MA621L-Mathematical Modelling

Model-6 : Thirsty Crow to Tsunami Model

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A Fable as a Modelling Problem

The Thirsty Crow (Aesop, about 600 BC)

A thirsty crow finds a pitcher. The water is too low for its beak. It drops pebbles in, one by one. The water rises, and the crow drinks.

Question

Is this only a story, or does it work?

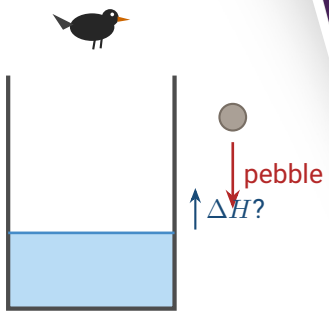
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It works. New Caledonian crows solve exactly this task in laboratory experiments (Jelbert et al., *PLOS ONE*, 2014). So whatever model we build can be tested against real birds.



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1. How is the pressure distributed in still water?
2. How much does the water rise for each pebble? How many pebbles are needed?
3. Must the pebble sink? How fast does it sink?
4. What does the water itself do while the pebble falls?
5. Which forces move the water?
6. How do the pebble and the water act on each other?
7. And at a much larger scale: what happens when an asteroid falls into the ocean?

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Each question needs a little more mathematics than the one before, and each answer must agree with the earlier, simpler ones.



From Story to Model: Pressure in Still Water

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Which quantities do we need to describe the pitcher problem? Which of them are fixed, and which do we want to find?

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Symbol	Meaning	Type	Value used
ρ_w	density of water	parameter	$1000\ \mathrm{kg/m^3}$
ρ_s	density of stone	parameter	$2500\ \mathrm{kg/m^3}$
g	gravity	parameter	$9.81\ \mathrm{m/s^2}$
A (width W)	cross-section of the pitcher	parameter	$W = 0.15$ m
H_0	initial water level	parameter	0.10 m
H_{beak}	level the crow can reach	parameter	0.14 m
V_p	volume of one pebble	parameter	disc πr^2, $r = 0.03$ m
H	water level	variable	unknown
n	number of pebbles	variable	unknown

Can We Simplify the Geometry?



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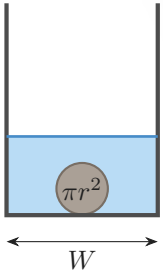
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- In a 2D slice the pitcher has width W and a pebble is a disc of radius r.
- “Volume” becomes area, and “cross-section” becomes width.
- The structure of the problem is the same; the arithmetic is lighter.
- This is an assumption, so we write it down.



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Before writing any equation: what are we taking for granted about the water, the pitcher and the pebbles?

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A1. Water is **incompressible**: its volume never changes.

A2. The pitcher has **vertical walls**, so the cross-section A is constant.

A3. Pebbles are **rigid, identical** and, when we count them, **fully under water**.

A4. We wait until everything is at rest: **statics**, no time variable.

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We will come back to each of these. A2 fails for a conical pitcher, A3 fails when the pebbles pile up above the water, and A4 fails as soon as we ask how fast the pebble sinks.

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The only combination with the units of pressure is

$$p = C\,\rho g h, \qquad C \text{ a pure number.}$$

Can Units Give the Constant?



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Question

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No. Units fix the *form* of the formula. To find C we need a derivation, which we do next.

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Question

Have we proved this?

Not yet. ΔH could also depend on a ratio without units, such as ρ_s/ρ_w, and units cannot rule that out. We will see that it does depend on it, but only through one question: does the pebble sink or float?

Is This Formula Believable?



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Someone claims that a pebble released at the surface of water of depth h reaches the bottom after

$$t = \sqrt{\frac{2h}{g}}\, e^{\rho_s/\rho_w}.$$

Without doing any mechanics, find two reasons to doubt it.

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Checking units and limits takes a minute and catches many wrong formulas.

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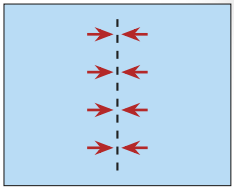
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- It **pushes**, and the push is **perpendicular** to the surface.
- **Pressure** p is this normal force per unit area.
- One atmosphere is about 10^5 Pa, the pressure of a 10 m column of water.
- We use **gauge pressure**, $p = p_{\text{abs}} - p_{\text{atm}}$, so that $p = 0$ at the free surface.



imaginary surface

Deriving the Pressure: Isolate a Slab

Take y upward, with the free surface at $y = H$ where $p = 0$. Cut out a thin horizontal slab of area A_s and thickness Δy.

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Which forces act on this slab in the vertical direction?

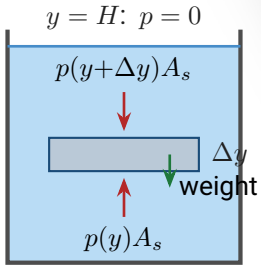


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- Pressure from below: $p(y)A_s$, upward.
- Pressure from above: $p(y + \Delta y)A_s$, downward.
- Weight: $\rho_w g A_s \Delta y$, downward. The side faces push horizontally and cancel.

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The slab is at rest, so the forces balance:

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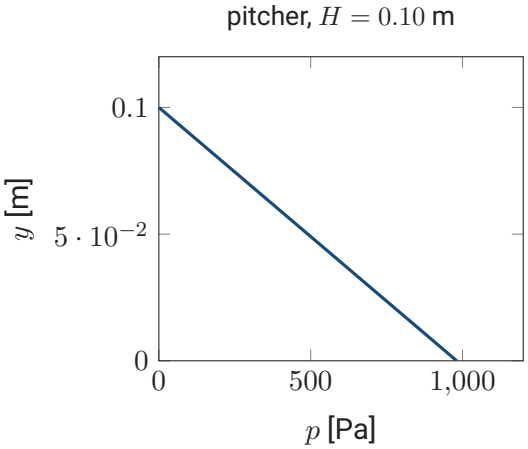
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So $C = 1$. Notice the two ingredients: physics gave a differential equation, and a boundary condition chose the solution.

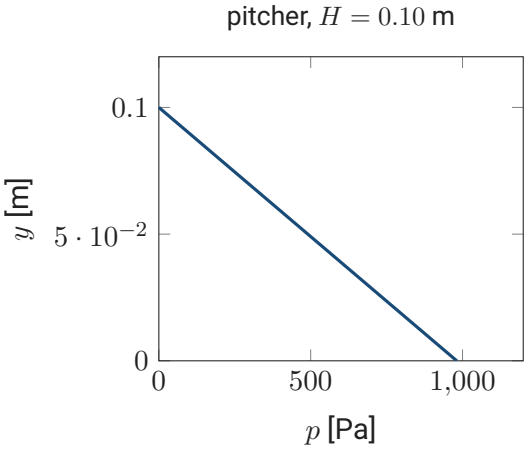
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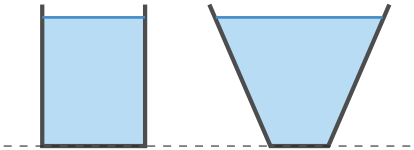


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How can we check $p = \rho_w g \times$depth without an experiment?

- Deeper water gives larger pressure: we feel it in our ears in a pool.
- The slab area A_s has dropped out, as it should.
- Size: 10 cm of water gives 981 Pa, about 1% of an atmosphere. Plausible.

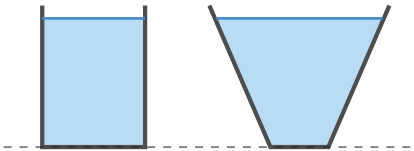
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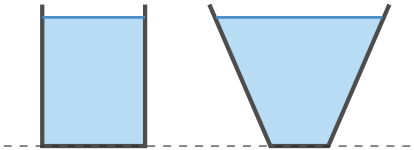


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The total *force* on the bottom is a different question (see the exercises).

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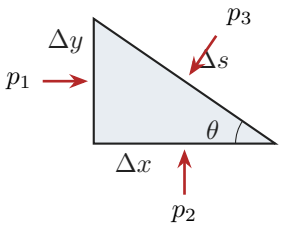
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The interface has no mass. If the pressures differed, a net force would act on something with no mass, so they must be equal.

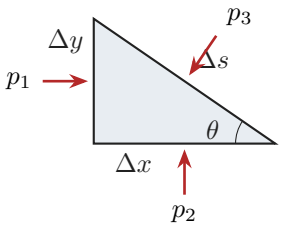
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Cut out a small wedge with angle θ at the bottom right corner. Its sides satisfy

$$\Delta x = \Delta s \cos\theta, \qquad \Delta y = \Delta s \sin\theta.$$

On each face the pressure pushes inward, perpendicular to the face. Let us balance the forces.

The Wedge: Forces in the x-Direction



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$$(p_1 - p_3) \, \Delta s \sin\theta = 0 \quad \Longrightarrow \quad p_1 = p_3.$$

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The weight of the wedge is $\frac{1}{2}\rho g\,\Delta x\,\Delta y$, so

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Pressure at a point is the same in every direction. So one function $p(\boldsymbol{x}, t)$ describes it completely.

Estimate Before You Compute



Problem (groups of 2–3, five minutes)

A pitcher holds about a litre and is about 12 cm tall. The water is 4 cm below the crow's reach. Pebbles are about 2 cm across.

Roughly how many pebbles does the crow need?

Use only volumes and common sense, and round every number to something easy.

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Hint

What volume of water must be "lifted"? What volume does one pebble add?

Estimate Before You Compute: One Solution



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Question

A pebble is rounder than a cube. Does this change the answer much?

Estimate Before You Compute: One Solution



- Cross-section of the pitcher: $\dfrac{10^{-3}\ \mathrm{m}^3}{0.12\ \mathrm{m}} \approx 0.008\ \mathrm{m}^2$.
- Volume needed to raise the water by 4 cm: $0.04 \times 0.008 \approx 3 \times 10^{-4}\ \mathrm{m}^3 = 0.3$ L.
- Volume of one pebble, treated as a cube: $(0.02\ \mathrm{m})^3 = 8$ mL.
- Number of pebbles: $300/8 \approx 40$.

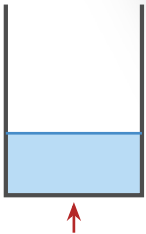
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A sphere 2 cm across has volume $\frac{\pi}{6}(0.02)^3 \approx 4$ mL, giving about 70 pebbles. The sensible answer is “a few dozen”: the order of magnitude is what counts.

Summary: Pressure in Still Water

1. A model starts with variables, parameters and written-down assumptions.
2. Units fix the form of a formula ($p = C\rho g h$, $\Delta H \sim V_p/A$); limits test it.
3. A force balance on a slab gives $\frac{dp}{dy} = -\rho_w g$, so $p = \rho_w g \times$ depth.
4. Pressure at a point is the same in every direction.
5. A quick estimate gives a target for the exact model.



Exercises: Pressure in Still Water



Exercise 1: U-tube

A U-tube contains water. 10 cm of oil ($\rho = 800\ \mathrm{kg/m^3}$) is poured into one arm. Find the height difference between the two free surfaces.

Exercise 2: Blood pressure

Blood pressure is quoted as "120 mm Hg" ($\rho_{\mathrm{Hg}} = 13600\ \mathrm{kg/m^3}$). Convert this to Pa and to metres of water. Why do giraffes need a high blood pressure?

Exercises: Pressure in Still Water (contd.)



Exercise 3: Wave speed from units

The speed c of waves on shallow water may depend on ρ, g and the depth h. Show that $c = C\sqrt{gh}$. Why can units not give C?

