

MA621L-Mathematical Modelling

Model-3 : Transport and Traffic Flow

Panchatcharam Mariappan¹

¹Associate Professor
Department of Mathematics and Statistics
IIT Tirupati, Tirupati

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Transport Equation

The Ant and the Dove Story

One quiet morning, an ant was collecting food along the edge of a river. Suddenly, it slipped and fell into the flowing water. The river began to carry it downstream. A dove, watching from a nearby tree, noticed the ant struggling. Acting quickly, the dove dropped a leaf into the water just ahead of the ant.

The leaf floated with the river's current. The ant climbed onto the leaf and was carried safely to the riverbank. The ant survived, not because the water stopped, but because something carried it safely across the stream.

Ant and Dove



Ant and Dove Story

The Ant and the Dove



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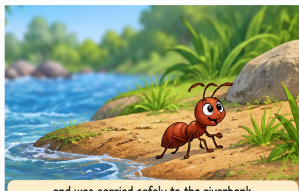
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The leaf floated with the river's current. The ant climbed onto the leaf...



...and was carried safely to the riverbank. The ant survived, not because the water stopped, but because something carried it safely across the stream.

A small act of kindness can make a big difference.

From the Story to a System

The story is a metaphor for **transport**:

- The **river flow** is the *velocity field* (constant, for now).
- The **leaf** is the carrier: a parcel of water.
- The **ant** is whatever the parcel carries: a pollutant, heat, a dye.

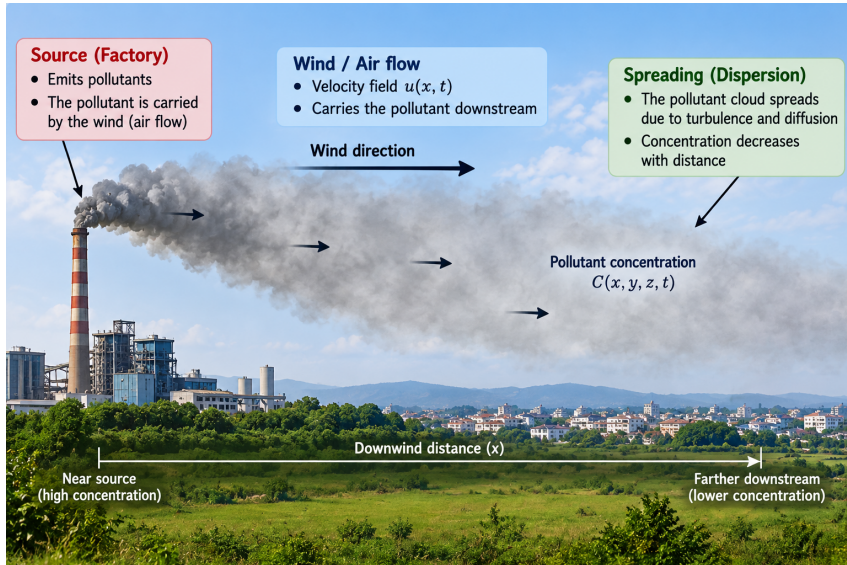
Now replace the ant with a pollutant.

What the story does not tell us

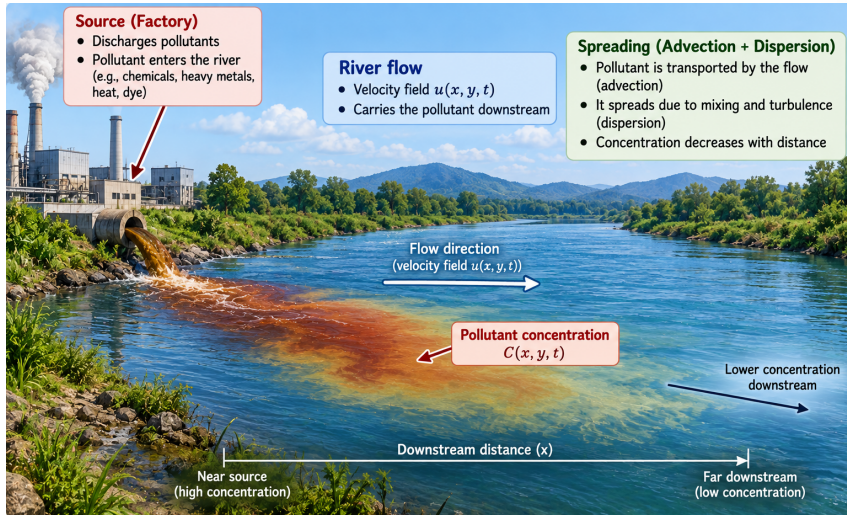
Where is the ant at time t ? When does it reach a given point downstream? Does it spread out on the way? The story has no numbers and no question. A model needs both.



Air Pollutant

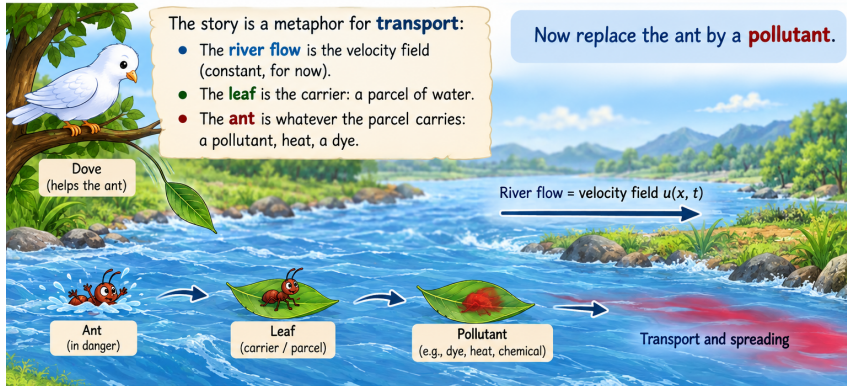


Water Pollutant



Pollutants are carried by the river flow and spread out as they move downstream.

