 0 to temperature T.
- We can interpret $Q = mcT$ as the "heat content" (stored heat) of the body at temperature T relative to $T = 0$.
- Heat content grows linearly with temperature (for fixed m and c).
- Choosing $T = 0$ is like choosing the zero level in gravitational potential energy.

Think: Heat is to temperature what weight is to height.

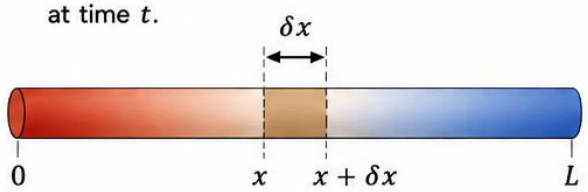
Derivation: Heat Equation

Apply It to the Slice — and Hit a Problem

$Q = m\,c\,T$

1 Take a thin slice of the rod

Consider the slice between x and $x + \delta x$ at time t.



Cross-sectional area = A (constant)

2 Mass of the slice

Volume of slice = $A\,\delta x$

Density = ρ (constant)

$m = \rho \times \text{volume} = \rho A\,\delta x$

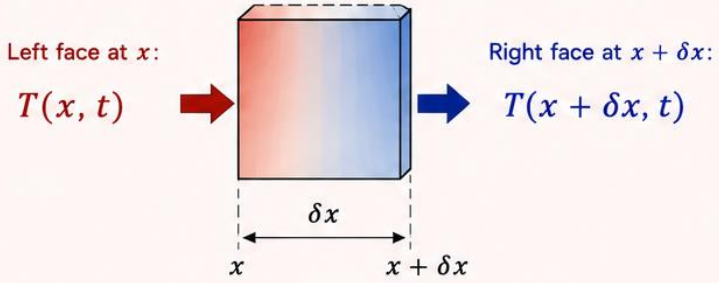
3 Try to use $Q = m\,c\,T$

If the slice had one temperature T, then

$$Q = c\,m\,T$$
$$= c\,(\rho A\,\delta x)\cdot T$$
$$= c\,\rho A\,\delta x \cdot T$$

? But now pause and ask: *T at which point?*

The temperature is not the same across the slice — that's the whole reason heat is flowing!



- $T(x, t)$ at the left face $\neq$ $T(x + \delta x, t)$ at the right face.
- Inside the slice, temperature varies continuously from $T(x, t)$ to $T(x + \delta x, t)$.
- The formula $Q = m\,c\,T$ assumes one temperature for the whole mass. Our slice does not have one.

This is the problem we must solve.
The integral is about to arrive as the solution to a problem we can feel — not as notation from the sky.

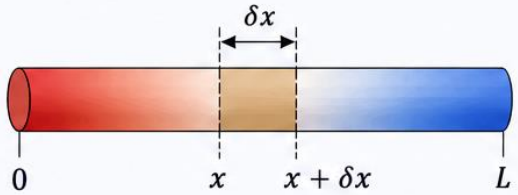
Takeaway: To find the heat content of the slice when temperature varies inside it, we must add up (integrate) the contributions from each tiny piece that has (almost) a uniform temperature.

Derivation: Heat Equation

From a Slice to an Integral: Building the Heat Content

1 The slice we want to study

Consider the slice between x and $x + \delta x$ at time t.

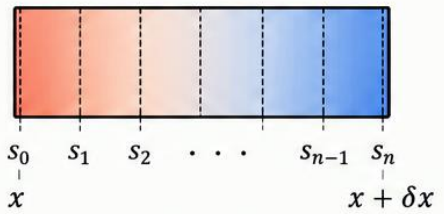


Cross-sectional area $= A$ (constant)

2 Divide the slice into thin sub-slices

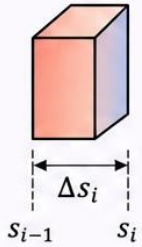
Partition $[x, x + \delta x]$ into n sub-slices:

$$x = s_0 < s_1 < \cdots < s_n = x + \delta x.$$



3 Each sub-slice is (almost) uniform

Each sub-slice is so thin that its temperature is approximately constant, say $T(s_i, t)$.



Mass of i-th sub-slice:

$$m_i = \rho A \, \Delta s_i$$

Heat content of i-th sub-slice:

$$Q_i \approx m_i \, c \, T(s_i, t) = c \rho A \, \Delta s_i \, T(s_i, t)$$

4 Sum over all sub-slices

The total heat content in the slice is the sum of the heat contents of all sub-slices:

$$Q(t) \approx c \rho A \sum_{i=1}^{n} T(s_i, t) \, \Delta s_i$$

This is a **Riemann sum** for the temperature function $T(s, t)$ over the interval $[x, x + \delta x]$.