Exercise 4: Conical pitcher

A pitcher has radius $R(y) = R_0 + \alpha y$. Does $p = \rho g \times$depth still hold? Find the total force on the bottom, and explain in three sentences why the cone and the cylinder of the previous slide do not contradict each other.



Archimedes and the Rising Level

What Force Does Still Water Exert on a Stone?



Question

A body B lies under still water, where $p = \rho_w g\,(H - y)$. The water pushes on every patch of its surface. How do we add up all these pushes?

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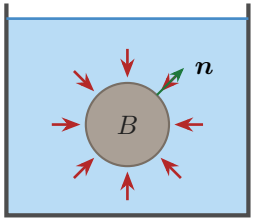
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Deeper patches feel more pressure, and their normals point partly downward, so their pushes point partly upward. The total should point up. But how big is it?



From the Slab to a Closed Box

For the slab we used only two faces: $p(y)A_s$ pushing up, $p(y+\Delta y)A_s$ pushing down.

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Now take a closed box of width w and height h (2D, per unit depth), with its top at depth d. What force does the water exert on each of its four faces?

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- Top, at depth d: $\rho_w g\, d\, w$, down.
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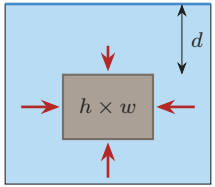
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$$F_{\text{up}} = \rho_w g\,(d+h)\,w - \rho_w g\, d\, w = \rho_w g\, hw = \rho_w g\, V$$

The depth d cancels; only the volume $V = hw$ remains.

Every Face in One Formula



Question

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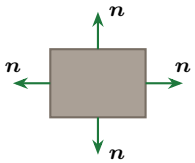
Question

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Give each face its **outward unit normal** n: length 1, perpendicular to the face, pointing out of the body.

Pressure pushes *inward*, opposite to n. So a face of length L with pressure p feels

$$-p\,\boldsymbol{n}\,L.$$



Checking the Rule on the Box



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Apply $-p\,\boldsymbol{n}\,L$ to each face of the box. Do we get back the forces we found before?

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bottom	$(0, -1)$	$(0,\ p_{\text{bot}}\,w)$: up
top	$(0, 1)$	$(0,\ -p_{\text{top}}\,w)$: down
left	$(-1, 0)$	$(\bar{p}\,h,\ 0)$: to the right
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$\bar{p}$: average pressure on a side face.

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$\bar{p}$: average pressure on a side face. Yes: the same four forces, now from one rule. Adding them up,

$$\boldsymbol{F} = -\sum_{\text{faces}} p\,\boldsymbol{n}\,L.$$

Slanted Faces: a Polygon



Question

Cut the corners off the box: it becomes a polygon with slanted faces. Does the rule still work?

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So face i, with length L_i, outward normal $\boldsymbol{n}_i$ and pressure p_i, feels $-p_i\,\boldsymbol{n}_i\,L_i$:

$$\boldsymbol{F} \approx -\sum_i p_i\,\boldsymbol{n}_i\,L_i.$$

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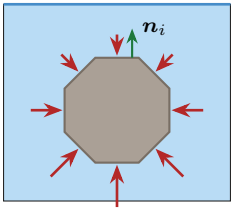
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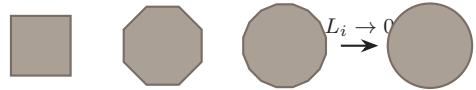
Lower faces: deeper, larger p_i, normals pointing down, so their pushes point up.



More and More Faces: From Polygon to Circle

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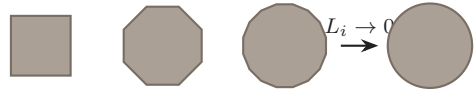
The stone is round. What happens if we use more and more faces, each smaller and smaller?



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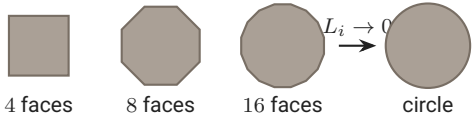
The polygon approaches the circle, each face L_i becomes a small arc dS, and the sum becomes an integral:

$$\boldsymbol{F} = -\sum_i p_i \, \boldsymbol{n}_i \, L_i \quad \xrightarrow{L_i \to 0} \quad \boxed{\boldsymbol{F} = -\oint_{\partial B} p \, \boldsymbol{n} \, dS}$$

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For the box, this integral is just the sum over its four faces: our first result is a special case.

The Stone in the Pitcher

For a stone B under still water, $p = \rho_w g\,(H - y)$, and

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Before computing anything: which way should $\boldsymbol{F}$ point?

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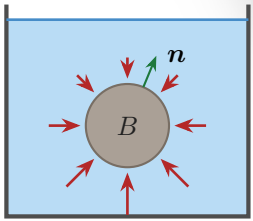
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But how big is the total? Two ways to find out follow.



First Answer: Replace the Stone by Water



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Imagine the stone replaced by water of the same shape. What happens to that piece of water?

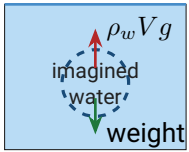
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The water outside cannot tell what is inside the boundary, so the stone feels the same force.



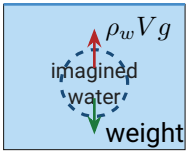
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Archimedes' principle

A fully submerged body feels an upward force equal to the weight of the water it displaces: $\boldsymbol{F}_{\text{buoy}} = \rho_w V g$.

What Is a Buoyant Force?



Question

Push a ball under water and let go: it shoots up. In a swimming pool you can lift a friend easily. What is pushing upward?

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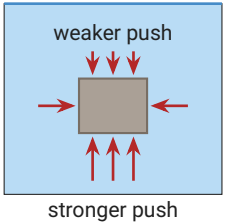
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Buoyant force

The net upward force that a fluid exerts on a body immersed in it.

It is not a new kind of force: it is the water pressure, added up over the surface of the body.

Where Does the Buoyant Force Come From?



Question

A block with top and bottom area S and height a has its top at depth d. Using $p = \rho_w g \times$depth, find the net force of the water on it.

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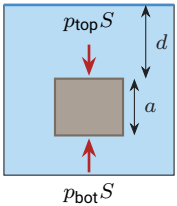
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$$F_{\text{up}} = \rho_w g\, (d+a)\, S - \rho_w g\, d\, S = \rho_w g\, aS = \rho_w g\, V$$

Does the force grow if we push the block deeper?
No: d cancels, since both pressures grow equally. Only $V = aS$ counts: the force is the weight of the displaced water.

Second Answer: Compute the Integral



Question

Can we evaluate $-\oint p\,\boldsymbol{n}\,dS$ directly, without the trick?

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Take any fixed unit vector $\boldsymbol{e}$ and use the divergence theorem:

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In still water $\nabla p = (0, -\rho_w g)$, so

$$\boldsymbol{F} = -\int_B \nabla p\,dV = (0,\ \rho_w g V).$$

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Same answer. But this version works for *any* pressure field, which we will need when the water moves.

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Question

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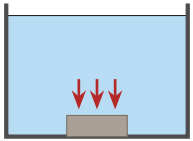
No. There is no water below it to push up;
the water above pushes it *down*.

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Our derivation assumed that

1. the body is surrounded by water on all sides;
2. the water is at rest.

For a sinking pebble the second assumption fails.

Will the Pebble Sink?



Question

A fully submerged body feels its weight $\rho_s V g$ downward and the buoyancy $\rho_w V g$ upward. When does it sink?

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Material	ρ_s/ρ_w	Sinks or floats?
granite pebble	≈ 2.5	sinks
glass marble	≈ 2.6	sinks
ice	0.92	floats
cork	≈ 0.25	floats

How Heavy Is the Pebble Under Water?



Question

Our pebble (2D, per metre of depth) has $V_p = \pi r^2 = 2.83 \times 10^{-3}\ \mathrm{m}^2$ and $\rho_s = 2500\ \mathrm{kg/m}^3$. How much weight is left under water?

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$$\begin{aligned} \text{weight} &= \rho_s V_p g \approx 69.4\ \mathrm{N/m}, \\ \text{buoyancy} &= \rho_w V_p g \approx 27.8\ \mathrm{N/m}, \\ \text{net downward force} &= (\rho_s - \rho_w) V_p g \approx 41.6\ \mathrm{N/m}. \end{aligned}$$

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Under water the pebble keeps only 60% of its weight. The combination $(\rho_s - \rho_w)g$ will appear again when the pebble moves.

True or False?



Question

“A stone deeper in the water feels a larger buoyant force.”

True or False?



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False. $\rho_w V g$ contains no depth: deeper down, the pressure on the top and on the bottom both grow by the same amount.

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Question

"The buoyant force depends on what the body is made of."

False. It depends only on the volume of water displaced. The material enters through the weight.

True or False? (contd.)



Question

“Every body in water feels a buoyant force.”

True or False? (contd.)



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False. It needs water underneath: remember the rubber disc pressed onto the bottom.

True or False? (contd.)



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Each of these mistakes disappears once we remember where the formula comes from: the integral of the pressure over the surface.

How Much of a Floating Body Is Under Water?



Question

A floating body displaces only a volume $V_d < V$. How large is V_d?

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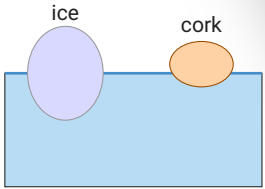
At equilibrium, buoyancy balances weight:

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How Much of a Floating Body Is Under Water?

Question

A floating body displaces only a volume $V_d < V$. How large is V_d?



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- Ice: 0.92, so nine tenths of an iceberg is under water.
- Cork: 0.25, so a cork displaces only a quarter of its volume.

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What real crows do

New Caledonian crows choose sinking objects over floating ones without being trained (Jelbert et al., 2014).

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What if a pebble is frozen inside the ice? Try it in the exercises.

Where Does the Pebble's Volume Go?



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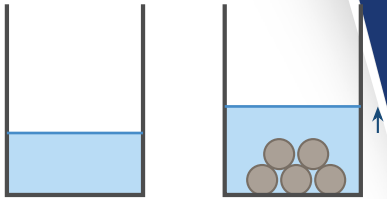
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$$V_p = \pi r^2 = 2.83 \times 10^{-3}\ \text{m}^2, \qquad \Delta H = \frac{\pi r^2}{W} = 0.0188\ \text{m}, \qquad n_{\min} = \left\lceil \frac{0.04}{0.0188} \right\rceil = \mathbf{3}.$$

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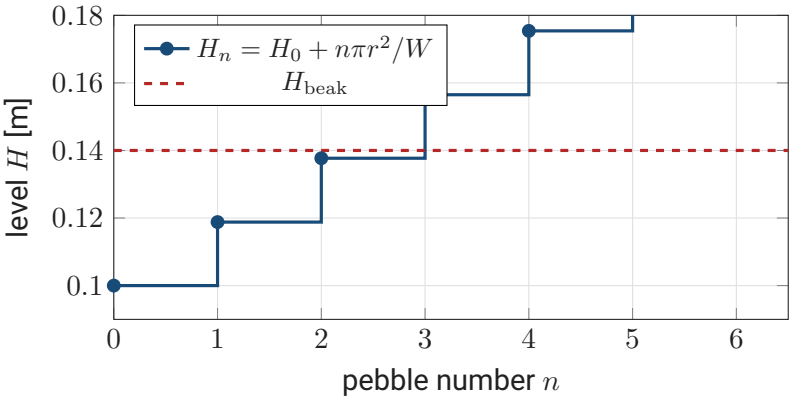
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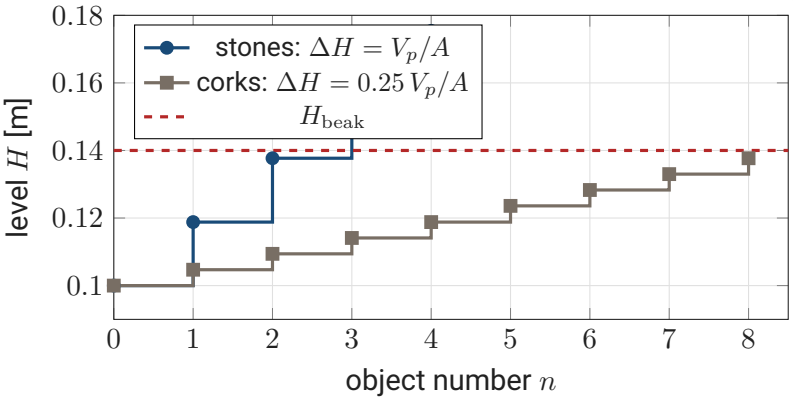
- The final state follows from volume conservation alone.
- Only the pebble's volume matters, not its shape: easy to test with coins and a glass.
- Whatever the splash does on the way, the level must end up here. Any more detailed model must reproduce this.