From the Story to a System

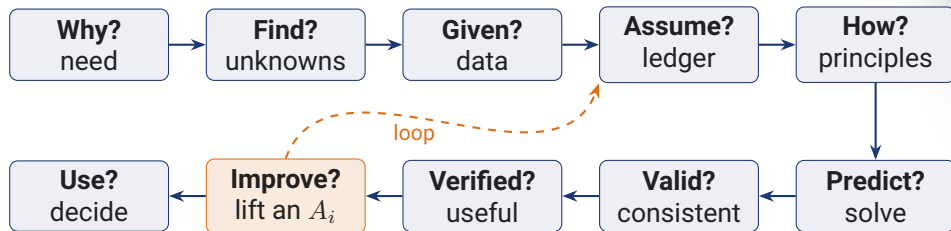


What the story does not tell us

Where is the ant at time t ? When does it reach a given point downstream?

Does it spread out on the way? The story has no numbers and no question. A model needs both.

Ten Questions Before One Equation (Dym, §1.2)



The list is not an algorithm; the questions recur. We now answer each one for the river problem, and the transport equation will appear at **How?**, not before.



Why? What are we looking for?

The need

A factory outfall discharges a dissolved chemical into a river for a short period after a process failure. A town draws its drinking water from an intake L km downstream.

The water utility must decide *when* to close the intake and *for how long*. Closing too early wastes supply; closing too late lets the chemical in.

This single sentence already fixes the scale of the model:

- we care about the reach from the outfall to the intake, not the whole river;
- we care about hours, not seconds or years;
- we care about the concentration *at one point*, $x = L$.



Find? What do we want to know?

- The linear density of the chemical $u(x, t)$ [kg m^{-1}]: mass per unit length of river, at position x and time t .
- The **arrival time** t_a of the chemical at the intake $x = L$.
- The **exposure window** $[t_a, t_e]$ during which $u(L, t) > 0$.
- The **concentration** $u(L, t)$ in that window, to compare with a safe limit.

Note

Only $u(L, \cdot)$ answers the question, but we cannot get it without $u(x, t)$ on the whole reach $0 \leq x \leq L$. The question forces a *distributed* model.



Given? What do we know?

Symbol	Meaning	Unit	Value (example)
c	mean river speed	m s^{-1}	0.5
L	outfall to intake distance	m	12 000
τ	duration of the discharge	s	1800 (30 min)
C_0	linear density at outfall while discharging	kg m^{-1}	known
$u_0(x)$	chemical already in the reach at $t = 0$	kg m^{-1}	0 (clean river)
$g(t)$	linear density entering at $x = 0$	kg m^{-1}	C_0 for $0 < t < \tau$, else 0

The data are of two kinds: an *initial* state u_0 on the reach, and an *inflow* history g at the upstream end. We will see that these are exactly the data the model can accept.



Assume? The Assumption Ledger

	Assumption	If we lift it
A1	Chemical is well mixed across the river; u depends on (x, t) only (1D).	2D/3D plume
A2	River speed c is constant and positive.	$c = c(x)$ or $c(x, t)$
A3	Speed does not depend on the chemical.	$c = c(u)$: nonlinear, shocks
A4	No diffusion or dispersion: chemical moves only with the water.	convection–diffusion
A5	Conservative substance: no decay, reaction or settling.	decay term $-ku$
A6	No sources or sinks on the reach except the outfall at $x = 0$.	source term $s(x, t)$
A7	Continuum: u is a smooth density along the river.	particle / stochastic model

Each assumption can be lifted on its own. The right-hand column is the **Improve?** step, written in advance.

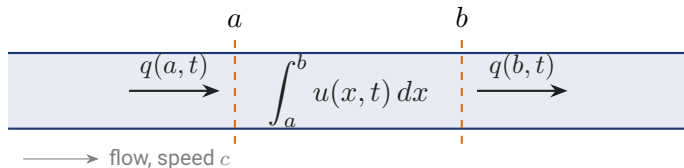


How? Which principle governs?

Conservation of mass. Dym's balance law for a quantity $Q(t)$ inside a boundary:

$$\frac{dQ}{dt} = q_{\text{in}} + g - q_{\text{out}} - c_{\text{cons}}.$$

Take the boundary to be the stretch $[a, b]$ of river. By A5 and A6 there is no generation or consumption inside it:



$$\frac{d}{dt} \int_a^b u(x, t) dx = q(a, t) - q(b, t), \quad q = \text{flux } [\text{kg s}^{-1}].$$

Conservation alone is one equation in two unknowns, u and q . We need a second, *constitutive*, statement.