5 Refine the subdivision

Make the sub-slices thinner and thinner (max $\Delta s_i \to 0$). The Riemann sum becomes an integral:

$$Q(t) = c \rho A \int_{x}^{x+\delta x} T(s, t) \, ds$$

Now we have the heat content of the slice when temperature varies across it.

6 What did we achieve?

- We started with the simple formula $Q = mcT$ for a uniform temperature.
- A small slice does not have a single temperature, so we split it into tiny pieces.
- Summing their heat contents leads to a Riemann sum, and in the limit, to an integral.
- The integral gives the correct heat content of the slice even when the temperature varies inside it.

Takeaway: When temperature is not uniform, we find the heat content by adding (integrating) the contributions from many tiny pieces, each of which has an (almost) uniform temperature. The integral is the natural answer to a real problem — not just notation.

Derivation: Heat Equation

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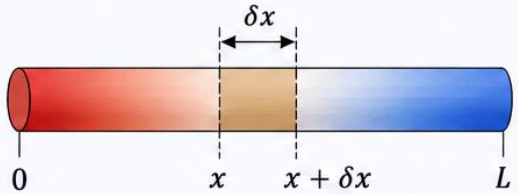

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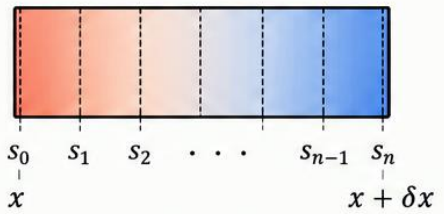

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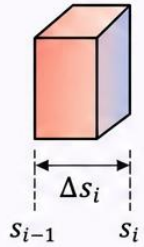

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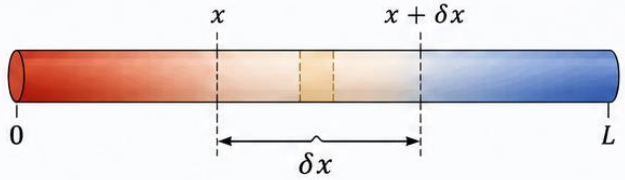
Derivation: Heat Equation

Heat Content of a Section: From Variable Temperature to an "Effective" Temperature

1 Heat content $Q(t)$ of the section between x and $x + \delta x$

Consider a uniform rod (cross-sectional area A, density ρ, specific heat c). Temperature varies with position x at time t: $T(x, t)$.



We want the total heat stored in this section at time t.

2 Heat content $Q(t)$ of the section

$$Q(t) = c\,\rho\,A \int_{x}^{x+\delta x} T(s, t)\, ds$$

This comes from summing the heat in infinitesimal slices and taking the limit (Riemann sum → integral).

3 By the Mean Value Theorem for integrals

Since $T(s, t)$ is continuous on $[x, x + \delta x]$, there exists some $\xi \in (x, x + \delta x)$ such that

$$\int_{x}^{x+\delta x} T(s, t)\, ds = T(\xi, t)\, \delta x$$

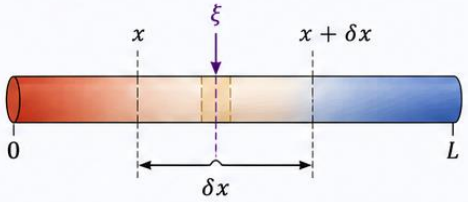
4 Hence,

$$Q(t) = c\,\rho\,A\,\delta x \cdot T(\xi, t)$$

The total heat in the section equals (material constant) × (volume of the section) × (temperature at some intermediate point ξ).

5 Temperature at the intermediate point ξ *represents* the entire section

Although temperature varies across the section, its heat content is exactly what it would be if the whole section of length δx were at the uniform temperature $T(\xi, t)$.



So the section behaves, for heat content, as if it were at temperature $T(\xi, t)$.

This "effective" temperature lies somewhere inside the section.

Derivation: Heat Equation

Mean Value Theorem for Integrals

Theorem (Mean Value Theorem for Integrals)

Let $f : [a, b] \to \mathbb{R}$ be a function such that

- f is continuous on the closed interval $[a, b]$.
- $a < b$ (i.e., the interval has positive length).

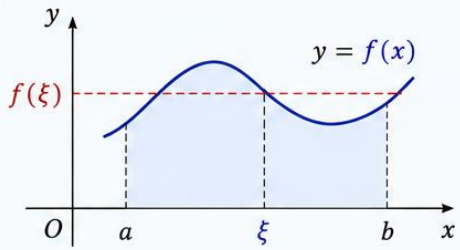
Then there exists **at least one point** $\xi \in (a, b)$ such that

$$\int_a^b f(x)\,dx = f(\xi)\,(b-a)$$

or, equivalently,

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

In words: The **average value** of f on $[a, b]$ is attained by f at some point ξ inside the interval.