The Level, Pebble by Pebble



One step per pebble. The third step crosses the beak line: the fable in one picture.

Stones Against Corks



Three stones do the job. Eight corks still fall short, and they cover the surface.

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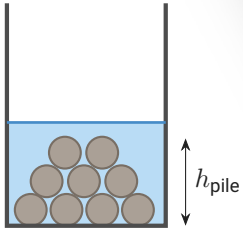
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Height of the pile after n pebbles:

$$h_{\text{pile}} = \frac{nV_p}{(1-\varphi)A}.$$



gaps: porosity φ

When Does the Pile Reach the Surface?



While the pile is below the water ($h_{\text{pile}} \leq H$), nothing changes: $H = H_0 + nV_p/A$.

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For our numbers with $\varphi = 0.25$: $n^* \approx 15$. The crow needs only 3, so it is safe. But what happens after n^*?

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Only the part below the water. Below the level H the pitcher holds the water plus the solid part of the pile, $(1-\varphi)AH$:

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So the crow can succeed only if $H_{\text{beak}} < H_0/\varphi$.

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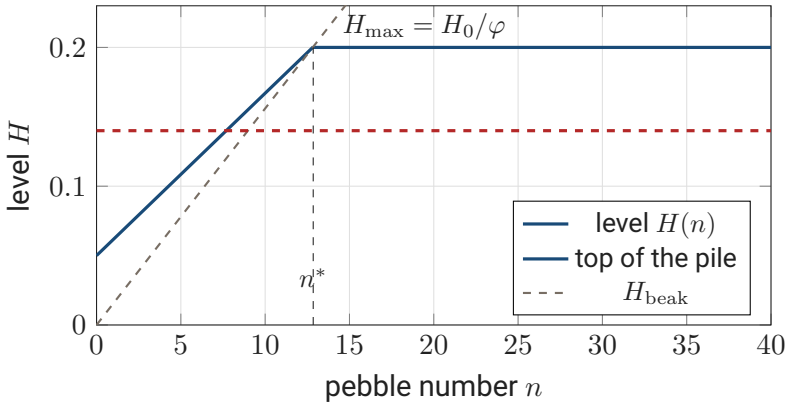
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H_0 [m]	$H_{\max} = H_0/\varphi$ ($\varphi = 0.25$)	The crow needs 0.14 m
0.10	0.40	succeeds easily
0.05	0.20	succeeds
0.03	0.12	**cannot succeed**

The Level Saturates



A straight line until the pile emerges at n^*, then a flat line at H_0/φ ($H_0 = 0.05$, $\varphi = 0.25$).

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A smaller φ: mix sizes, so that small pebbles fill the gaps between large ones.

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If crows “understand” this physics, what should they prefer in an experiment?

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In experiments (Jelbert et al., 2014, and later studies) crows drop objects into a tube of water to raise a floating treat. They:

- prefer sinking objects to floating ones;
- prefer large stones to small ones;
- prefer tubes of water to tubes of sand;
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A 2600-year-old story, tested in 2014: the model makes predictions, the experiment decides.

Summary: Archimedes and the Rising Level



1. Archimedes' principle, derived twice: the buoyant force is $\rho_w V g$, upward.
2. A body sinks if $\rho_s > \rho_w$; a floating body displaces only $V\rho_s/\rho_w$, so stones beat corks.
3. Level law: $H_n = H_0 + nV_p/A$; for our pitcher, 3 pebbles.
4. With a porous pile the pile emerges at n^*, and the level can never exceed $H_{\max} = H_0/\varphi$.
5. Real crows behave as the model predicts.

Exercises: Archimedes and the Rising Level



Exercise 5: A floating body

Prove Archimedes' principle for a floating body: use the divergence theorem on the wetted part and atmospheric pressure on the dry part, and show $\boldsymbol{F} = \rho_w V_d \, g$.

Exercise 6: Ice cube

Show that a melting ice cube does not change the level in a full glass. What happens if a pebble is frozen inside the ice?

Exercises: Archimedes and the Rising Level (contd.)



Exercise 7: Conical pitcher

For a pitcher with cross-section $A(y) = A_0(1 + y/L)^2$, set up the equation for H_n (a cubic; do not solve it). Which shape gives the largest rise per pebble?

Exercise 8: A narrow pile

The pile covers only a fraction β of the base. Find n^* and $H_{\max}$ again.

Exercise 9: Measure φ

Pack coins into a glass of water, measure the rise of the level, and find the porosity φ.



How Fast Does the Pebble Sink?

From “How Much” to “How Fast”



The level law tells us where the water ends up. It says nothing about time.

Question

Does it take the pebble a tenth of a second or ten seconds to reach the bottom? What would we need to know?

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We need Newton’s second law for the pebble: its mass, and every force acting on it while it falls through the water.

Which Forces Act on a Sinking Pebble?

Pebble: volume V_p, density ρ_s, mass $m = \rho_s V_p$, speed $V(t)$ measured downward.

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List the forces.

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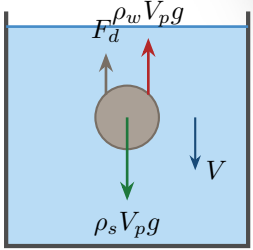
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$$m\frac{dV}{dt} = (\rho_s - \rho_w)V_p\, g - F_d(V)$$

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We computed the weight and the buoyancy exactly. Can we compute $F_d(V)$ the same way?

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Model	What is closed
sinking pebble	the wake $\rightarrow$ drag law $F_d(V)$
blood flow in an artery (1D)	velocity profile $\rightarrow$ wall friction
turbulent flow	small eddies $\rightarrow$ turbulent stresses

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The same surface integral is split in two: one part we derive, the other part we must close.

Two Candidate Drag Laws

For a sphere of diameter d moving at speed V there are two classical laws.

Linear (Stokes) drag

$$F_d = 3\pi\mu\, d\, V$$

Valid when viscosity dominates; μ is the viscosity in Pa s.

Quadratic drag

$$F_d = \tfrac{1}{2}\rho_w C_d A_f V^2$$

Valid when the pebble must push water out of the way; $C_d \approx 0.5$–1, A_f the frontal area.

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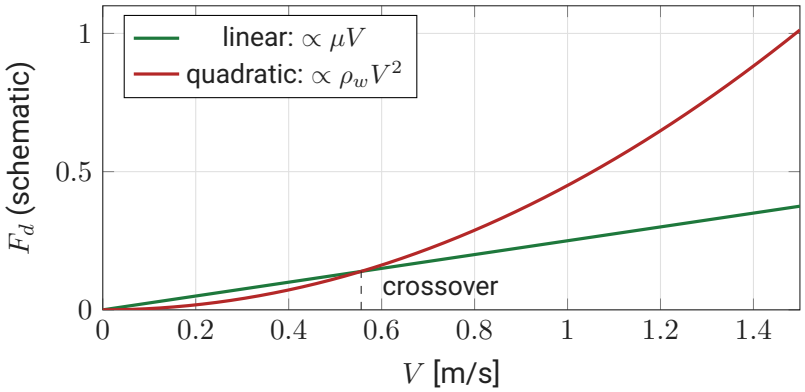
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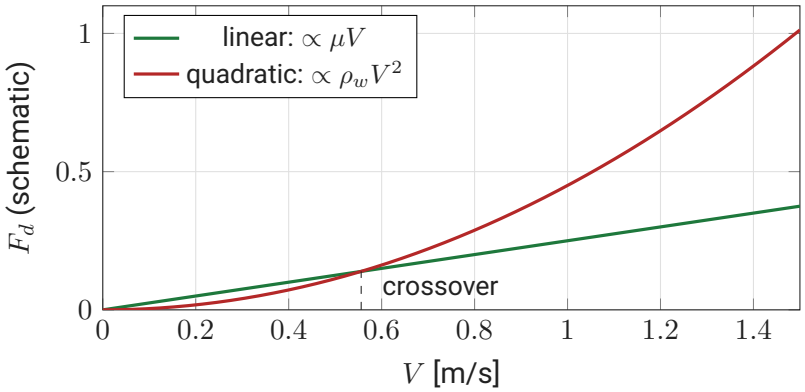
Question

Which one should we use for our pebble?

When Is Each Law Larger?



When Is Each Law Larger?



At low speed the linear law dominates, at high speed the quadratic one. The crossover is where the two are equal, and that is a statement about a ratio.

The Reynolds Number



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Divide the quadratic drag by the linear drag (dropping constants). What ratio do you get?

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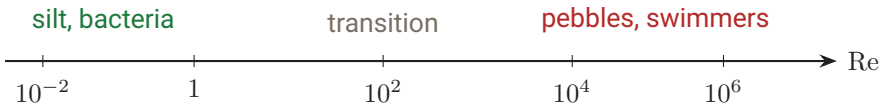
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Rule of thumb

$\mathrm{Re} \lesssim 1$: linear drag. $\mathrm{Re} \gtrsim 10^3$: quadratic drag with C_d roughly constant.



Which Regime Is Our Pebble In?



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No: we guess, choose the law, solve, and then check the regime.

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$$\mathrm{Re} = \frac{1000 \cdot 10^{-4} \cdot 10^{-5}}{10^{-3}} = 10^{-3} \quad \Longrightarrow \quad \text{linear drag.}$$

Same equation, same pitcher, different drag law. The sizes decide the model.

Setting Up the Equation

Insert the quadratic drag and divide by $m = \rho_s V_p$:

$$\frac{dV}{dt} = g' - kV^2, \qquad g' = g\,\frac{\rho_s - \rho_w}{\rho_s}, \qquad k = \frac{\rho_w C_d A_f}{2\rho_s V_p}.$$

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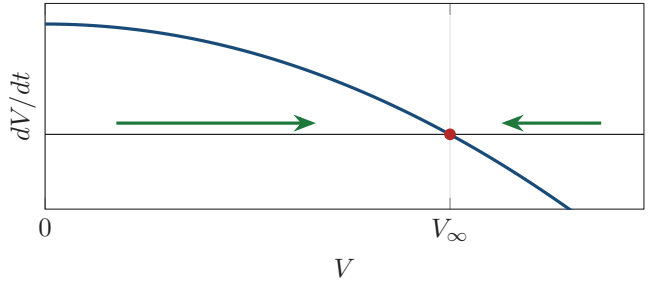
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Question

Before solving: what kind of equation is this, and what can we learn without solving it?