How? Closing the model with a constitutive law

By A4 the chemical moves only because the water moves, so the mass crossing a point per unit time is

$$q(x, t) = c u(x, t).$$

(This is Dym's traffic relation $q = \rho v$ in another setting.) Substituting, and writing $q(a) - q(b) = - \int_a^b q_x dx$ for smooth u (A7),

$$\int_a^b (u_t + c u_x) dx = 0 \quad \text{for every } [a, b].$$

An integral that vanishes over every interval has a vanishing integrand (if continuous):

The transport equation

$$u_t + c u_x = 0, \quad 0 < x < L, \quad t > 0.$$



How? A second route: follow the water

The same law, seen by the leaf rather than by the river bank. By A2 and A4 the chemical in $[a, b]$ at time t has moved to $[a + c \delta t, b + c \delta t]$ at time $t + \delta t$, and by A5–A6 none is lost:

$$\int_a^b u(x, t) dx = \int_{a+c\delta t}^{b+c\delta t} u(x, t + \delta t) dx.$$

Differentiating with respect to b :

$$u(b, t) = u(b + c \delta t, t + \delta t).$$

The concentration at b at time t is found at $b + c \delta t$ a moment later, as it should be when the water carries it at speed c . Since b is arbitrary, replace it by x :

$$u(x, t) = u(x + c \delta t, t + \delta t).$$



How? A second route: follow the water

Then

$$\frac{u(x + c \delta t, t + \delta t) - u(x, t)}{\delta t} = 0.$$

Let $\delta t \rightarrow 0$ and put $\mathbf{v} = (c, 1)$ in the (x, t) plane:

$$D_{\mathbf{v}}u = 0 \implies (c, 1) \cdot (u_x, u_t) = 0 \implies u_t + c u_x = 0.$$

Reading the result

u does not change along the direction $(c, 1)$ in the (x, t) plane. That is the whole content of the equation, and it tells us how to solve it.

Two derivations, one Eulerian (fixed window) and one Lagrangian (moving parcel), same model. Agreement of independent routes is a first sign of consistency.

The Model M



Initial-boundary value problem

$$u_t + cu_x = 0, 0 < x < L, t > 0$$

$$u(x, 0) = u_0(x), 0 \leq x \leq L \quad \text{initial state}$$

$$u(0, t) = g(t), t \geq 0, \quad \text{inflow}$$

Where may data be given?

With $c > 0$ information travels downstream only. A condition at the inflow end $x = 0$ is needed; a condition at the outflow end $x = L$ may *not* be imposed. The value there is an output of the model; it is what we want to predict.



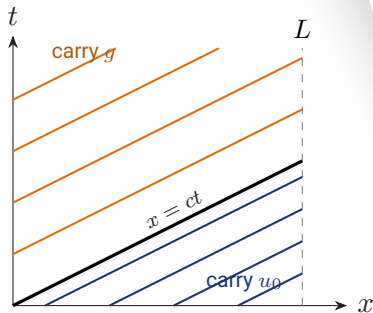
Predict? Characteristics

Along the lines $x - ct = \text{const}$ we have

$$\frac{d}{dt}u(\xi + ct, t) = cu_x + u_t = 0,$$

so u is constant on each line.

- Lines starting on $t = 0$ carry u_0 .
- Lines starting on $x = 0$ carry g .
- The line $x = ct$ separates the two regions.



$$u(x, t) = \begin{cases} u_0(x - ct), & x \geq ct, \\ g(t - x/c), & x < ct. \end{cases}$$

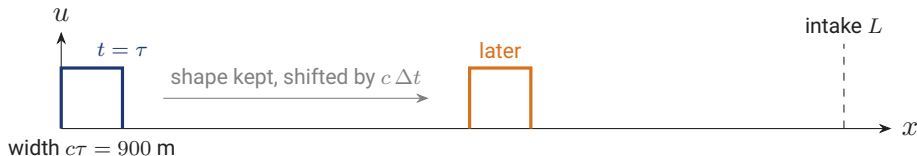


Predict? Answering the question

With $u_0 = 0$, the intake sees $u(L, t) = g(t - L/c)$: the outfall signal, delayed by L/c and otherwise unchanged.

For $c = 0.5 \text{ m s}^{-1}$, $L = 12 \text{ km}$, $\tau = 30 \text{ min}$

- Arrival time: $t_a = L/c = 24\,000 \text{ s} = 6 \text{ h } 40 \text{ min}$ after the discharge starts.
- Exposure window: $[t_a, t_a + \tau] = [6 \text{ h } 40, 7 \text{ h } 10]$.
- Concentration at the intake: C_0 , the same as at the outfall.
- Total mass passing the intake: $\int c u(L, t) dt = c C_0 \tau$, equal to the mass discharged.





Predict? When there is no formula

For $c = c(x)$, a source term, or real data for g , we compute. The first-order **upwind** scheme on a grid $x_i = i\Delta x$, $t^n = n\Delta t$:

$$u_i^{n+1} = u_i^n - \nu (u_i^n - u_{i-1}^n), \quad \nu = \frac{c \Delta t}{\Delta x}.$$

- *Upwind*: the difference looks upstream, where the information comes from.
- Stable only if $\nu \leq 1$ (CFL condition): in one step the water must not travel further than one cell.
- For $\nu = 1$ the scheme is exact: $u_i^{n+1} = u_{i-1}^n$ is the characteristic shift.
- For $\nu < 1$ the scheme smears sharp fronts. That smearing is *numerical* diffusion, not physics. Do not confuse it with lifting A4.



Valid? Is the model consistent with its principles and assumptions?