Conditions are essential:

- Continuity on $[a, b]$ guarantees the existence of such a point ξ.
- Without continuity, the conclusion may fail.

Applied to the Heat Content Problem

Assume for each fixed t, the temperature function $T(s, t)$ is continuous with respect to s on $[x, x + \delta x]$.

By the Mean Value Theorem for integrals, there exists some $\xi \in (x, x + \delta x)$ such that

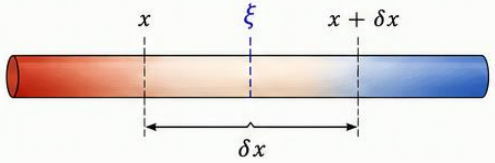
$$\int_x^{x+\delta x} T(s, t)\,ds = T(\xi, t)\,\delta x$$

$\Rightarrow$

Heat content of the section

$$Q(t) = c\,\rho\,A \int_x^{x+\delta x} T(s, t)\,ds$$

$$= c\,\rho\,A\,\delta x\,T(\xi, t)$$



Even though temperature varies across $[x, x + \delta x]$, the entire section has the same heat content as if it were at the uniform temperature $T(\xi, t)$ everywhere.

Note: The point ξ is not necessarily the midpoint $\frac{x+x+\delta x}{2} = x + \frac{\delta x}{2}$.
It is some (possibly unknown) point inside the interval whose existence is guaranteed by the theorem.

Observation

General Case: When the Specific Heat Varies with Temperature

We started with the law of heating

$$Q = c\ m\ \Delta T$$

- c = specific heat (constant)
- m = mass (constant)
- $\Delta T = T_2 - T_1$

But c can depend on temperature:

$$c = c(T)$$

(measured experimentally)

The general formula is

$$Q = m \int_{T_1}^{T_2} c(T)\, dT$$

- m = mass (kg)
- $c(T)$ = specific heat as a function of temperature [J/(kg·K)]
- T_1 = initial temperature (K)
- T_2 = final temperature (K)
- Q = heat added to raise the temperature from T_1 to T_2

Why this matters — the "constant → varying → integrate" upgrade

1 Honest physics

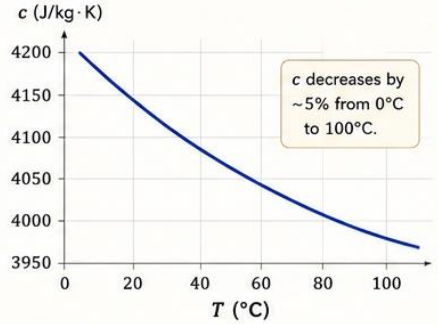
Specific heat is not truly constant. It changes with temperature. The integral gives the exact heat.

2 Same move we'll use in $Q(t)$

We replaced a constant by a varying quantity and integrated to get the total. You'll do the same upgrade in space for heat rate → $Q(t)$ on the next slide.

What we do	In temperature (this slide)	In space (next slide)
Constant case	c is constant $Q = c\ m\ \Delta T$	Heat rate is uniform $Q = q\ L$ (length L)
Variable case	c varies with T $Q = m \int_{T_1}^{T_2} c(T)\, dT$	Heat rate varies with x $Q = \int_{x_1}^{x_2} q(x)\, dx$
What we did	Replace constant → function of T, then integrate over T	Replace constant → function of x, then integrate over x

Example: $c(T)$ for water (liquid)



WORKED EXAMPLE

Heat 1.00 kg of water from 20°C to 80°C. Use the (approximate) data in the table.

T (°C)	$c(T)$ [J/(kg·K)]
20	4182
40	4173
60	4157
80	4136

Approximate the integral using the trapezoidal rule.

$$Q \approx m \sum_{i=1}^{n} \frac{c_i + c_{i-1}}{2} (T_i - T_{i-1})$$

Here, $\Delta T = 20$°C for each step.

$$Q \approx 1.00 \left[\frac{4182+4173}{2}(20) + \frac{4173+4157}{2}(20) + \frac{4157+4136}{2}(20) \right]$$

$$= 1.00\ (4177.5 + 4165.0 + 4146.5)$$

$$= 12\ 489\ \text{J} \approx \boxed{12.5\ \text{kJ}}$$

Compare to constant-c

If we (incorrectly) used $c = 4180$ J/(kg·K),

$Q = c\ m\ \Delta T$

$= 4180 \cdot 1.00 \cdot 60$

$= 12\ 540$ J

(error ≈ 0.4%)

Takeaway: The integral $Q = m \int_{T_1}^{T_2} c(T)\, dT$ is the honest, general statement of the heating law.