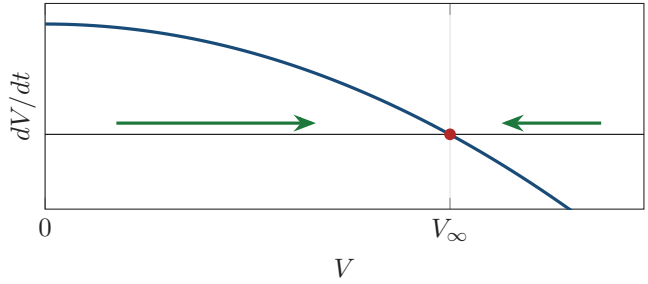
Learning from the Graph of dV/dt



Question

Where is dV/dt positive, where negative? What happens to V in the long run?

Learning from the Graph of dV/dt



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Where is dV/dt positive, where negative? What happens to V in the long run?

Below $V_\infty = \sqrt{g'/k}$ the pebble speeds up; above it, it slows down. Every solution approaches V_∞: a stable equilibrium, the **terminal velocity**.

The Terminal Velocity



Question

At which speed is the acceleration zero?

The Terminal Velocity



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Set $dV/dt = 0$:

$$\boxed{V_\infty = \sqrt{\frac{2(\rho_s - \rho_w)V_p\, g}{\rho_w C_d A_f}}}$$

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$$V_\infty = \sqrt{\frac{2 \cdot 1.5 \cdot 2.83 \times 10^{-3} \cdot 9.81}{0.06}} \approx 1.2\ \mathrm{m/s}.$$

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Now check the regime: $\mathrm{Re} = 1000 \cdot 1.2 \cdot 0.06/10^{-3} \approx 7 \times 10^4$. Quadratic drag was the right choice.

Solving the Equation Exactly



Question

Solve $\frac{dV}{dt} = g'\left(1 - \frac{V^2}{V_\infty^2}\right)$ with $V(0) = 0$.

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Separate variables and use partial fractions:

$$\int_0^V \frac{dV}{1 - V^2/V_\infty^2} = \frac{V_\infty}{2} \ln \frac{V_\infty + V}{V_\infty - V} = g't.$$

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Solve for V:

$$\boxed{V(t) = V_\infty \tanh\left(\frac{t}{\tau}\right), \qquad \tau = \frac{V_\infty}{g'}}$$

Checking the Solution

$$V(t) = V_\infty \tanh\left(\frac{t}{\tau}\right), \qquad x(t) = V_\infty \tau \ln\cosh\left(\frac{t}{\tau}\right) \text{ (depth reached)}$$

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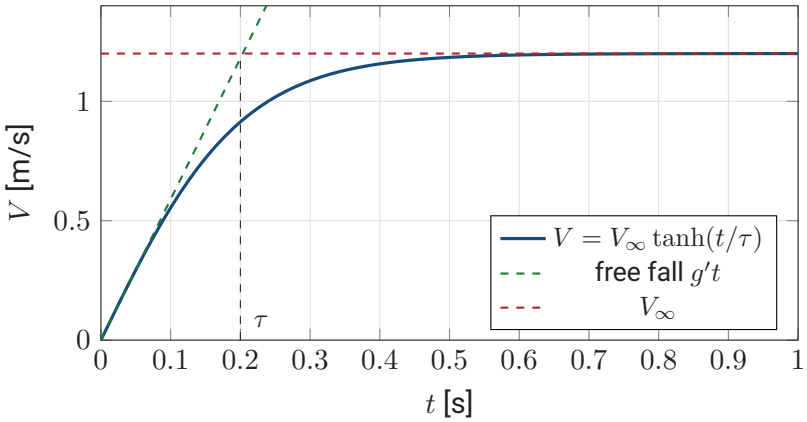
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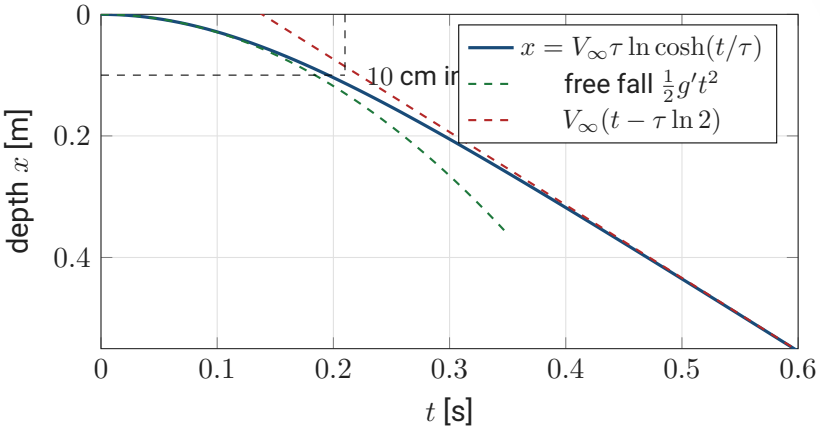
Both limits are what the graph of dV/dt predicted before we solved anything.

The Speed over Time



$\tau = V_\infty / g' \approx 0.2$ s. After one τ the pebble has 76% of its terminal speed.

The Depth over Time



First a parabola (free fall), then a straight line (terminal speed).

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$$\frac{dV}{dt} = g' - \frac{V}{\tau_s}, \qquad \tau_s = \frac{m}{3\pi\mu d}, \qquad V(t) = V_\infty\big(1 - e^{-t/\tau_s}\big), \quad V_\infty = g'\tau_s.$$

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- A linear equation gives an exponential approach; the quadratic one gives a $\tanh$. A different drag law changes the shape of the curve.
- For silt $V_\infty \sim 10^{-4}$ m/s: it takes hours to settle through 10 cm. The crow's pebble needs a fraction of a second.

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The level rises almost instantly. That is why the static level law describes what the crow sees.

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Glue two identical pebbles side by side (twice the volume, twice the frontal area). Does V_∞ change? What if you glue them one behind the other (twice the volume, same frontal area)?

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Shape did not matter for the final level, but it matters for the speed.

Does Dropping from Higher Help?



Problem (groups of 2–3, four minutes)

The crow can drop a pebble gently at the surface, or from 30 cm above, so that it enters the water at about 2.4 m/s, faster than V_∞. Using only the graph of dV/dt, sketch $V(t)$ in the water for both cases. Does the high drop reach the bottom much sooner?

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Entering above V_∞, the drag is larger than the driving force, so the pebble *slows down* towards V_∞. It arrives a little sooner, but the extra speed is gone within about τ. Dropping from high up gains almost nothing, apart from a bigger splash.

How Many Parameters Really Matter?



Question

The equation contains g, ρ_s, ρ_w, C_d, A_f, V_p. Can we get rid of them?

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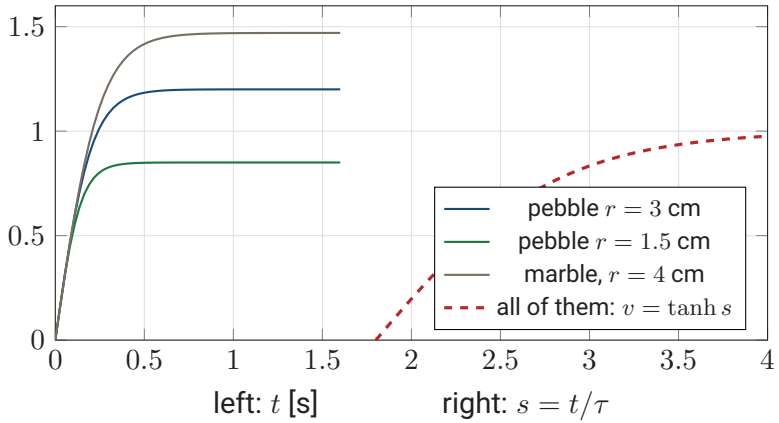
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Every pebble, in every liquid, follows the same curve. The parameters only set the two scales V_∞ and τ.

All Pebbles on One Curve



Three pebbles, three curves. In scaled variables they become one curve.

Why Scale an Equation?



Question

What did we gain by removing the parameters?

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What did we gain by removing the parameters?

1. **Fewer parameters**: here none are left, so one experiment describes all pebbles.
2. **Comparison**: data from different pebbles should fall on one curve. If they do not, the model is wrong.
3. **Permission to neglect**: when we scale the equations of the moving water, the sizes of the terms will tell us which ones can be dropped.

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Question

Our pebble model is simple. Which effects does it ignore?

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- **Constant buoyancy and constant** C_d: the true force depends on the history of the wake.
- **Added mass**: to accelerate the pebble we must also accelerate the water around it, so it behaves as if heavier, $m \to m + m_a$. This matters at the start and at impact.
- **The splash**: the violent deformation of the surface at entry. No model with one variable can describe it.
- **The rising level**: it changes the fall depth slightly. Negligible here, but worth stating.

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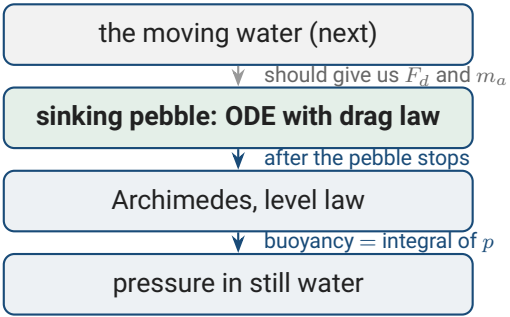
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To capture these effects we must describe the water itself.

How the Models Fit Together So Far



Each model contains the one below it as a special case.

Summary: How Fast Does the Pebble Sink?



1. Newton's law: $m\dot{V} = (\rho_s - \rho_w)V_p g - F_d(V)$. Everything is derived except the drag, which needs a closure.
2. The Reynolds number $\mathrm{Re} = \rho V d/\mu$ chooses the drag law; our pebble has $\mathrm{Re} \sim 10^4$, so the drag is quadratic.
3. The graph of dV/dt shows a stable terminal velocity; the exact solution is $V = V_\infty \tanh(t/\tau)$.
4. $V_\infty \approx 1.2$ m/s and $\tau \approx 0.2$ s: sinking is almost instant.
5. Scaling puts every pebble on the curve $v = \tanh s$.

Exercises: How Fast Does the Pebble Sink?



Exercise 10: The $\tanh$ solution

Fill in the partial-fraction steps leading to $V = V_\infty \tanh(t/\tau)$. Find the time to reach 99% of V_∞ for our pebble.

Exercise 11: Starting too fast

Solve the quadratic model with $V(0) = 2V_\infty$ (a $\coth$ appears). Compare the approach to V_∞ from above and from below.

Exercise 12: A rising cork

A cork is released at depth h. Find its terminal rise speed, and show that its speed at the surface no longer depends on h when $h \gg V_\infty^2/g'$.

Exercises: How Fast Does the Pebble Sink? (contd.)



Exercise 13: Muddy water

Use Stokes' terminal velocity $V_\infty = (\rho_s - \rho_w) g d^2/(18\mu)$ for a silt grain. Check that $\mathrm{Re} \ll 1$, and find how long 10 cm of water stays muddy. Explain in one sentence why the crow's trick is fast but muddy water clears slowly.