1. **Dimensions.** $[u_t] = \text{kg m}^{-1} \text{s}^{-1}$ and $[c u_x] = \text{m s}^{-1} \cdot \text{kg m}^{-2} = \text{kg m}^{-1} \text{s}^{-1}$.
2. **Mass balance.** $\frac{d}{dt} \int_0^L u \, dx = c u(0, t) - c u(L, t)$: what enters minus what leaves (A5, A6).
3. **Limiting case.** $c \rightarrow 0$ gives $u_t = 0$: nothing moves. Correct for a still pond without diffusion.
4. **Shape.** The profile is translated, never spread: exactly what A4 says.

Valid? Well-posedness

5. **Well-posedness** (Hadamard). The solution exists and is unique by characteristics, and

$$\sup |u - \tilde{u}| \leq \max \left(\sup |u_0 - \tilde{u}_0|, \sup |g - \tilde{g}| \right),$$

so small errors in the data give small errors in the prediction.

A subtlety

Our g jumps at $t = 0$ and $t = \tau$, so u is discontinuous along two characteristics and $u_t + cu_x = 0$ fails there in the classical sense. The *integral* balance still holds. The integral form is the more basic model.



Verified? Are the predictions good for the purpose?

Test. Release a harmless tracer dye at the outfall for a known time; record its concentration at the intake. **Typical tracer results on real rivers:**

- The cloud arrives at roughly the time $L/\bar{c}$ predicts, using the mean speed.
- It arrives *spread out*: lower peak, earlier leading edge, and a long trailing tail.

Verdict for our question

- Peak concentration: the model *overestimates* it. Safe side.
- Arrival: the model says 6 h 40 min; the true leading edge comes earlier.
- Window: the model gives 30 min; the tail lasts longer. **Unsafe side.**

The error that matters is set by **Use?**, not by the size of the error. For an intake closure the window is the critical output, and A4 is the assumption to lift first.



Improve? Lift one assumption at a time

Lift	New model	What changes
A4	$u_t + c u_x = D u_{xx}$	spreading $\sim \sqrt{2Dt}$; Péclet number cL/D
A5	$u_t + c u_x = -k u$	$u(L, t) = g(t - L/c) e^{-kL/c}$
A2	$u_t + c(x) u_x = 0$	curved characteristics; $t_a = \int_0^L dx/c(x)$
A3	$u_t + (f(u))_x = 0$	nonlinear: shocks and fans (traffic)
A6	$u_t + c u_x = s(x, t)$	tributaries, lateral inflow
A1	$u_t + \mathbf{v} \cdot \nabla u = 0$	plume near the outfall before mixing

Each line is a new (S, Q, M) and must be re-validated. Implement the loop

model $\rightarrow$ validate $\rightarrow$ verify $\rightarrow$ improve $\rightarrow$ predict.

The transport equation is the base case of all of them.



Use? How will we exercise the model?

- **Early warning.** Close the intake at t_a — (safety margin) and reopen after $t_a + \tau +$ (margin); the margins come from the **Verified?** step.
- **Monitoring design.** A sensor at distance d upstream of the intake gives d/c of warning time. How far upstream must it be?
- **Inverse use.** Two sensors, a known distance apart, measure arrival times: this identifies c . With decay, the ratio of peaks identifies k .

The same M , other systems S

Traffic density on a road, a solute band in a chromatography column, phonon energy in a crystal, age-structured populations ($u_t + u_a = -\mu u$). One conservation law, one flux law.



Velten's (S, Q, M)

The Model as a Triple (S, Q, M) (Velten, Def. 1.4.1)

- S A reach of river from an outfall to a drinking-water intake, carrying a dissolved chemical.
- Q When, for how long, and at what concentration does the chemical reach the intake?
- M $u_t + c u_x = 0$ on $0 < x < L$, $u(x, 0) = u_0(x)$, $u(0, t) = g(t)$, together with A1–A7.

State variable and reduced system (Velten, Defs. 1.5.1–1.5.2)

- State variable: $u(x, t)$.
- Reduced system: $S_r = \{ c, L, u_0, g \}$, with g fixed by C_0 and τ .

Everything else about the river (its depth, width, bed, fish, temperature) has been discarded because Q does not need it.



Position in SQM Space (Velten, §1.7)

Axis	Classification
S	Physical; natural system with a technical input; deterministic; continuous; one-dimensional (A1); environmental hydraulics.
Q	Mechanistic; instationary; distributed; direct problem (the Use? inverse version is a parameter identification problem); management model; macroscopic, reach scale.
M	Linear; analytical (closed form by characteristics), numerical when $c = c(x)$ or data are measured; autonomous; continuous; first-order hyperbolic partial differential equation.

Compare

Lifting A4 keeps S and Q and changes M (to a parabolic PDE). Lifting A3 changes M from linear to nonlinear. Lifting A1 changes the dimension on the S axis. Improvement is movement in SQM space.



Exercises



1. Verify that $u(x, t) = g(t - x/c) e^{-kx/c}$ solves $u_t + c u_x = -ku$ with $u(0, t) = g(t)$. Two sensors at $x_1 < x_2$ record peaks P_1, P_2 . Find k .
2. The river widens so that $c(x) = c_0/(1 + x/\ell)$. Find the characteristics and the arrival time at $x = L$. Compare with L/c_0 .
3. Show that the total mass passing the intake equals the mass discharged, for any g with compact support.
4. Implement the upwind scheme for the spill with $\nu = 1, 0.5$ and 1.2 . Explain each result.
5. Place the convection–diffusion model in SQM space. Which entries of the table change?
6. Write the ten questions for vehicle density on a single-lane road with a traffic signal.