It reduces to $Q = c\ m\ \Delta T$ when c is constant.

The move you're learning (constant → varying → integrate) is the same move we'll use in space for $Q(t)$.

Observation

Differentiating the Heat Content with Respect to Time

1. Leibniz Rule (Differentiation under the Integral Sign)

Let $f(s,t)$ and $\frac{\partial f}{\partial t}(s,t)$ be continuous on the rectangle $[a,b] \times [t_0, t_1]$.

Let a and b be constants (independent of t).

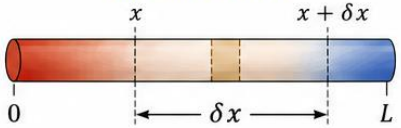
Then

$$\frac{d}{dt}\left[\int_a^b f(s,t)\,ds\right] = \int_a^b \frac{\partial f(s,t)}{\partial t}\,ds$$

Why it holds (intuition):

If the integrand and its partial time derivative are continuous, the limit defining the derivative can be passed under the integral sign when the limits are fixed.

Rod and Section



We consider the section between x and $x + \delta x$ of a uniform rod with area A, density ρ, specific heat c. Temperature: $T(s,t)$.

2. Differentiate the integral form under the integral sign

Heat content of the section

$$Q(t) = c\,\rho\,A\int_x^{x+\delta x} T(s,t)\,ds$$

$\Rightarrow$ Differentiate both sides with respect to t.

$$\frac{dQ}{dt} = c\,\rho\,A\,\frac{d}{dt}\left[\int_x^{x+\delta x} T(s,t)\,ds\right]$$

$\Rightarrow$ By Leibniz rule (limits x, $x+\delta x$ are fixed):

$$\frac{dQ}{dt} = c\,\rho\,A\int_x^{x+\delta x} \frac{\partial T(s,t)}{\partial t}\,ds$$

3. Apply the Integral Mean Value Theorem (again)

The function $\frac{\partial T(s,t)}{\partial t}$ is continuous in s on $[x, x+\delta x]$ (for each fixed t).

By the Integral Mean Value Theorem,

there exists $\xi \in (x, x+\delta x)$ such that

$$\int_x^{x+\delta x} \frac{\partial T(s,t)}{\partial t}\,ds = \frac{\partial T(\xi,t)}{\partial t}\,\delta x$$

$\Rightarrow$ Hence,

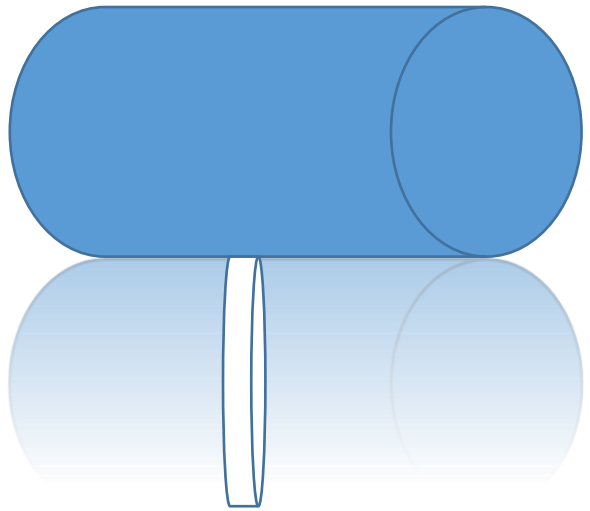
$$\frac{dQ}{dt} = c\,\rho\,A\,\delta x\,\frac{\partial T(\xi,t)}{\partial t}, \quad \xi \in (x, x+\delta x)$$

(The time rate of change of heat content depends on the time rate of change of temperature at some intermediate point ξ in the section.)

Takeaway: We first express the heat content as an integral of $T(s,t)$, then differentiate under the integral sign (Leibniz rule), and finally use the Integral Mean Value Theorem again to represent the result using the derivative of temperature at some intermediate point ξ.

Derivation: Heat Equation

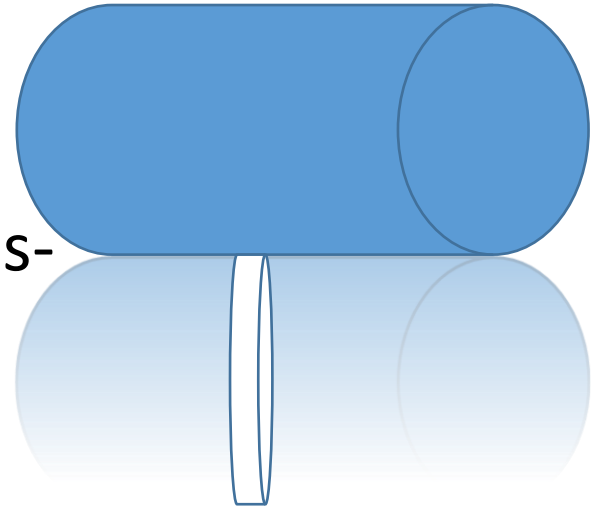
- **Rate of change of heat content** = $c\rho A\delta x \frac{\partial T(\xi,t)}{\partial t}$
- $\xi \in (x, x+\delta x)$
- c proportionality factor or specific heat constant, depends on metal