Exercise 14: Scaling the linear model

Scale the linear-drag equation and show that it also has no parameters left. If both drag terms are kept, which single ratio decides which one matters?



From Particles to Fields: What Does the Water Do?

Questions the Pebble Model Cannot Answer



Question

So far the water acted on the pebble through a few numbers: a pressure, a buoyant force, a drag. Which questions about the pitcher remain open?

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All of these depend on *what the water does at every point*. We need a new kind of variable.

New Unknowns: Fields



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A value at every point and every time:

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Question

But water is made of molecules. What does “the velocity at a point” even mean?

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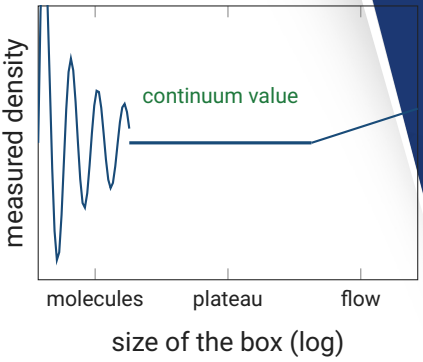


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Six orders of magnitude separate the two scales. In between, “density at a point” and “velocity at a point” make sense, and we may use calculus.



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air in a nanochannel	molecules $\sim$ channel	no
the upper atmosphere	free path $\sim$ km	no
blood in a 3 mm artery	cells $8\ \mu$m $\ll 3$ mm	yes
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The same test gives different answers. The continuum is an assumption we can check.

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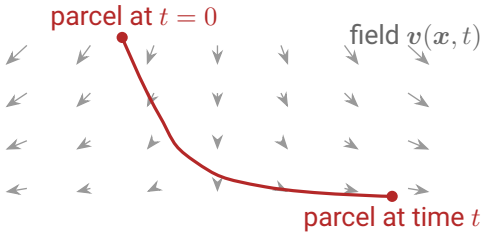
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The link: the parcel now at $\boldsymbol{x}$ moves with the local velocity,

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}(\boldsymbol{x}(t), t).$$

Its solution curves are the **pathlines**.

A Pathline in a Velocity Field



Grey arrows: the velocity field at one instant (the view from the bridge). Red curve: the path of one parcel (the view from the boat).

Pathlines, Streamlines, Streaklines



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Neither. They are **streaklines**: the positions, at one moment, of all droplets that left the same nozzle. One velocity field, three different families of curves.

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Two reasons for a change: the field changes where you are ($\partial_t f$), and you are carried to a place where it is different ($(\boldsymbol{v} \cdot \nabla) f$, called **advection**).

Acceleration Without Time Dependence



Question

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Yes, it speeds up in the narrows. Here $\partial \boldsymbol{v}/\partial t = 0$, but

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The acceleration of the water, $D\boldsymbol{v}/Dt$, is **nonlinear** in $\boldsymbol{v}$. This term is the main source of difficulty in fluid dynamics.

Walking Towards a Bonfire



Question

At dusk the air cools everywhere, and you walk towards a bonfire. Does the temperature you feel rise or fall?

Walking Towards a Bonfire



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It can do either:

$$\frac{DT}{Dt} = \underbrace{\frac{\partial T}{\partial t}}_{<0\text{: dusk}} + \underbrace{(\boldsymbol{v}\cdot\nabla)T}_{>0\text{: towards the fire}},$$

and the two terms compete.

Two Sensors in a River



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Neither alone. The fixed sensor measures $\partial \boldsymbol{v}/\partial t = 0$; the drifting one measures $D\boldsymbol{v}/Dt$. The advection term is the *difference* between the two readings.

Worked Example: A Probe in a Vortex

The water rotates, $\boldsymbol{v} = \omega(-y,\, x)$, and the temperature is $T = T_0 + \beta y$.

Question

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- On the right ($x > 0$) the probe is carried upward into warmer water, so its reading rises.
- On the left it is carried downward and cools.
- A fixed probe reads the same temperature forever.

How Does the Mass in a Region Change?

Fix a region Ω with outward normal $\boldsymbol{n}$.
It contains mass $M(t) = \int_\Omega \rho \, dV$.

Question

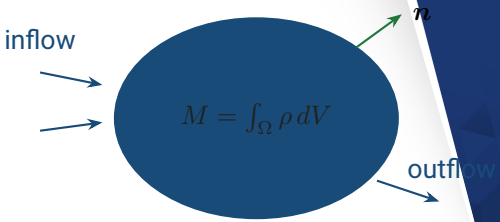
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Mass cannot appear or disappear.
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Only by water flowing through the boundary:

$$\frac{dM}{dt} = -\oint_{\partial\Omega} \rho \, (\boldsymbol{v} \cdot \boldsymbol{n}) \, dS.$$

The minus sign: outflow ($\boldsymbol{v} \cdot \boldsymbol{n} > 0$) decreases M.

How Much Water Crosses a Small Surface?



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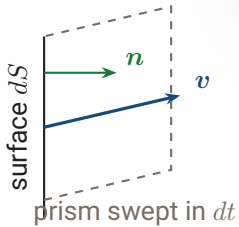
The water that crosses fills a slanted prism with base dS and height $(\boldsymbol{v} \cdot \boldsymbol{n})\, dt$:

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Only the normal component moves water across; the tangential part just slides along the surface.

From a Balance on a Region to an Equation at a Point



Use the divergence theorem on the boundary term:

$$\int_\Omega \frac{\partial \rho}{\partial t}\, dV = -\oint_{\partial\Omega} \rho\, \boldsymbol{v} \cdot \boldsymbol{n}\, dS = -\int_\Omega \nabla \cdot (\rho \boldsymbol{v})\, dV.$$

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If an integral vanishes over every region, what about the integrand? If it is continuous, it is zero at every point:

The continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0}$$

What Does $\nabla \cdot \boldsymbol{v}$ Mean?



Expand $\nabla \cdot (\rho \boldsymbol{v}) = \boldsymbol{v} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{v}$ and recognise the material derivative:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \boldsymbol{v}.$$

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A parcel keeps its mass, so its density falls exactly when its volume grows. Hence

$$\nabla \cdot \boldsymbol{v} = \frac{1}{\mathrm{Vol}} \frac{d\,\mathrm{Vol}}{dt}\bigg|_{\text{parcel}} :$$

the divergence is the rate at which a small parcel of fluid expands.

A Garden Hose



Question

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Take the nozzle as the region. The flow is steady, so $dM/dt = 0$: what flows in must flow out.

$$\rho v_1 A_1 = \rho v_2 A_2 \quad \Longrightarrow \quad v_2 = 2v_1.$$

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The same balance that gave us a PDE is also a quick tool for direct calculation.

The Same Equation Elsewhere

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

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Nothing in the derivation was special to water. What else could ρ be?

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System	ρ means	Conserved quantity
water in the pitcher	mass per volume	mass
air around a fast aircraft	mass per volume	mass
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For traffic, the velocity depends on the density, $v(\rho)$: a closure again.
“Conservation law plus closure” is a pattern we will meet many times.

Water Does Not Compress



Question

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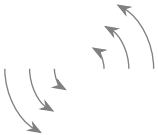
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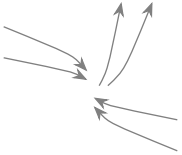
Parcels can change shape, but never volume.

Earlier we used "the volume of water never changes" for the whole pitcher. Now the same statement holds at every point. Integrating it over the pitcher gives back the level law (see the exercises).

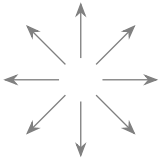
Three Flows: Which Ones Can Be Water?



(a) $\boldsymbol{v} = \omega(-y, x)$



(b) $\boldsymbol{v} = (\alpha x, -\alpha y)$



(c) $\boldsymbol{v} = (\alpha x, \alpha y)$

Problem (in pairs, five minutes)

For each flow compute $\nabla \cdot \boldsymbol{v}$ and $D\boldsymbol{v}/Dt$. Which flows are possible for water? What does each acceleration mean?

Three Flows: Answers



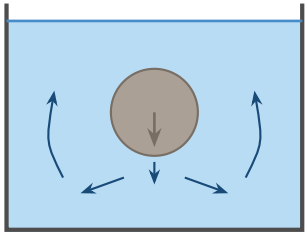
Flow	$\nabla \cdot \boldsymbol{v}$	$D\boldsymbol{v}/Dt$	Possible for water?
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- (b) squeezes in one direction and stretches in the other, keeping the volume. This is what the water under a falling pebble does.
- (c) creates volume out of nothing: impossible for water, possible for an expanding gas.
- All three flows are steady, yet all three accelerate.

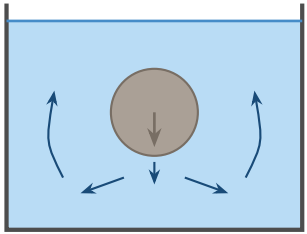
Where Does the Water Under the Pebble Go?



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The pebble moves down. The water below it cannot be compressed. Where does it go?

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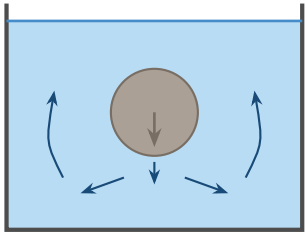


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- It escapes sideways, like flow (b).
- Then it moves up along the walls.
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One condition, $\nabla \cdot \boldsymbol{v} = 0$, already organises the whole flow, before we have written any force.

Summary: From Particles to Fields



1. The water is described by fields $\boldsymbol{v}(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$; the continuum idea is a checkable assumption about scales.
2. Lagrangian view (follow the water) and Eulerian view (watch a point), linked by $\dot{\boldsymbol{x}} = \boldsymbol{v}(\boldsymbol{x}, t)$.
3. The material derivative $Df/Dt = \partial_t f + (\boldsymbol{v} \cdot \nabla) f$ is the change following the water; its advection term is nonlinear.
4. A mass balance on a region, the divergence theorem and "zero over every region" give $\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$.
5. For water: $\nabla \cdot \boldsymbol{v} = 0$.

Exercises: From Particles to Fields



Exercise 15: Stagnation flow

For $\boldsymbol{v} = (\alpha x, -\alpha y)$, solve the pathline equations, show that pathlines are hyperbolas $xy = \text{const}$, and check that a small rectangle of water keeps its area.

Exercise 16: Acceleration

Compute $D\boldsymbol{v}/Dt$ for the stagnation flow. Where near a falling pebble does the water move like this?

Exercise 17: Shallow water

Water in a channel has depth $h(x,t)$ and depth-averaged speed $u(x,t)$. Balance the mass in $[x, x+\Delta x]$ and derive $\partial_t h + \partial_x (hu) = 0$.

Exercises: From Particles to Fields (contd.)



Exercise 18: The level law again

Integrate $\nabla \cdot \boldsymbol{v} = 0$ over the water in the pitcher. Use $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on the walls and on the pebble, and $\boldsymbol{v} \cdot \boldsymbol{n} = \dot{H}$ on a flat surface. Follow a pebble as it crosses the surface, and recover $H_n = H_0 + nV_p/A$.

Exercise 19: Is the continuum safe?

Find the side of a cube that holds 10^6 water molecules. Compare it with the pebble and with the thinnest sheet of a splash (10^{-4} m).