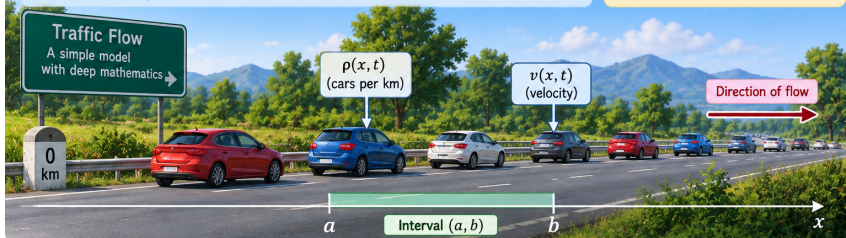
Burgers' Equation

Car Traffic

Traffic Flow Model

Consider traffic flow on a single-lane road (no overtaking).

Let $\rho(x, t)$ denote the density of cars (vehicles per km) and $v(x, t)$ the velocity of the cars at position x and time t .



Density of cars

$\rho(x, t)$ = number of cars per km at position x and time t .

The number of cars in the interval (a, b) at time t is

$$\int_a^b \rho(x, t) dx$$

Velocity of cars

$v(x, t)$ = velocity of the cars (km/h) at position x and time t .

The number of cars that pass through position x at time t (the flux) is

$$\rho(x, t) v(x, t)$$

Inviscid Burgers' equation

$$u_t + uu_x = 0$$

Also known as Bateman–Burgers equation, introduced by Harry Bateman (1915), studied by J. M. Burgers (1948).

Assumptions

- Single-lane traffic (no overtaking).
- Cars move with velocity $v(x, t)$ depending on position and time.
- Conservation of the number of cars.
- Under suitable assumptions (e.g., $v = \rho$ or identifying u with velocity), we obtain the quasilinear equation

$$u_t + uu_x = 0.$$



From the River to the Road

In the river model the carrier did not care what it carried: the water moved at speed c whether or not the chemical was there (assumption A3).

Now let the carried quantity set its own speed. On a road the “ant” is a car, and the “leaf” is the space in front of it: the more cars there are, the slower each one drives.

Lifting A3

Transport: $q = cu$ with c fixed. Traffic: $q = \rho v(\rho)$.

One change in the flux law turns a linear PDE into a nonlinear one, and the new model can do something the old one never could: produce a shock.

We answer Dym’s ten questions again, and the inviscid Burgers’ equation appears at **How?**.





Why? A Decision on a Heavy-Traffic Road

A main road carries heavy traffic past a junction; vehicles from the side road cannot get in or out. The authority must choose:

Option A: build a new road

- Splits the demand between two roads.
- Adds capacity permanently.
- Costly: land, money, years.

Option B: install a signal

- Cheap and quick.
- Gives the side road a turn.
- Stops the main road every cycle: queues form.

The question the model must answer

Does the queue formed during each red *clear* during the next green? If yes, a signal is enough. If it grows from cycle to cycle, only more capacity, a new road, will help.

Why? What are we looking for?

Why?



The need

A single-lane road leads to a traffic signal. The traffic engineer must choose the red and green times.

- While the light is red, a queue forms. How fast does its tail move back up the road? Will it block the junction behind?
- When the light turns green, how long before the last car in the queue starts moving? Is the green long enough to clear the queue?

The question is about queues of tens of cars over minutes. We do not need to know what any single driver does. This points to a *macroscopic* model (Dym, §6.1).



Find? What do we want to know?

- Traffic density $\rho(x, t)$ [veh km⁻¹] at position x and time t .
- Speed of the traffic $v(x, t)$ [km h⁻¹] and flow rate $q(x, t)$ [veh h⁻¹].
- Position $x_s(t)$ and speed $\dot{x}_s$ of the **tail of the queue** during red.
- The time at which the **last car of the queue** starts to move after green.

Three variables, one conservation law

ρ , v and q are linked by $q = \rho v$ (Dym, eq. 6.7). That is one equation for three unknowns, so the model will need a second relation between v and ρ .



Given? What do we know?

Symbol	Meaning	Unit	Value (example)
$v_{\max}$	speed on an empty road	km h^{-1}	60
$\rho_{\max}$	jam density, bumper to bumper	veh km^{-1}	150 (≈ 6.7 m per car)
ρ_A	density of arriving traffic	veh km^{-1}	30
T_r, T_g	red and green durations	s	design variables
$\rho_0(x)$	density on the road at $t = 0$	veh km^{-1}	from counts

$v_{\max}$ and $\rho_{\max}$ are properties of the road and its drivers. T_r and T_g are what the engineer controls. Keep the two kinds apart: the first are *parameters*, the second are *inputs*.



Assume? The Assumption Ledger

Assumption	If we lift it
B1 Single lane, no overtaking; ρ depends on (x, t) only.	multilane models
B2 No entry or exit ramps: cars are neither created nor destroyed.	source term $s(x, t)$
B3 Continuum: enough cars that ρ is a smooth density (Dym, §6.2.4).	car-following, cellular automata
B4 Speed depends only on the local density: $v = v(\rho)$.	look-ahead: viscous term
B5 $v(0) = v_{\max}$, $v'(\rho) \leq 0$, $v(\rho_{\max}) = 0$ (Dym, eq. 6.11).	—
B6 Linear speed–density law (Greenshields).	other $v(\rho)$
B7 Uniform road: $v_{\max}$, $\rho_{\max}$ do not depend on x .	bottlenecks, $v_{\max}(x)$
B8 Drivers are identical and react instantly.	reaction time, second-order models

B3 is the boldest: cars are discrete. Dym argues it is acceptable once many cars occupy the stretch of road we look at.