Rate of change of heat content = net rate of heat conducted in and out of section

Derivation: Heat Equation

Rate of change of heat content = net rate of heat conducted in and out of section



- Heat flux $J(x,t)$: The rate of heat passing through a cross-section, per unit area, per unit time

Fourier's Law

More is the area exposed → More will be the heat transferred

Fourier's Law

More is the area exposed → More will be the heat transferred

$$J \propto A$$

Fourier's Law

More is the temperature difference → More will be the heat transfer.

Fourier's Law

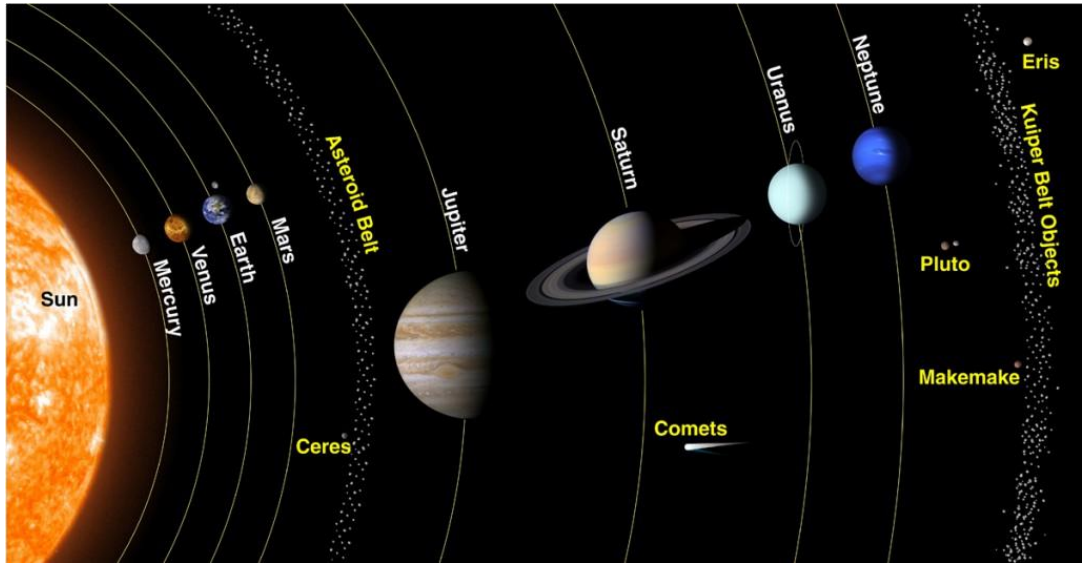
More is the temperature difference → More will be the heat transfer.

$$J \propto dT$$

Fourier's Law

Farther is the heat source – Lesser is the heat transferred

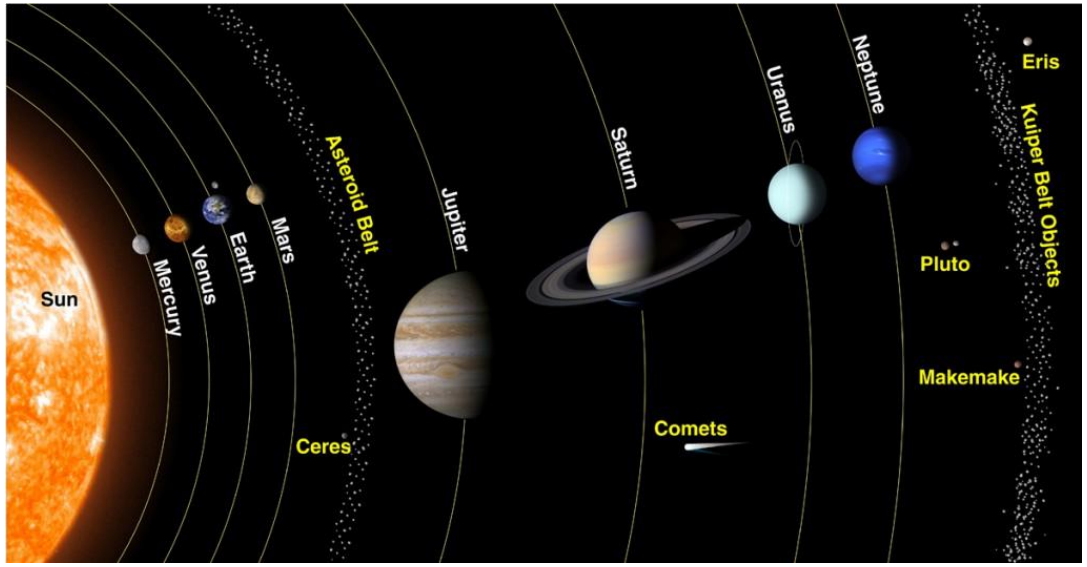
Rank	Planet	Mean Temperature (C)
1	Venus	464
2	Mercury	167
3	Earth	15
4	Mars	-65
5	Jupiter	-110
6	Saturn	-140
7	Uranus	-195
8	Neptune	-200
9	Pluto	-225