Which Forces Move the Water? The Navier–Stokes Equations

Newton's Law for a Region of Water



Question

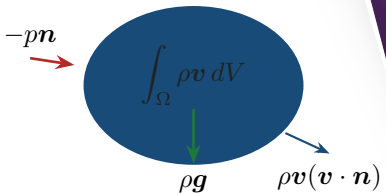
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Newton's Law for a Region of Water

Question

The momentum in a fixed region Ω is $\int_\Omega \rho \boldsymbol{v}\, dV$. What can change it?

- **Gravity**, acting on every parcel: $\int_\Omega \rho\, \boldsymbol{g}\, dV$.
- **Pressure** from the surrounding water, on the boundary: $-p\,\boldsymbol{n}$ per unit area. (For now we keep only pressure.)
- **Flow through the boundary**: water entering or leaving brings its momentum with it, at rate $\rho \boldsymbol{v}\,(\boldsymbol{v}\cdot\boldsymbol{n})\, dS$.



From the Balance to Euler's Equation

$$\frac{d}{dt}\int_\Omega \rho \boldsymbol{v}\, dV = -\oint_{\partial\Omega} \rho \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n})\, dS - \oint_{\partial\Omega} p\,\boldsymbol{n}\, dS + \int_\Omega \rho\,\boldsymbol{g}\, dV$$

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We did exactly this for mass. Which steps turn the balance into an equation at a point?

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Question

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The divergence theorem, then "zero over every region means zero at every point", then subtract v times the continuity equation.

Reading Euler's Equation



Euler's equation (1757)

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p + \rho \boldsymbol{g}$$

Mass times acceleration of a parcel $=$ push from high to low pressure $+$ weight.

Question

What does the equation give when the water is at rest?

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What does the equation give when the water is at rest?

$\boldsymbol{v} = 0$ gives $\nabla p = \rho \boldsymbol{g}$, that is $dp/dy = -\rho g$: the pressure law we started with.

Why Does the Velocity Appear Twice?



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In the flow term $\rho \boldsymbol{v}\,(\boldsymbol{v}\cdot\boldsymbol{n})$ the velocity appears twice. Why?

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Why do firefighters lean forward when holding a hose?

The jet takes momentum out of the nozzle, so the hose is pushed backwards. No pressure argument is needed: it is the flow term alone.

A Quick Consequence: Bernoulli



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Bernoulli needs a steady flow without friction. A splash is neither.

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Friction along the pebble's surface, and with it a large part of the drag.

Measuring Friction in a Fluid



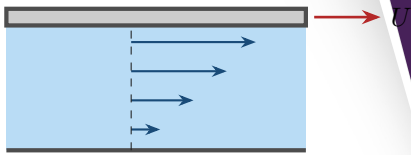
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Experiments show (and for water very accurately)

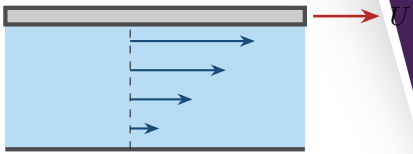
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with μ the **viscosity**: water 10^{-3}, honey about 10, air 2×10^{-5} Pa s.

Measuring Friction in a Fluid

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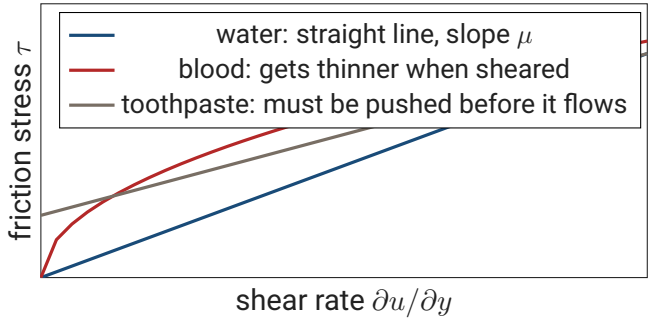
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$$\tau = \mu \frac{U}{h},$$

with μ the **viscosity**: water 10^{-3}, honey about 10, air 2×10^{-5} Pa s.

In general: the friction stress on a surface is μ times the rate at which the tangential velocity changes across it, $\tau = \mu \, \partial u/\partial y$.

Not Every Fluid Is Like Water



The force appears in the same place in the equations; only the closure changes with the material.

The Navier–Stokes Equations



Add the friction force to the momentum balance. For incompressible flow with constant μ it becomes a Laplacian of each velocity component (the full derivation is an exercise):

Incompressible Navier–Stokes equations

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g}, \qquad \nabla \cdot \boldsymbol{v} = 0$$

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The Laplacian measures how much $\boldsymbol{v}$ at a point differs from the average of its neighbours. Viscosity pulls every parcel towards that average: it spreads momentum out and smooths the flow.

Where Is the Equation for the Pressure?



Question

In 2D the unknowns are u, v, p and the equations are two momentum equations and continuity. Which equation tells us how p changes in time?

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In still water the pressure was whatever held the water up. Now it is whatever keeps the water from being compressed: under the falling pebble, the pressure is what pushes the water sideways.

Gap or Viscosity?



Question

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Only through the combination μ/h. To test the law $\tau = \mu U/h$ we must *vary* U or h and check that the force changes as predicted. A material law is tested by how the response changes, not by a single number.

What Happens at the Boundaries?



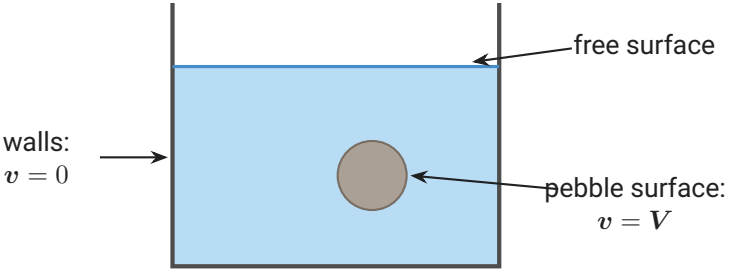
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Three kinds of boundary, each with its own physics.

At a Solid Wall



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At a Solid Wall



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- It cannot flow through the wall: $\boldsymbol{v} \cdot \boldsymbol{n} = 0$.
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The moving-wall version is how the pebble will talk to the water.

At the Free Surface: the Kinematic Condition



Question

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Describe the surface as $F(\boldsymbol{x}, t) = 0$, for example $F = y - \eta(x, t)$. Following a parcel, F must stay zero:

Kinematic condition

$$\frac{DF}{Dt} = 0 \text{ on } F = 0 \qquad \Longleftrightarrow \qquad \frac{\partial \eta}{\partial t} + u\,\frac{\partial \eta}{\partial x} = v.$$

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The position of the surface is an *unknown*: the equations must find the domain together with the solution. This is a **free-boundary problem**.

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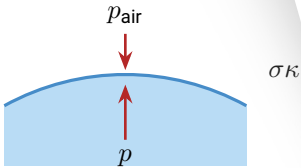
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- A curved surface can hold a pressure jump (soap bubbles, droplets).
- For the pitcher we would like to drop it and use $p = 0$. Is that justified? The next slides decide.

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$$\frac{D\hat{\boldsymbol{v}}}{D\hat{t}} = -\hat{\nabla}\hat{p} + \frac{1}{\mathrm{Re}}\,\hat{\nabla}^2\hat{\boldsymbol{v}} + \frac{1}{\mathrm{Fr}^2}\,\hat{\boldsymbol{g}}$$

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$\mathrm{Re} = \dfrac{\rho U L}{\mu}$ (inertia/viscosity), $\mathrm{Fr} = \dfrac{U}{\sqrt{gL}}$ (inertia/gravity), $\mathrm{Bo} = \dfrac{\rho g L^2}{\sigma}$ (gravity/surf

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Compute Re, Fr and Bo for the pebble in water ($\sigma = 0.072$ N/m). What do they tell us?

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$\mathrm{Fr} \approx 1$ means the splash rises about as high as the pitcher is wide: the splash and the level rise are both important.

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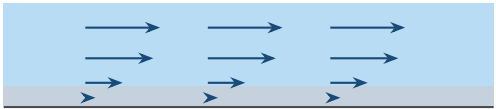
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Viscosity survives in thin **boundary layers**, of thickness $\delta \sim L/\sqrt{\mathrm{Re}}$, along walls and around the pebble.



The drag coefficient C_d of the pebble is produced in this thin layer.

Reynolds Numbers in Nature

Flow	Re	Dominant effect
a swimming bacterium	10^{-5}	viscosity only
silt settling in the pitcher	10^{-3}	viscosity
blood in a capillary	10^{-3}–10^{-2}	viscosity
blood in the aorta	10^{3}	inertia, pulsating
our pebble	$\mathbf{7 \times 10^{4}}$	inertia, plus boundary layers
a human swimmer	10^{6}	inertia, turbulent wake
an aircraft wing	10^{8}	inertia, thin boundary layers

One number spans thirteen orders of magnitude and tells us which model to use.

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Without solving anything, the numbers tell us that the fable works for a crow but not for an ant.

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A film studio builds a pitcher 10 times larger. Which number must match the real one so that the splash looks right? What does that mean for the camera?

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The splash is a contest between inertia and gravity, so match $\mathrm{Fr} = U/\sqrt{gL}$. Then speeds grow by $\sqrt{10}$, and times grow by $\sqrt{10}$ as well: film at normal speed and play back about 3 times faster.

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Re then comes out 30 times too large: with water we cannot match both numbers at once.

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The ODE for the pebble is a summary of the PDE for the water, integrated over the pebble's surface.

The Complete Model of the Water

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\begin{cases}
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Question

What is still missing?

The pebble. The pitcher in this model is empty. Notice also that the domain $\Omega(t)$ moves with the surface: a free-boundary problem even before the pebble arrives.

Summary: The Navier–Stokes Equations



1. A momentum balance on a region gives Euler's equation $\rho\, D\boldsymbol{v}/Dt = -\nabla p + \rho \boldsymbol{g}$.
2. The plate experiment gives friction $\tau = \mu\, \partial u/\partial y$, which adds $\mu \nabla^2 \boldsymbol{v}$: the Navier–Stokes equations.
3. The pressure is the Lagrange multiplier of $\nabla \cdot \boldsymbol{v} = 0$.
4. Boundary conditions: no-slip at walls; at the free surface, the kinematic condition $DF/Dt = 0$ and $p = 0$.
5. Scaling with Re, Fr, Bo shows which terms matter; small viscosity survives in boundary layers; the fable fails for an ant.

Exercises: The Navier–Stokes Equations



Exercise 20: Flow in a channel

Steady flow in a channel of width $2h$ is driven by a pressure gradient G. Reduce Navier–Stokes to $\mu u''(y) = -G$, solve it with no-slip at the walls, and find the velocity profile and the flow rate. (This is the basic model of blood flow in a vessel.)

Exercise 21: Stream function

Show that $\boldsymbol{v} = (\partial_y \psi, -\partial_x \psi)$ satisfies $\nabla \cdot \boldsymbol{v} = 0$ for every ψ. Find ψ for the stagnation flow.

Exercises: The Navier–Stokes Equations (contd.)



Exercise 22: Boundary layers

Estimate $\delta \sim L/\sqrt{\mathrm{Re}}$ for the pebble and for a silt grain. For which one is "a thin layer with frictionless flow outside" a sensible picture?

Exercise 23: The ant's pitcher

Find the terminal velocity of a 0.2 mm sand grain (check Re first), then compute Re, Fr, Bo. Describe the physics at ant scale in three sentences.

Exercise 24: The friction term

Starting from the stress $\sigma = -pI + \mu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$, compute $\nabla \cdot \sigma$ and use $\nabla \cdot \boldsymbol{v} = 0$ to obtain $-\nabla p + \mu \nabla^2 \boldsymbol{v}$.