How? Conservation of cars

The number of cars in the interval (a, b) at time t is

$$\int_a^b \rho(x, t) dx.$$

The number of cars passing the point x per unit time is $\rho(x, t) v(x, t)$. By B2 the count in (a, b) changes only through its ends:

$$\frac{d}{dt} \int_a^b \rho(x, t) dx = \rho(a, t) v(a, t) - \rho(b, t) v(b, t).$$

This is Dym's balance law with no generation and no consumption, the same principle that gave the transport equation.



How? Conservation of cars

Integrate in time over $[t_1, t_2]$ and assume ρ, v smooth (B3):

$$\begin{aligned}\int_{t_1}^{t_2} \int_a^b \frac{\partial \rho}{\partial t} dx dt &= \int_{t_1}^{t_2} (\rho(a, t)v(a, t) - \rho(b, t)v(b, t)) dt \\ &= - \int_{t_1}^{t_2} \int_a^b \frac{\partial}{\partial x} (\rho v) dx dt,\end{aligned}$$

so

$$\int_{t_1}^{t_2} \int_a^b (\rho_t + (\rho v)_x) dx dt = 0.$$

Since $t_1 < t_2$ and $a < b$ are arbitrary,

$$\rho_t + (\rho v)_x = 0, \quad x \in \mathbb{R}, t > 0.$$



How? Closing the model: speed and density

By B4, $v = v(\rho)$. B5 fixes the ends: an empty road gives $v = v_{\max}$, a jammed road gives $v = 0$ at $\rho = \rho_{\max}$. The simplest law with these properties (B6) is linear:

$$v(\rho) = v_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right), \quad 0 \leq \rho \leq \rho_{\max}.$$

The flow–density relation is then Dym's *fundamental diagram*:

$$q(\rho) = \rho v(\rho) = v_{\max} \left(\rho - \frac{\rho^2}{\rho_{\max}} \right), \quad q_{\max} = \frac{v_{\max} \rho_{\max}}{4} \text{ at } \rho = \frac{\rho_{\max}}{2}.$$

The model is the Lighthill–Whitham–Richards (LWR) equation

$$\rho_t + \left[v_{\max} \rho \left(1 - \frac{\rho}{\rho_{\max}} \right) \right]_x = 0, \quad x \in \mathbb{R}, \quad t > 0.$$



How? From LWR to Burgers' equation

Expand the derivative of the flux:

$$\rho_t + q'(\rho) \rho_x = 0, \quad q'(\rho) = v_{\max} \left(1 - \frac{2\rho}{\rho_{\max}} \right).$$

Define $u = q'(\rho)$. Since u is linear in ρ , $u_t = -\frac{2v_{\max}}{\rho_{\max}} \rho_t$ and $u_x = -\frac{2v_{\max}}{\rho_{\max}} \rho_x$.
Multiplying the equation by $-2v_{\max}/\rho_{\max}$:

Inviscid Burgers' equation (Bateman 1915; Burgers 1948)

$$u_t + u u_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(x, 0) = v_{\max} \left(1 - \frac{2\rho_0(x)}{\rho_{\max}} \right).$$

Choosing units with $v_{\max} = 1$ gives $u = 1 - 2\rho/\rho_{\max}$; the scaling is a convenience, not an assumption.

What is u ? Not the speed of the cars

$$v = v_{\max} \left(1 - \frac{\rho}{\rho_{\max}}\right), \quad u = v_{\max} \left(1 - \frac{2\rho}{\rho_{\max}}\right).$$

- v is how fast a **car** moves.
- $u = q'(\rho)$ is how fast a **change in density** moves: the speed of a wave in the traffic.
- $u \leq v$ always. For $\rho > \rho_{\max}/2$, $u < 0$: waves travel *backwards* while every car moves forwards.



ρ	v	u
0	$v_{\max}$	$v_{\max}$
$\rho_{\max}/2$	$v_{\max}/2$	0
$\rho_{\max}$	0	$-v_{\max}$

Every driver has seen this: brake lights spreading back towards you through a line of moving cars.



Predict? Characteristics now cross

As before, $\frac{d}{dt}u(x(t), t) = 0$ along $\frac{dx}{dt} = u$. So u is constant on straight lines

$$x = x_0 + u_0(x_0)t,$$

but now the slope of each line depends on the data it carries.

- If u_0 increases with x (density falls ahead), the lines fan out: the traffic thins smoothly.
- If u_0 decreases with x (denser traffic ahead), faster lines catch slower ones. They first cross at

$$t_b = -\frac{1}{\min_x u'_0(x)},$$

after which the formula would give two values of u at one point. A **shock** forms: a jump in density.

The transport equation could never do this: its characteristics were parallel.



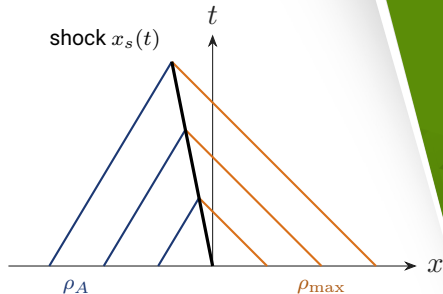
Predict? Red light: the queue as a shock

Arriving traffic ρ_A for $x < 0$ meets stopped cars $\rho_{\max}$ for $x > 0$. Cars are conserved across the jump, so its speed is (Rankine–Hugoniot)

$$\dot{x}_s = \frac{q(\rho_A) - q(\rho_{\max})}{\rho_A - \rho_{\max}} = -v_{\max} \frac{\rho_A}{\rho_{\max}}.$$

Numbers: $\rho_A = 30$, $\rho_{\max} = 150$, $v_{\max} = 60$ km/h:

- the queue tail moves back at 12 km/h;
- after a 60 s red it is 200 m long and holds 30 cars.