Fourier's Law

Farther is the heat source – Lesser is the heat transferred

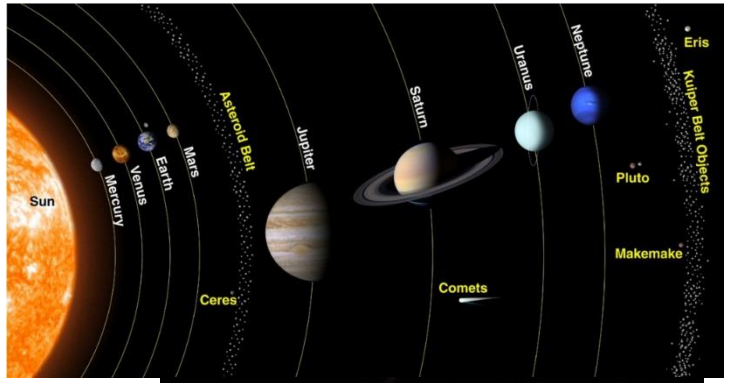
$$J \propto \frac{1}{dx}$$



Fourier's Law

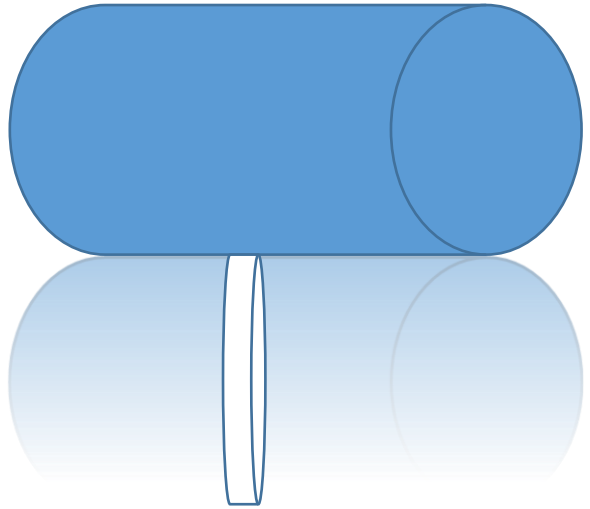
$$J \propto A \frac{dT}{dx}$$

$$J = -kA \frac{dT}{dx}$$



Derivation: Heat Equation

- **Rate of change of heat content** = $c\rho A\delta x \frac{\partial T(\xi,t)}{\partial t}$
- **Net rate of heat conducted in and out of cross section**
- $J(x,t) - J(x+\delta x,t) = kA\frac{dT(x+\delta x,t)}{dx} - kA\frac{dT(x,t)}{dx}$
- $c\rho A\delta x \frac{\partial T(\xi,t)}{\partial t} = kA\frac{dT(x+\delta x,t)}{dx} - kA\frac{dT(x,t)}{dx}$
- $c\rho A\delta x \frac{\partial T(\xi,t)}{\partial t} = kA\left[\frac{dT(x+\delta x,t)}{dx} - \frac{dT(x,t)}{dx}\right]$
- $c\rho \frac{\partial T(\xi,t)}{\partial t} = k\frac{\left(\frac{dT(x+\delta x,t)}{dx} - \frac{dT(x,t)}{dx}\right)}{\delta x}$
- $c\rho \frac{\partial T(x,t)}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$



Heat Equation

$$c\rho\frac{\partial T(x,t)}{\partial t}=k\frac{\partial^2 T}{\partial x^2}$$

$$c\rho\frac{\partial T(x,t)}{\partial t}=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)$$

