INDIAN INSTITUTE OF TECHNOLOGY TIRUPATI
Department of Mathematics and Statistics
MA633L - Numerical Analysis
Root Finding Methods

Note: Usual Notations are used. Questions may have some typos or grammatical mistakes as proofreading is not done for this.

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1. The growth of a population can often be modelled over short periods of time by assuming that the population grows continuously with time at a rate proportional to the number present at that time with no immigration. Suppose that $N(t)$ denotes the number in the population at time t, λ denotes the constant birth rate of the population with initial population N_0 and immigration is permitted at a constant rate v, then population growth is given by
$$N(t) = N_0 e^{-\lambda t} + \frac{v}{\lambda}(e^{\lambda t} - 1)$$
If the initial population is 1 million, immigrated people in the first year is 435000 and the total population at the end of the first year is 1.6 million, find the birth rate of this population. Using the Newton-Raphson method with a starting value of 1. Use this λ value to predict the population at the end of the second year, assuming that the immigration rate during this year remains at 435,000 individuals per year.

2. Determine the number of iterations necessary to solve $f(x) = x^3 + 4x^2 - 10 = 0$ with accuracy 10^{-3} using the Bisection method on the interval $[1, 2]$

3. Use the Bisection method and the Regula-Falsi method (three iterations) to find the root of the function $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$.

4. Let $f(x) = 3(x + 1)(x - 0.5)(x - 1)$. Using the Bisection method and the Regula-Falsi method on the following intervals compute the root (first three iterations)

   (a) $[-2, 1.5]$

   (b) $[-1.25, 2.5]$

5. Use the Bisection method and the Regula-Falsi method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on the following intervals

   (a) $[0, 1]$

   (b) $[1, 3.2]$

   (c) $[3.2, 4]$

6. Use the Bisection method and the Regula-Falsi method to find solutions accurate to within 10^{-2} for $x^4 - 2x^3 - 4x^2 + 4x + 4 = 0$ on the following intervals

   (a) $[-2, 1]$

   (b) $[0, 2]$

(c) $[2,3]$

(d) $[-1,0]$

7. Use the Bisection method and the Regula-Falsi method to find an approximation to within 10^{-5} to the following problems

   (a) $3x - e^x = 0, [1,2]$

   (b) $2x + 3\cos x - e^x = 0, [0,1]$

   (c) $x^2 - 4x + 4 - \ln x = 0, [1,2], [2,4]$

   (d) $x + 1 - 2\sin(\pi x) = 0, [0, 0.5], [0.5, 1]$

   (e) $x - \frac{1}{2^x} = 0, [0,1]$

   (f) $e^x - x^2 + 3x - 2 = 0, [0,1]$

   (g) $2x\cos(2x) - (x+1)^2 = 0, [-3,-2], [-1,0]$

   (h) $x\cos x - 2x^2 + 3x - 1 = 0, [0.2, 0.3], [1.2, 1.3]$

   (i) $e^x - 2 = \cos(e^x - 2), [0.5, 1.5]$

   (j) $x^3 + x - 4 = 0, [1,4], [$

   (k) $x^3 - x - 1 = 0, [1,2]$

   (l) $(x+2)(x+1)^2 x(x-1)^3(x-2) = 0, [-1.5, 2.5], [-0.5, 2.4], [-0.5, 3], [= 3, -0.5]$

   (m) $(x+2)(x+1)x(x-1)^3(x-2) = 0, [-3, 2.5], [-2.5, 3], [-1.75, 1.5], [1.5, 1.75]$