Pebble and Water Together: Fluid–Structure Interaction

How Does the Water Push the Pebble?

A rigid disc k has centre $\boldsymbol{X}_k(t)$, velocity $\boldsymbol{V}_k(t)$ and mass $m = \rho_s V_p$:

$$m\,\frac{d\boldsymbol{V}_k}{dt} = m\,\boldsymbol{g} + \boldsymbol{F}_k, \qquad \frac{d\boldsymbol{X}_k}{dt} = \boldsymbol{V}_k.$$

Question

What is $\boldsymbol{F}_k$, the force of the water on the disc?

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The stress of the water, added up over the disc's surface:

$$\boldsymbol{F}_k = \oint_{\partial B_k} \left[-p\,\boldsymbol{n} + \mu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)\boldsymbol{n} \right] dS, \qquad \boldsymbol{n} \text{ pointing into the water.}$$

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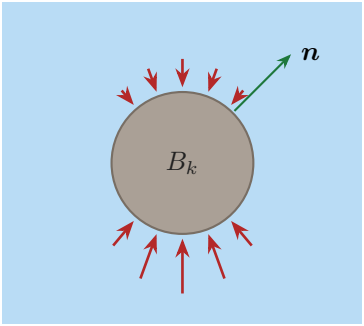
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At high Re the pressure part dominates: it contains the buoyancy and most of the drag. Pressure on a circle points through the centre, so we ignore rotation.

Where Is the Pressure Larger?

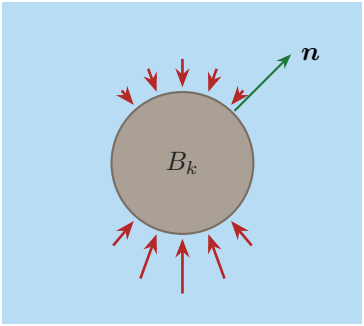


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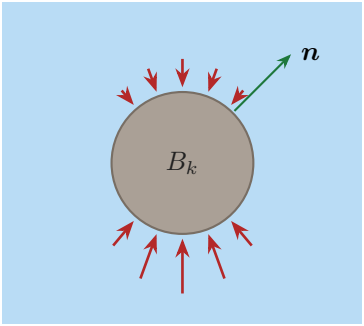
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This is the same integral as in Archimedes' principle, now with the pressure of the moving water.

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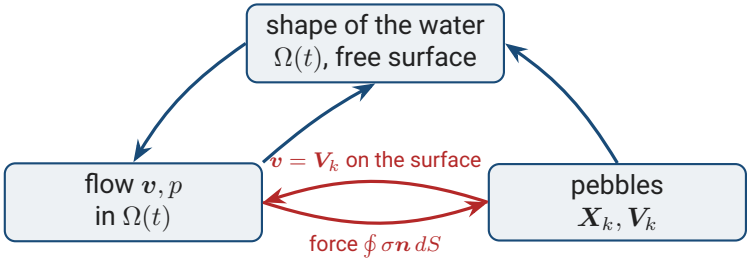
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Water → pebble (dynamic condition)

$$m\dot{\boldsymbol{V}}_k = m\boldsymbol{g} + \oint_{\partial B_k} \sigma\, \boldsymbol{n}\, dS$$

Information flows in both directions. That is what “interaction” means.

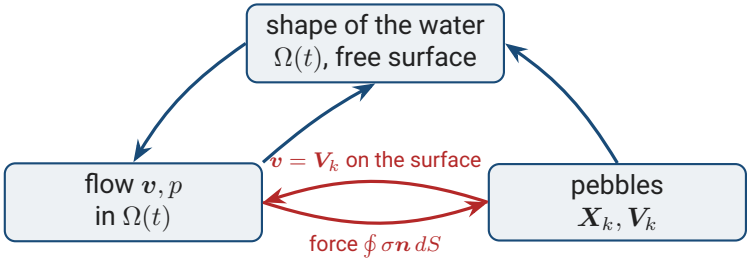
Can We Solve the Water First?



Question

Could we compute the flow first, and then move the pebble through it?

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No. The shape of the water depends on the pebble and the surface; the flow depends on that shape; the pebble depends on the flow. Every part needs every other part.

The Complete Model

The thirsty-crow FSI system

$$
\begin{cases}
\rho_w \dfrac{D\boldsymbol{v}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{v} + \rho_w \boldsymbol{g}, \quad \nabla \cdot \boldsymbol{v} = 0 & \text{in } \Omega(t) \text{ (unknown)} \\
\dfrac{DF}{Dt} = 0, \quad p = 0 & \text{free surface} \\
\boldsymbol{v} = \boldsymbol{0} \text{ (walls)}, \qquad \boldsymbol{v} = \boldsymbol{V}_k \ (\partial B_k) & \text{no-slip} \\
m\dot{\boldsymbol{V}}_k = m\boldsymbol{g} + \oint_{\partial B_k} \sigma \boldsymbol{n} \, dS, \quad \dot{\boldsymbol{X}}_k = \boldsymbol{V}_k & \text{pebbles} \\
\text{no overlap of pebbles and walls; initial values of } \boldsymbol{v}, F, \boldsymbol{X}_k, \boldsymbol{V}_k &
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$$

A system of PDEs for the water and ODEs for the pebbles, coupled on moving boundaries whose position is part of the answer. This is the mathematics hidden in a children's story.

What Happens When Pebbles Meet?



Question

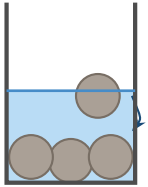
The model says “no overlap”. What should happen when a pebble lands on another?

What Happens When Pebbles Meet?

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The model says “no overlap”. What should happen when a pebble lands on another?

- They may touch but not overlap: $|\boldsymbol{X}_j - \boldsymbol{X}_k| \geq r_j + r_k$.
- On impact the approach velocity is removed (no bounce), or reversed with a restitution coefficient e.
- Along the contact: free sliding, or friction $|F_t| \leq \mu_c |F_n|$.



With no bounce and no friction, a new pebble slides off the pile and settles low. e and μ_c are closures.

Can We Solve This System Exactly?



Question

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What makes this system hard?

1. **Nonlinearity**: $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$, hard even in a fixed box (turbulence).
2. **Free boundary**: the domain $\Omega(t)$ is unknown.
3. **Coupling**: ODEs and PDEs exchange data on moving surfaces; even existence of solutions is a research question for many such systems.
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4. **Contacts**: velocities jump when pebbles collide.

We cannot solve it on paper. But we can state it exactly, check that its limits agree with our simpler models, and learn from those limits.

The Same System in Other Places



Question

Keep the water equations and the two coupling conditions, but replace the rigid pebble by something else. What problems do we get?

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Replace the pebble by…	New problem	Field
an elastic membrane	artery wall with blood	haemodynamics
an elastic beam	a flag in the wind	aeroelasticity
a floating hull	a ship in waves	marine engineering
a wing section	wing flutter	aircraft design

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Only the equation of the solid changes. For an artery, Newton's ODE for the disc becomes an elasticity PDE for the wall.

When Is a Shortcut Enough?



Question

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Without the pebble the water is simply still. The shortcut gives a pebble falling through still water with buoyancy: fine for a first estimate of the sinking time (with a drag law added, that is our ODE model).

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It fails for everything the pebble *causes*: the splash, the waves, the rising level, and the drag itself. Whether coupling is needed depends on the question we ask.

Check 1: Freeze Everything



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- The force on the pebble:

$$\boldsymbol{F}_k = -\oint_{\partial B_k} p\,\boldsymbol{n}\,dS = -\int_{B_k} \nabla p\,dV = \rho_w V_p\, g \;\text{ upward.}$$

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Archimedes' principle, recovered from the full model with the same divergence-theorem step.

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Split the stress integral into its still-water part and the rest:

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- The still-water part is the buoyancy; the rest *is* the drag.
- In the ODE model we had to guess $F_d = \frac{1}{2}\rho_w C_d A_f V^2$. The full model computes it: C_d is an output here, an input there.

Check 3: The Level Law



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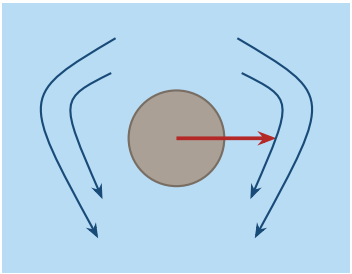
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- While a pebble crosses the surface, the level rises exactly as fast as its submerged volume grows.
- Once it is fully under water, $\dot{H} = 0$.
- Adding up n pebbles: $H_n = H_0 + nV_p/A$.

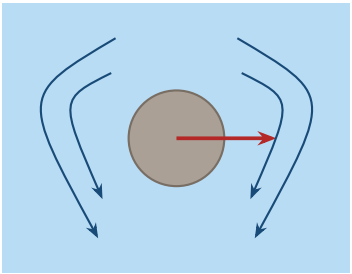
Why Is It Hard to Accelerate a Ball Under Water?



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To move the ball you must also set the surrounding water in motion. That water has kinetic energy, proportional to U^2:

$$T_{\text{water}} = \tfrac{1}{2}\, m_a\, U^2,$$

which defines the **added mass** m_a.

The Added Mass of a Disc



For a circular disc the flow of the surrounding water can be computed exactly, and the energy gives

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For our stone ($\rho_s/\rho_w = 2.5$) this adds 40% to the inertia at the start of the fall, and nothing once the terminal speed is reached.

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- Stone: 0.4, the pebble mostly follows its own weight.
- Artery wall in blood: about 1, wall and blood are tightly locked together.

This one ratio tells us in advance how hard the coupled problem will be.

Ranking the Inertia



Question

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The lighter the body, the more its motion belongs to the water. That is why bubbles wobble so much as they rise.

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- The body accelerates as if alone; the water answers one step later with the added-mass reaction $-m_a \dot{V}$.
- This is a feedback loop with gain about $m_a/m = \rho_w/\rho_s$.
- For a stone (< 1) the error dies out; for a light body or a vessel wall ($\gtrsim 1$) it grows at every step and the computation blows up.

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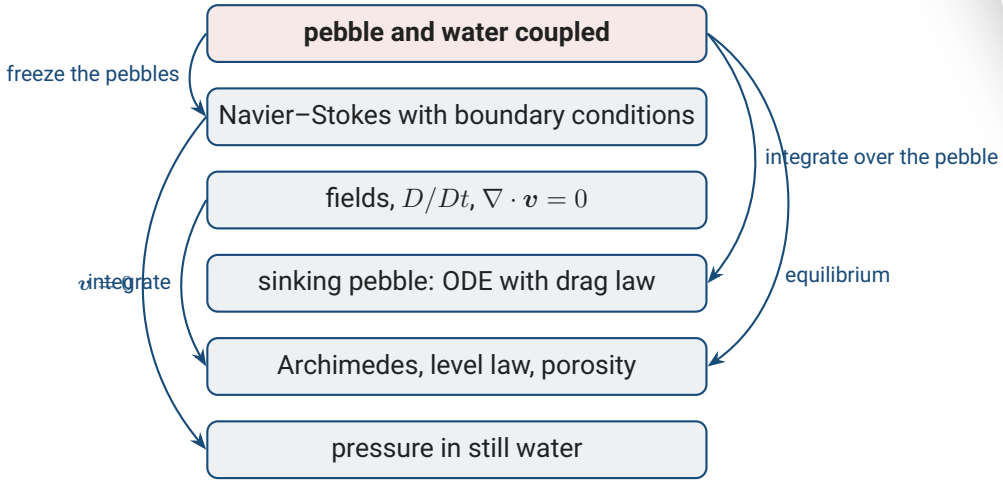
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A number from the model, ρ_w/ρ_s, predicts a failure of the computation. Understanding the equations comes before computing them.