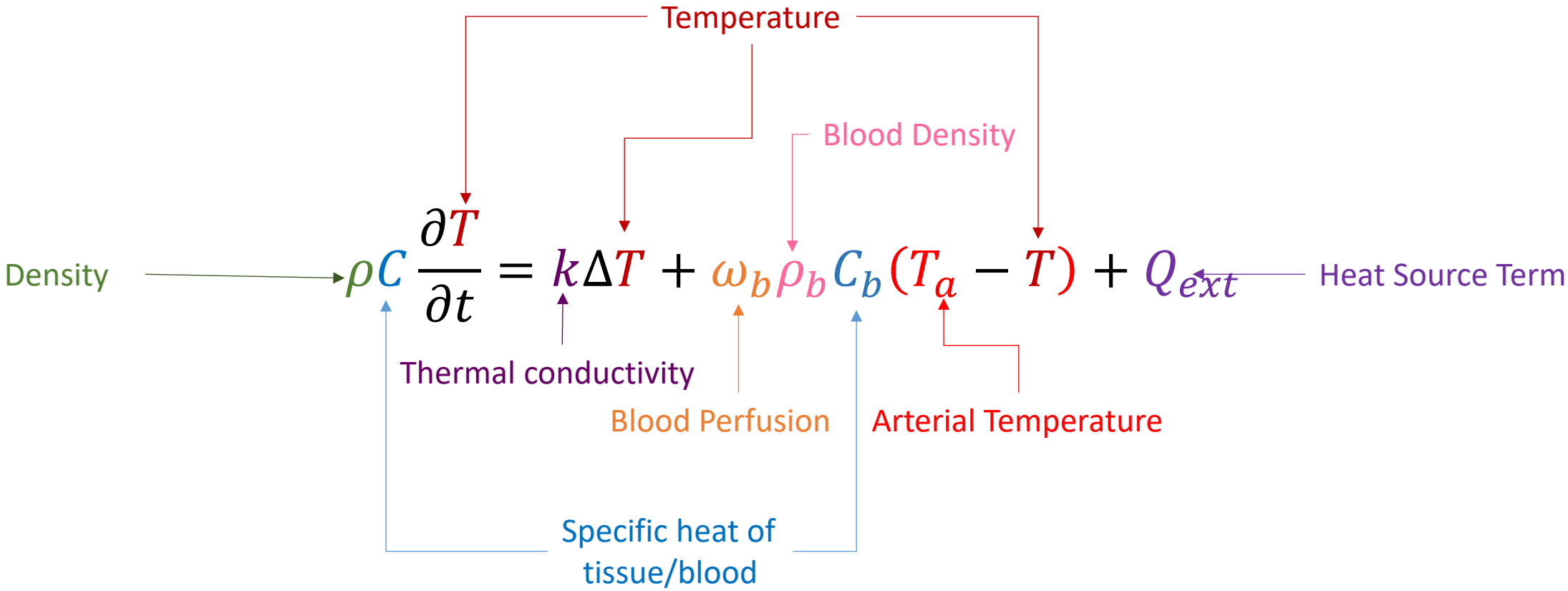
$$c\rho\frac{\partial T(x,t)}{\partial t}=k\nabla^2 T$$

Bioheat Equation

$$c\rho \frac{\partial T(x,t)}{\partial t} = k\nabla^2 T$$

$$c\rho \frac{\partial T(x,t)}{\partial t} = k\Delta T + \mathrm{Q_{ext}} + \mathrm{Q}_{sink}$$

Mathematical Model

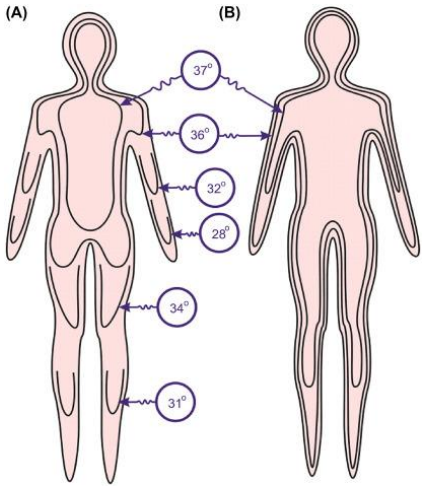


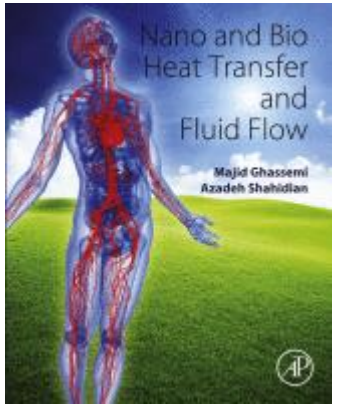
Bioheat Equation: Exercises

- https://www.sciencedirect.com/topics/engineering/bioheat-transfer

$$\rho C \frac{\partial T}{\partial t} = k\Delta T + Q_{perf} + Q_m + Q_{others}$$

"Since the body does not operate with 100% efficiency, only a fraction of the metabolic rate is applied to work and the remainder shows up as heat. The mechanical efficiency associated with metabolic energy utilization is zero for most activities except when the person is performing external mechanical work, such as in walking upstairs, lifting something to a higher level, or cycling on an exercise machine. When external work is dissipated into heat in the human body, the mechanical efficiency is negative. An example of negative mechanical efficiency is walking downstairs. Storage of energy takes place whenever there is an imbalance of production and dissipation mechanisms. In many instances, such as astronauts in space suits or military personnel in chemical defense garments, energy storage is forced due to the lack of appropriate heat exchange with the environment."