   (n) $e^{-x} - x - 2 = 0, [1,2], [-2,-1]$

8. Use the Bisection method to find an approximation within 10^{-5} to the first positive value of x with $x = \tan x$.

9. Use the Bisection method to find an approximation within 10^{-5} to the first positive value of x with $x = 2\sin x$.

10. Use the Bisection method to find an approximation to $\sqrt{3}$ correct to within 10^{-5}

11. Use the Bisection method to find an approximation to $\sqrt[3]{25}$ correct to within 10^{-5}

12. The function $f(x) = \sin(\pi x)$ has zeros at every integer. Show that when $-1 < a < 0$ and $2 < b < 3$, the Bisection method converges to L where

$$L = \begin{cases} 0 & \text{if } a + b < 2 \\ 2 & \text{if } a + b > 2 \\ 1 & \text{if } a + b = 2 \end{cases}$$

13. A trough of length L has a cross-section in the shape of a semicircle with radius r. When filled with water to within a distance h of the top, the volume V of water is

$$V = L\left[0.5\pi r^2 - r^2 \sin^{-1}\left(\frac{h}{r}\right) - h\sqrt{r^2 - h^2}\right]$$

Suppose $L = 10m, r = 1m$ and $V = 12.4m^2$. Find the depth of water in the trough to within $0.01m$. Guess your interval.

14. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate ω.
$$\frac{d\theta}{dt} = \omega < 0$$
At the end of t seconds, the position of the object is given by
$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{\omega t} - e^{-\omega t}}{2} - \sin(\omega t)\right)$$
Suppose the particle has moved $1.7m$ in $1s$. Find the root within 10^{-5} the rate ω at which θ changes.

15. Show that $g(x) = \frac{1}{3}(x^2 - 1)$ has a unique fixed point on the interval $[-1, 1]$.

16. Show that $g(x) = \frac{1}{3^x}$ has at least one fixed point on the interval $[0, 1]$.

17. Show that $g(x) = x^3 + 4x^2 - 10$ has unique root in the interval $[1, 2]$. Which of the following functions, guarantees the convergence of fixed point iterations on the interval $[1, 2]$? Perform the first four iterations for each function f with starting value $x_0 = 1$.
    (a) $f_1(x) = x - x^3 - 4x^2 + 10$
    (b) $f_2(x) = \sqrt{\frac{10}{x} - 4x}$
    (c) $f_3(x) = \frac{1}{2}\sqrt{10 - x^3}$
    (d) $f_4(x) = \sqrt{\frac{10}{4 + x}}$
    (e) $f_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

18. Show that $g(x) = x^4 + 2x^2 - x - 3$ has unique root in the interval $[0, 1.5]$. Which of the following functions, guarantees the convergence of fixed point iterations on the interval $[0, 1.5]$? Perform the first four iterations for each function f with starting value $x_0 = 0$
    (a) $f_1(x) = (3 + x - 2x^2)^{1/4}$
    (b) $f_2(x) = \sqrt{\frac{x + 3 - x^4}{2}}$
    (c) $f_3(x) = \sqrt{\frac{x + 3}{2 + x^2}}$
    (d) $f_4(x) = \frac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1}$

19. Show that $g(x) = x^5 - 7$ has unique root in the interval $[0, 2]$. Which of the following functions, guarantees the convergence of fixed point iterations on the interval $[0, 2]$? Perform the first four iterations for each function f with starting value $x_0 = 1$

(a) $f_1(x) = x\left(1 + \dfrac{7 - x^5}{x^2}\right)^3$

(b) $f_2(x) = x - \dfrac{x^5 - 7}{x^2}$

(c) $f_3(x) = x - \dfrac{x^5 - 7}{5x^4}$

(d) $f_4(x) = x - \dfrac{x^5 - 7}{12}$

20. Show that $g(x) = x^3 - 27$ has unique root in the interval $[2, 3]$. Which of the following functions, guarantees the convergence of fixed point iterations on the interval $[2, 3]$? Perform the first four iterations for each function f with starting value $x_0 = 1$