Predict? Red light: the queue as a shock

Check: the queue gained $(\rho_{\max} - \rho_A) \times 0.2 \text{ km} = 24 \text{ cars net}$, exactly the $q(\rho_A) \times 1 \text{ min} = 1440 \text{ veh/h} \times 1 \text{ min} = 24 \text{ cars}$ that arrived.





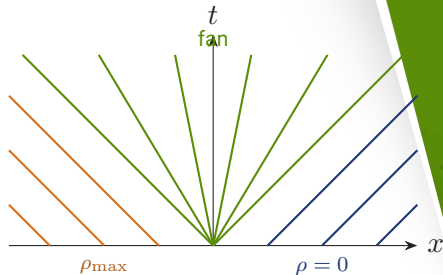
Predict? Green light: the queue as a fan

Jam $\rho_{\max}$ for $x < 0$, empty road for $x > 0$:
 u jumps *up* from $-v_{\max}$ to $v_{\max}$. The lines spread out from the origin, and

$$u = \frac{x}{t}, \quad \rho(x, t) = \frac{\rho_{\max}}{2} \left(1 - \frac{x}{v_{\max} t} \right)$$

for $-v_{\max} t \leq x \leq v_{\max} t$.

- The first car leaves at $v_{\max}$.
- The “start moving” signal travels back at $v_{\max}$: the last car of a 200 m queue moves after $0.2/60 \text{ h} = 12 \text{ s}$.
- At the stop line $x = 0$: $\rho = \rho_{\max}/2$, flow = $q_{\max}$.





Valid? Is the model consistent?

1. **Dimensions.** $[\rho_t] = \text{veh km}^{-1}\text{h}^{-1}$; $[(\rho v)_x] = \text{veh km}^{-1} \cdot \text{km h}^{-1} \cdot \text{km}^{-1} = \text{veh km}^{-1}\text{h}^{-1}$.
2. **Conservation.** The shock speed was checked against a car count on the red-light slide.
3. **Limiting case.** For light traffic, $\rho \ll \rho_{\max}$, the equation becomes $\rho_t + v_{\max}\rho_x \approx 0$: our transport equation, with the cars as the “ant”.
4. **Physical range.** If $0 \leq \rho_0 \leq \rho_{\max}$ then $0 \leq \rho \leq \rho_{\max}$ for all t (values are only moved along characteristics or across shocks), so $v \geq 0$: no car ever reverses.
5. **Fundamental diagram.** Concave, zero at both ends, maximum $q_{\max} = v_{\max}\rho_{\max}/4 = 2250 \text{ veh/h}$ for our numbers. Compare with a measured lane capacity before trusting it.



Valid? Well-posedness and the form of the law

- **Existence.** A smooth solution exists only up to t_b . After that we need *weak* solutions, which satisfy the integral balance of the **How?** slides but not the PDE pointwise. Integrating over $[t_1, t_2] \times (a, b)$ was not a formality: it is the form of the law that survives shocks.
- **Uniqueness.** Weak solutions are not unique. The physical one satisfies the entropy (Lax) condition $u_L > u_R$ across a shock: characteristics run *into* it. A jump with $u_L < u_R$ is rejected: at a green light it would be a queue that never moves off. The fan is chosen instead.
- **Which quantity is conserved?** For smooth u , $u_t + uu_x = 0$ can also be written $(u^2/2)_t + (u^3/3)_x = 0$, which gives a *different* shock speed. The physics decides: cars are conserved, so the flux must be $q(\rho)$. The modelling fixes the mathematics here, not the other way round.



Verified? Are the predictions good for the purpose?

Test. Film a signalised approach, count cars per metre from the frames, and track the queue tail against time.

- **What LWR gets right:** queues grow backwards during red, discharge from the stop line at roughly a steady saturation flow, and the “start” wave runs back through the queue. These are the quantities the signal design needs.
- **What it gets wrong:**
 - measured (ρ, q) points scatter widely around any single curve $q(\rho)$;
 - stop-and-go waves appear in dense traffic with no signal at all. In LWR a uniform flow stays uniform, so these “phantom jams” cannot occur;
 - real shocks are not sharp: drivers look ahead and brake gradually.

Verdict

Fit for signal timing and queue length. Not fit for explaining congestion that appears from nowhere: for that, lift B4 or B8.



Improve? Lift one assumption at a time

Lift	New model	What changes
B4	$v = v(\rho) - \frac{\varepsilon}{\rho} \rho_x \Rightarrow u_t + uu_x = \varepsilon u_{xx}$	viscous Burgers'; shocks become smooth layers of width $\sim \varepsilon / (u_L - u_R)$
B6	$v = v_0 \ln(\rho_{\max} / \rho)$ (Greenberg), etc.	new $q(\rho)$; u no longer linear in ρ
B2	$\rho_t + (\rho v)_x = s(x, t)$	on-ramps, off-ramps
B7	$v_{\max}(x), \rho_{\max}(x)$	lane drops, bottlenecks
B8	second-order models for v	reaction time; stop-and-go waves
B3	car-following (Dym, §6.3); cellular automata	discrete cars, randomness

Lifting B4 is instructive: a driver who slows when density *ahead* is higher adds a diffusion term, the same step that took the river from transport to convection–diffusion.