How All the Models Fit Together



Every arrow is something we have shown, not an analogy.

How Do We Trust a Complicated Model?



Question

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Make it reproduce the simpler models. It must show:

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- the level law $H_n = H_0 + nV_p/A$ once the water has calmed down;
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If it cannot, its sophistication is worthless. And the experiments exist: real crows behave as the simple model predicts, including failure when the water starts too low.

One Pebble, from Beak to Bottom



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Follow one pebble from the crow's beak to the bottom. Which equations are in charge at each stage?

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Follow one pebble from the crow's beak to the bottom. Which equations are in charge at each stage?

1. **In the air**: $m\dot{\boldsymbol{V}} = m\boldsymbol{g}$, plain Newton.
2. **Impact**: the surface deforms violently; the kinematic condition redraws $\Omega(t)$; the added mass matters most.
3. **Descent**: buoyancy plus drag; the speed approaches V_∞.
4. **Near the bottom**: incompressibility squeezes the water out sideways; the pressure rises sharply.
5. **Contact**: the pebble stops, or slides into place on the pile.
6. **Afterwards**: viscosity damps the waves, the surface flattens, and the level has risen by exactly V_p/A.

What We Built from One Fable



Question	Mathematics	Result
pressure in still water?	units, first-order ODE	$p = \rho g \times$ depth
rise per pebble?	surface integrals, divergence theorem	$H_n = H_0 + nV_p/A$, $H_{\max} = H$
how fast does it sink?	nonlinear ODE, scaling	$V = V_\infty \tanh(t/\tau)$
what does the water do?	fields, balances on regions	D/Dt, $\nabla \cdot \boldsymbol{v} = 0$
which forces move it?	PDEs, closures, scaling	Navier–Stokes; Re, Fr, Bo
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The story was small. Each new question made the mathematics grow, one assumption at a time.

Exercises: Pebble and Water Together



Exercise 25: Three bodies

Write the full system for two pebbles and one cork. What changes for the floating cork? (Its wet surface meets the free surface.)

Exercise 26: The rising bubble

From $(m+m_a)\dot{V} =$ buoyancy $-$ weight with $m \to 0$, show that a released bubble starts with acceleration $2g$ (use $m_a = \frac{1}{2}\rho_w V$ for a sphere).

Exercises: Pebble and Water Together (contd.)



Exercise 27: Energy balance

Multiply the pebble equation by $\boldsymbol{V}_k$ and the momentum equation by $\boldsymbol{v}$, integrate and add. Identify the rate of change of kinetic energy, the work of gravity, the loss by viscosity (what is its sign?) and the boundary terms that cancel.

Exercise 28: Filming the fable

For a pitcher 10 times larger, use Fr to find the playback factor, and show that Re cannot be matched at the same time with water.

Exercise 29: An elastic ball

Replace the rigid pebble by a ball modelled as a mass on a spring. Which lines of the full system change? Predict the difference in the splash, using energy.



Ledger and Classification

Thirsty Crow: Dym's Questions



Question	Thirsty-crow model
Why?	Explain why dropping pebbles lets the crow drink, and when it cannot.
Find?	Level after n pebbles; pebbles needed; sinking time; the flow and the splash.
Given?	ρ_w, ρ_s, g, μ; pitcher width W; H_0, H_{beak}; pebble radius r; porosity φ.
How?	Hydrostatics; Archimedes and volume balance; Newton with drag; Navier–Stokes coupled to rigid bodies.
Valid?	Every richer model reduces to the simpler ones: $\boldsymbol{v} = 0$ gives $p = \rho g h$; equilibrium gives Archimedes; integrating over the pebble gives the drag ODE.
Verified?	Level law with a glass and coins; $H_{\max} = H_0/\varphi$ against the crow experiments (Jelbert et al., 2014).

Thirsty Crow: Assumption Ledger



- A1: Water is incompressible.
- A2: The pitcher has vertical walls (constant cross-section A); 2D slice of width W.
- A3: Pebbles are rigid and identical; fully under water when counted (relaxed by the porosity φ).
- A4: Statics for the level law; time enters with the sinking pebble.
- A5: Drag law $F_d = \frac{1}{2}\rho_w C_d A_f V^2$ with constant C_d ($\mathrm{Re} \sim 10^4$).
- A6: Water is Newtonian; no-slip on walls and pebbles.
- A7: Surface tension neglected ($\mathrm{Bo} \approx 480$); not true for an ant.
- A8: Pebbles are discs without spin; contacts without bounce or friction.

Velten SQM. S physical, natural; Q mechanistic, from stationary (still water, level law) to instationary and distributed (moving water); M algebraic (level law), ODE (sinking), nonlinear PDE coupled to ODEs on a free boundary (full model), deterministic.

Summary

Model	Structure	Key result
still water	$dp/dy = -\rho g$	$p = \rho g \times$ depth
Archimedes	$\boldsymbol{F} = -\oint p\,\boldsymbol{n}\,dS$	$\boldsymbol{F} = \rho_w V g$; sinks iff $\rho_s > \rho_w$
level law	volume balance	$H_n = H_0 + nV_p/A$; $n_{\min} = 3$
porous pile	solid fraction $1 - \varphi$	$H_{\max} = H_0/\varphi$
sinking pebble	$m\dot{V} = (\rho_s - \rho_w)V_p g - F_d$	$V = V_\infty \tanh(t/\tau)$
mass	balance on a region	$\nabla \cdot \boldsymbol{v} = 0$
momentum	balance on a region + friction	Navier–Stokes; Re, Fr, Bo
coupled	ODE + PDE on a free boundary	FSI system; $m_a = \rho_w \pi r^2$

Take-home

Before trusting a complicated model, make it reproduce its simpler relatives.



From the Thirsty Crow to Tsunamis

Can an Asteroid Flood the Earth Like the Crow?



The crow raised the water by dropping stones into a pitcher.

Question

A 1 km asteroid falls into the ocean. Using the crow's level law $\Delta H = V/A$, how much does the sea level rise?

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- Volume of the asteroid: $V = \frac{4}{3}\pi(500\ \mathrm{m})^3 \approx 5 \times 10^8\ \mathrm{m}^3$.
- Area of the oceans: $A \approx 3.6 \times 10^{14}\ \mathrm{m}^2$.
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So is an asteroid impact harmless?

No. The danger is not the volume but the *energy*, and the energy leaves as waves.

Pebble and Asteroid Side by Side



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Before any new equation, compare the two impacts. Which numbers change the most?

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Quantity	Pebble in pitcher	Asteroid in ocean (1 km)
size	6 cm	1 km
speed	~ 1 m/s	~ 20 km/s
density ratio ρ_s/ρ_w	2.5	~ 3 (rocky asteroid)
water depth	10 cm	~ 4 km
Froude number $v/\sqrt{gD}$	~ 1	~ 200
Mach number v/c_{water}	$\sim 10^{-3}$	~ 13
kinetic energy	~ 0.1 J	$\sim 3 \times 10^{20}$ J ($\approx 75{,}000$ Mt TNT)

Pebble and Asteroid Side by Side

Question

Before any new equation, compare the two impacts. Which numbers change the most?

Quantity	Pebble in pitcher	Asteroid in ocean (1 km)
size	6 cm	1 km
speed	~ 1 m/s	~ 20 km/s
density ratio ρ_s/ρ_w	2.5	~ 3 (rocky asteroid)
water depth	10 cm	~ 4 km
Froude number $v/\sqrt{gD}$	~ 1	~ 200
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The density ratio is almost the same. The speed, the energy and the Mach number are completely different.

Which Assumptions Still Hold?



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- **Navier–Stokes with a free surface**: holds, and viscosity is negligible.

From Impact to the Coast



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6. **Coast**: in shallow water the wave grows, breaks and runs up onto land.

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The bigger the crater, the faster the cost grows: doubling R costs 16 times the energy.

The Radius of the Crater



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- For a 1 km asteroid, $R_c \approx 10$ km.
- The ocean is only about 4 km deep, so the crater reaches the sea floor.
- The ratio R_c/h separates deep-water from shallow-water impacts.

How Fast Do the Waves Travel?

Small waves of wavenumber k on water of depth h satisfy

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Source	λ	kh	Behaviour
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An impact wave spreads out into a train of waves and loses height faster than an earthquake tsunami.

How Does the Wave Shrink?



Question

A ring wave spreads out. If its energy is shared along a circle of circumference $2\pi r$, how does its height fall with r?

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Energy per unit length $\propto \eta^2 \propto 1/r$, so $\eta \propto 1/\sqrt{r}$.

Dispersion spreads the energy over a longer and longer train of waves, so impact waves fall off faster, closer to

$$\eta(r) \sim \eta_0 \frac{R_c}{r}.$$

What Happens Near the Coast?



Question

Near the coast the water gets shallower. What happens to the height of the wave?

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It grows (Green's law):

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It grows (Green's law):

$$\eta \propto h^{-1/4}.$$

Then the wave breaks and runs up onto land.

Rough size for a 1 km asteroid: about 10 m in deep water 1000 km away, tens of metres at the coast. We will check these numbers carefully.

Has This Happened?



Question

Do we have evidence of impact tsunamis, or of solid masses making giant waves?

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- **Chicxulub** (66 million years ago): a ~ 10 km asteroid; a global tsunami (Range et al., 2022). It ended the age of the dinosaurs.
- **Eltanin** (2.5 million years ago): a ~ 1–4 km asteroid in the deep Southern Ocean, the only known deep-ocean impact (Gersonde et al., 1997).
- **Lituya Bay, Alaska** (1958): a landslide fell into a bay and the water ran 524 m up the opposite slope. A solid mass entering water: the closest relative of the crow's pebble.

Two Models, Two Answers



Question

Is an impact tsunami more or less dangerous than an earthquake tsunami of the same energy?

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- Ward and Asphaug (2000) predicted dangerous impact tsunamis.
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Question

Two models, two answers. Which assumption differs?

What We Will Formulate Next



1. Energy and scaling: E, ε, R_c, the Froude and Mach numbers.
2. The initial shape of the water: a crater $\eta_0(r)$, for example a parabola with a rim.
3. Small water waves: derive $\omega^2 = gk\tanh(kh)$, with its deep- and shallow-water limits.
4. The ring wave and its decay law $\eta(r) \sim \eta_0 R_c/r$.
5. The shallow-water equations from balances on a region, and Green's law at the coast.
6. The assumption ledger, the Velten classification, and a crow-to-tsunami comparison.

Exercises: From the Crow to Tsunamis



Exercise 30: Sea-level rise

Repeat the level-law calculation for a 10 km asteroid. Is the rise still negligible?

Exercise 31: Crater size

Compute $R_c = (\varepsilon E/\rho_w g)^{1/4}$ for a 1 km asteroid at 20 km/s, with $\varepsilon = 0.01$ and $\varepsilon = 0.15$. Does the crater reach the floor of a 4 km deep ocean in both cases?

Exercise 32: Two kinds of tsunami

Using $\omega^2 = gk\tanh(kh)$, compute the phase speed ω/k for $\lambda = 200$ km and $\lambda = 25$ km in 4 km of water. How long does each take to cross 5000 km of ocean?



Thank You