(a) $f_1(x) = \dfrac{20x + \dfrac{21}{x^2}}{21}$

(b) $f_2(x) = x - \dfrac{x^3 - 21}{31x^2}$

(c) $f_3(x) = x - \dfrac{x^4 - 21x}{x^2 - 21}$

(d) $f_4(x) = \sqrt{\dfrac{21}{x}}$

21. Use the fixed point method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$ with $x_0 = 1$.

22. Use the fixed point method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$ with $x_0 = 1$.

23. For each of the following equations, determine an interval $[a, b]$ on which the fixed point method will converge. Estimate the number of iterations necessary to obtain approximations accurate within 10^{-5}

(a) $x = \dfrac{2 - e^x + x^2}{3}$

(b) $x = \dfrac{5}{x^2} + 2$

(c) $x = \sqrt{\dfrac{e^x}{3}}$

(d) $x = 5^{-x}$

(e) $x = 6^{-x}$

(f) $x = 0.5(\sin x + \cos x)$

(g) $2 + \sin x - x = 0$

(h) $x^3 - 2x - 5 = 0$

(i) $3x^2 - e^x = 0$

(j) $x - \cos x = 0$

24. Find all zeros (accurate within 10^{-4}) of $g(x) = x^2 + 10\cos x$ by using the fixed point method for appropriate iteration function f

25. Use the fixed point method to determine a solution accurate to within 10^{-4} for $x = \tan x$ for $x \in [4, 5]$

26. Use the fixed point method to determine a solution accurate to within 10^{-2} for $x + 2\sin(\pi x) = 0$ for $x \in [1, 2]$ and $x_0 = 1$

27. An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m})$$

where $g = 9.81m/s^2$ and k represents the coefficient of air resistance in N. Suppose $s_0 = 91.44m$ and $m = 0.113kg$ and $k = 0.1N$. Find within $0.01s$, the time it takes this mass to hit the ground.

28. Let $f(x) = x^2 - 6$ and $x_0 = -1$. Using the Newton-Raphson method, find x_2.

29. Let $f(x) = -x^3 - \cos x$ and $x_0 = 1$. Using the Newton-Raphson method, find x_2. What will happen if $x_0 = 0$?

30. Let $f(x) = x^2 - 6$ and $x_0 = 3$. Using the Newton-Raphson method, find x_3.

31. Let $f(x) = x^2 - 6$ and $x_0 = 3, x_1 = 2$. Using the Secant method, find x_3.

32. Let $f(x) = x^2 - 6$ and $a = 3, b = 2$. Using the Regula-Falsi method, find c.

33. Let $f(x) = -x^3 - \cos x$ and $x_0 = -1, x_1 = 0$. Using the Secant method, find x_3.

34. Let $f(x) = -x^3 - \cos x$ and $a = -1, b = 0$. Using the Regula-Falsi method, find c.

35. Use the Newton-Raphson method, the Secant method, the Bisection method and the Regula-Falsi method to find the solution accurately within 10^{-4} for the following problems. Take x_0 as any value on the specified interval for the Newton-Raphson method. Take any two values on the given interval for the Secant method and take $[a, b]$ as the given interval for the Regula-Falsi method