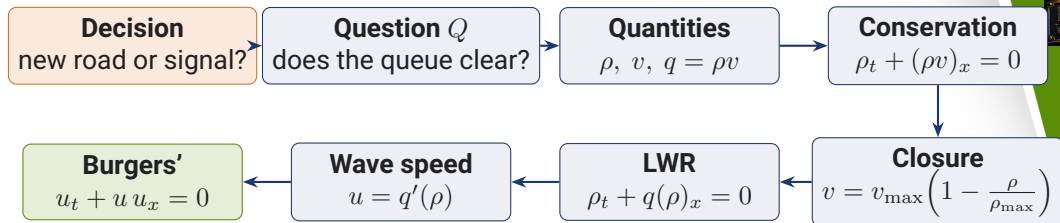
Use? How will we exercise the model?

- **Signal timing.** Choose T_r so that the queue $v_{\max}\rho_A T_r / \rho_{\max}$ does not reach the junction behind; choose T_g long enough to discharge it at $q_{\max}$.
- **Capacity.** The road cannot carry more than $q_{\max}$; demand above it must queue.
- **Control.** Ramp metering and variable speed limits change the inputs to keep ρ below $\rho_{\max}/2$, where the flow is largest: an optimal control problem.
- **Inverse use.** Loop detectors give (ρ, q) pairs; fitting $q(\rho)$ identifies $v_{\max}$ and $\rho_{\max}$.

The same M , other systems S

Burgers' equation appears in gas dynamics, shallow-water waves, and as Burgers' own 1948 model for turbulence: wherever the speed of a signal depends on the signal.

Building Burgers' Equation from the Decision



Where each option enters the model

- **Signal:** a switching condition at the stop line. Red: no cars pass, $\rho = \rho_{\max}$ there (a shock runs back). Green: the queue discharges as a fan.
- **New road:** the same equation with different data. The demand $q(\rho_A)$ on the main road falls because part of it moves to the new road.

The Model as a Triple (S, Q, M)

S A single-lane road approaching a traffic signal, with its drivers.

Q How fast does the queue grow during red and clear during green?

M $\rho_t + (\rho v(\rho))_x = 0$ with Greenshields' $v(\rho)$, in integral (weak) form with the entropy condition; equivalently $u_t + uu_x = 0$; plus B1–B8.

State variable and reduced system

State variable $\rho(x, t)$ (or u). Reduced system $S_r = \{v_{\max}, \rho_{\max}, \rho_0, T_r, T_g\}$.

S Physical; technical system with human behaviour; treated as deterministic; a discrete system described by a continuous model (B3); 1D.

Q Mechanistic conservation law with a phenomenological closure $v(\rho)$ (grey box); instationary; distributed; direct; management; macroscopic.

M Nonlinear (quasilinear) first-order hyperbolic conservation law; analytical (characteristics, shocks, fans) and numerical; autonomous; continuous.



Transport and Burgers' Side by Side



	River (transport)	Road (Burgers')
Conserved quantity	mass of chemical	number of cars
Flux law	$q = c u$	$q = \rho v(\rho)$
Equation	$u_t + c u_x = 0$	$u_t + u u_x = 0$
Type	linear	quasilinear
Characteristics	parallel	slopes depend on data
Profiles	translate unchanged	steepen or spread
Smooth data	stay smooth	can break at t_b
Needs weak form?	only for rough data	yes, generically

Same principle at **How?**, different closure. Every difference in the table comes from one assumption, A3.

Mathematical Aside: A Quasilinear Cauchy Problem



Example 1

Show that the PDE

$$uu_x + u_y = \frac{1}{2}$$

with initial data

$$\Gamma : \begin{cases} x = x_0(s) = s \\ y = y_0(s) = s \\ u = u_0(s) = \frac{s}{4} \end{cases} \quad s \in [0, 1]$$

has a unique solution.

Solution: Here $a = u$, $b = 1$. The transversality condition on Γ :

$$\frac{dx_0}{ds} b - \frac{dy_0}{ds} a = 1 - \frac{s}{4} \neq 0 \quad \text{for } s \in [0, 1].$$

Mathematical Aside: A Quasilinear Cauchy Problem

The characteristic system is

$$C : \begin{cases} \frac{dx}{dt} = u \\ \frac{dy}{dt} = 1 \\ \frac{du}{dt} = \frac{1}{2} \end{cases}$$

with initial conditions $x(s, 0) = y(s, 0) = s$, $u(s, 0) = s/4$. Solving,

$$\begin{cases} x(s, t) = s + st/4 + t^2/4 \\ y(s, t) = s + t \\ u(s, t) = s/4 + t/2 \end{cases}$$



Mathematical Aside: A Quasilinear Cauchy Problem

Rewriting s, t in terms of x, y :

$$\begin{cases} s = \frac{4x - y^2}{4 - y} \\ t = \frac{4(y - x)}{4 - y} \end{cases}$$

and the unique solution is

$$u = \frac{8y - 4x - y^2}{4(4 - y)}, \quad y \neq 4.$$

Link to the model

The solution blows up as $y \rightarrow 4$: the projected characteristics stop covering the plane smoothly. This is the same mechanism as the breaking time t_b in the traffic model.





Thank You