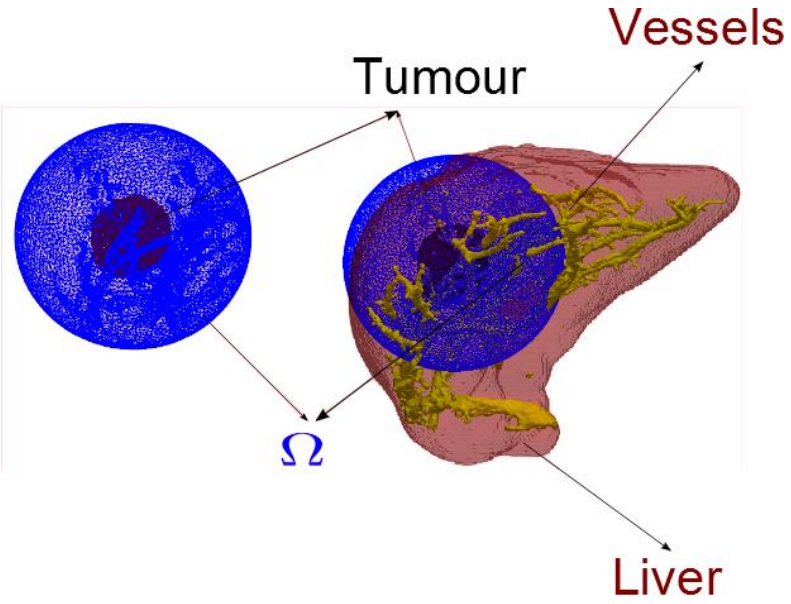




Majid Ghassemi and Azadeh Shahidian, Nano and Bio Heat Transfer and Fluid Flow, 2017, Chapter 3.

Domain of our Problem

- What is the domain of the problem?
- Is the system closed?
- Is the problem well posed?
- Do we need initial conditions?
- Do we need boundary conditions?

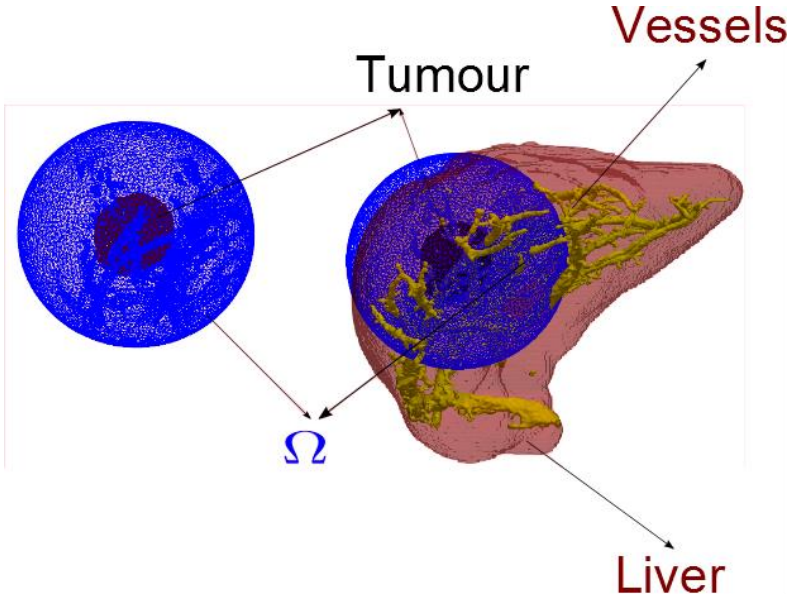


Bioheat Equation[1]

$$\rho C \frac{\partial T}{\partial t} = k\Delta T + \omega_b \rho_b C_b (T_a - T) + Q_r \; on \; \Omega$$

$$h_c T + k \frac{\partial T}{\partial \vec{n}} = h_c T_\infty \text{ on vessels boundary}$$

$$T = T_0 \text{ on } \partial\Omega$$



1. H. H. Pennes, Analysis of tissue and arterial blood temperature in the resting human forearm, J. Appl. Physio. 85(1):93-102, 1948