    (a) $x^3 - 2x^2 - 5 = 0, [1, 4]$

    (b) $x^3 + 3x^2 - 1 = 0, [-3, -2]$

    (c) $x - \cos x = 0, [0, \pi/2]$

    (d) $x - 0.8 - 0.2\sin x = 0, [0, \pi/2]$

    (e) $e^x + 2^{-x} + 2\cos x - 6 = 0, [1, 2]$

    (f) $\ln(x - 1) + \cos(x - 1) = 0, [1.3, 2]$

(g) $2x\cos(2x) + (x-2)^2 = 0, [2,3]$

(h) $2x\cos(2x) + (x-2)^2 = 0, [3,4]$

(i) $(x-2)^2 - \ln(x) = 0, [1,2]$

(j) $(x-2)^2 - \ln(x) = 0, [e,4]$

(k) $e^x - 3x^2 = 0, [0,1]$

(l) $e^x - 3x^2 = 0, [3,5]$

(m) $\sin x - e^{-x} = 0, [0,1]$

(n) $\sin x - e^{-x} = 0, [3,4]$

(o) $\sin x - e^{-x} = 0, [6,7]$

(p) $3xe^x = 0, [1,2]$

(q) $2x + 3\cos x - e^x = 0, [0,1]$

(r) $x^2 - 4x + 4 - \ln x = 0, [1,2]$

(s) $x^2 - 4x + 4 - \ln x = 0, [2,4]$

(t) $x + 1 - 2\sin(\pi x) = 0, [0, 0.5]$

(u) $x + 1 - 2\sin(\pi x) = 0, [0.5, 1]$

(v) $0.5\cos(2x) - x\sin(x) + 0.25x^2 + 0.5 = 0, [\pi/2, \pi]$

(w) $230x^4 + 18x^3 + 9x^2 - 221x - 9 = 0, [-1, 0]$

(x) $230x^4 + 18x^3 + 9x^2 - 221x - 9 = 0, [0, 1]$

(y) $\tan(\pi x) - 6 = 0, [0, 0.48]$

(z) $\frac{1}{x} - \tan x = 0, [0, \pi/2]$

36. Use the Newton-Raphson method, the Secant method, the Bisection method and the Regula-Falsi method to find the solution accurate within 10^{-4} for the following problems. Take x_0 as any value on the specified interval for the Newton-Raphson method. Take any two values on the given interval for the Secant method and take $[a, b]$ as the given interval for the Regula-Falsi method

(a) $\dfrac{1}{x} - 2^x = 0, [0,1]$

(b) $\dfrac{2}{x} + e^x + 2\cos x - 6 = 0, [1,3]$

(c) $\dfrac{x^3 + 4x^2 + 3x + 5}{2x^3 - 9x^2 + 18x - 2}, [0,4]$

(d) $x - \tan x = 0, [1,2]$

(e) $x^8 - 36x^7 + 546x^6 - 4536x^5 + 22449x^4 - 67284x^3 + 118124x^2 - 109584x + 40320 = 0, [5.5, 6.5]$

(f) $x^{20} - 1 = 0, [0, 10]$

(g) $x^{19} + 10^{-4} = 0, [-0.75, 0.5]$

(h) $\tan x - 30x = 0, [1, 1.57]$
(i) $x^2 - (1-x)^10, [0, 1]$
(j) $x^3 + 10^{-4}, [0.75, 0.5]$
(k) $xe^{-x^2} = 0, [-1, 4]$
(l) $2x\cos(2x) - (x-2)^2 = 0, [2, 3]$
(m) $2x\cos(2x) - (x-2)^2 = 0, [3, 4]$
(n) $e^x - 3x^2 = 0, [0, 1]$
(o) $e^x - 3x^2 = 0, [3, 5]$

37. Use the Newton-Raphson method and the modified Newton-Raphson method for multiple roots to find the solution accurately within 10^{-4} for the following problems. Take x_0 as any value on the specified interval.

(a) $x^2 - 2xe^{-x} + e^{-2x} = 0, [0, 1]$
(b) $\cos(x + \sqrt{2}) + x(x/2 + \sqrt{2}) = 0, [-2, -1]$
(c) $x^3 - 3x^2(2^{-x}) + 3x(4^{-x}) + 8^{-x} = 0, [0, 1]$
(d) $e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3 = 0, [-1, 0]$
(e) $1 - 4x\cos x + 2x^2 + \cos 2x = 0, [0, 1]$
(f) $x^2 + 6x^5 + 9x^4 - 2x^3 - 6x^2 + 1 = 0, [-3, -2]$
(g) $\sin 3x + 3e^{-2x}\sin x - 3e^{-x}\sin 2x - e^{-3x} = 0, [3, 4]$
(h) $e^{3x} - 27x^6 - 27x^4 e^x - 9x^2 e^{2x} = 0, [3, 5]$
(i) $e^{6x} + 1.441e^{2x} - 2.079e^{4x} - 0.330 = 0, [-1, 0]$
(j) $x = \tan x$ for $x \in [4, 5]$

38. The equation $4x^2 - e^x - e^{-x} = 0$ has two positive solutions. Use the Newton-Raphson method to approximate the solution to within 10^{-5} with the following values of $x_0 \in -10, -5, -3, -1, 0, 1, 3, 5, 10$

39. The equation $x^2 - 100\cos x = 0$ has two solutions. Use the Newton-Raphson method to approximate the solution to within 10^{-5} with the following values of $x_0 \in -100, -50, -25, 25, 50, 100$

40. Medical studies have established that a bungee jumper's chances of sustaining a significant vertebrae injury increases significantly if the free-fall velocity exceeds 36 m/s after 4 s of free fall. From a bungee-jumping study, it was identified that the fall velocity of a bungee-jumper is given by

$$v(t) = \sqrt{\frac{gm}{c}} \tanh\left(\sqrt{\frac{gc}{m}} t\right)$$

where m is the mass and c is the drag coefficient. A bungee-jumping company requested you to determine the mass at which this criterion is exceeded given a drag coefficient of $0.25kg/m$. Compute it using the Newton-Raphson method with a good guess of the initial value. Assume that $g = 9m/s^2$.

41. The saturation concentration of dissolved oxygen in freshwater can be calculated with the equation

$$\ln o_{sf} = -139.34411 + \frac{1.575701 \times 10^5}{T_a} - \frac{6.642308 \times 10^7}{T_a^2} + \frac{1.243800 \times 10^{10}}{T_a^3} - \frac{8.621946 \times 10^{11}}{T_a^4}$$

where o_{sf} is the saturation concentration of dissolved oxygen freshwater at 1 atm and T_a is the absolute temperature (K). If oxygen concentration $o_{sf} = 10$, using the Secant Method compute T_a in $°C$.

42. You buy a $10,00,000$ vehicle for $1,75,000$ per year for 7 years. Then determine the interest you are paying using the Newton-Raphson method with a good initial guess. From school mathematics, you might know the compound interest calculation formula. The annual payment of A is given by

$$A = P\frac{i(1+i)^n}{(1+i)^n - 1}$$

where P is worth, n is the number of years and i is the interest rate.

43. The accumulated value of a savings account based on regular periodic payments can be determined from the annuity due equation,

$$A = P\frac{(1+i)^n - 1}{i}$$

In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at Rs. 750,000 upon retirement in 20 years and can afford to put Rs. 1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly? Use the Newton-Raphson method with an appropriate initial guess.

44. Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = P\frac{1 - (1+i)^{-n}}{i}$$

known as an ordinary annuity equation. In this equation, A is the amount of the mortgage, P is the amount of each payment, and i is the interest rate per period for the n payment periods. Suppose that a 30-year home mortgage in the amount of Rs. 135,000 is needed and the borrower can afford house payments of at most Rs. 1000 per month. What is the maximal interest rate the borrower can afford to pay? Use the Newton-Raphson method with an appropriate initial guess.

45. A drug administered to a patient produces a concentration in the bloodstream given by $c(t) = Ate^{-t/3}$ milligrams per millilitre, t hours after A units have been injected. The maximum safe concentration is 1 mg/mL. Answer the following questions using the Newton-Raphson method

    (a) What amount should be injected to reach this maximum safe concentration, and when does this maximum occur?

    (b) An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/mL. Determine, to the nearest minute, when this second injection should be given.

    (c) Assume that the concentration from consecutive injections is additive and that 75% of the amount originally injected is administered in the second injection. When is it time for the third injection?

46. Find the root of the following function using the Newton-Raphson method

    (a) $f(x) = 3^{3x+1} - 7.5^{2x}$

    (b) $f(x) = 2^{x^2} - 3.7^{x+1}$

47. Player A will shut out (win by a score of 21-0) player B in a tennis game with probability

$$P = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21}$$

    where p denotes the probability A will win any specific rally (independent of the server). Find, the minimal value of p that will ensure that A will shut out B in at least half the matches they play.

48. In the design of all-terrain vehicles, it is necessary to consider the failure of the vehicle when attempting to negotiate two types of obstacles. One type of failure is called hang-up failure which occurs when the vehicle attempts to cross an obstacle that causes the bottom of the vehicle to touch the ground. The other type of failure is called nose-in failure and occurs when the vehicle descends into a ditch and its nose touches the ground. The maximum angle α that can be negotiated by a vehicle when β is the maximum angle at which hang-up failure does not occur satisfies the equation

$$A \sin\alpha\cos\alpha + B\sin^2\alpha - C\cos\alpha - E\sin\alpha = 0$$

    where $A = l\sin\beta_1, B = l\cos\beta_1, C = (h + 0.5D)\sin\beta_1 - 0.5D\tan\beta_1$ and $E = (h + 0.5D)\cos\beta_1 - 0.5D$ If $l = 89m, h = 49mmD = 55m$ and $\beta_1 = 11.5°$, find the angle α. What will be α if D is reduced to $D = 30m$? Use the Newton-Raphson method to compute these values.

49. Let $f(x) = e^{-x} - x - 1$. Use the Newton-Raphson method to compute the root of this equation with $x_0 = 1$. This equation has a multiple root at 0 with a multiplicity of 2.

Use the modified Newton-Raphson method for multiple roots and compute the roots. Compare that the Newton-Raphson method takes 15 iterations to get 10^{-5} accuracy whereas the modified Newton-Raphson method takes only 3 iterations.

50. Use fixed point iterations to locate the root of $f(x) = \sin(\sqrt{x}) - x$ with initial guess $x_0 = 0.5$. Find x_5.

51. Determine the highest real root of $f(x) = x^3 - 6x^2 + 11x - 6.1$ using Newton-Raphson method (three iterations) with $x_0 = 3.5$, the Secant method (three iterations) with $x_0 = 2.5, x_1 = 3.5$.

52. Determine the lowest positive root of $f(x) = 7\sin(x)e^{-x} - 1$ using Newton-Raphson method (three iterations) with $x_0 = 0.3$, the Secant method (three iterations) with $x_0 = 0.5, x_1 = 0.4$

53. Use the Newton-Raphson method to determine the root of the following functions for the given initial guesses (do five iterations at least)

    (a) $f(x) = x^5 - 16.05x^4 + 88.75x^3 - 192.0375x^2 + 116.35x + 31.6875$ using $x_0 = 0.5825$.

    (b) $f(x) = -2 + 6x - 4x^2 + 0.5x^3$ using $x_0 = 4.5$

    (c) $f(x) = -2 + 6x - 4x^2 + 0.5x^3$ using $x_0 = 4.43$

    (d) $f(x) = \tanh(x^2 - 9)$ using $x_0 = 3.2$

    (e) $f(x) = 0.0074x^4 - 0.284x^3 + 3.355x^2 - 12.183x + 5$ using $x_0 = 16.15$

    (f) $f(x) = e^{-0.5x}(4 - x) - 2$ using $x_0 = 2$

    (g) $f(x) = e^{-0.5x}(4 - x) - 2$ using $x_0 = 6$

    (h) $f(x) = e^{-0.5x}(4 - x) - 2$ using $x_0 = 8$

    (i) $f(x) = e^{0.5x} - 5 + 5x$ using $x_0 = 0.7$

    (j) $f(x) = x(x^2 - 1)(x - 3)e^{-0.5(x-1)^2}$ using $x_0 = 0.4$

    (k) $f(x) = e^{2x} - e^x - 2$ using $x_0 = 0.5$

    (l) $f(x) = 4\sin x - e^x$ using $x_0 = 0.25$

    (m) $f(x) = e^{-x} - \sin x$ using $x_0 = 0$

    (n) $f(x) = x - e^{-x^2}$ using $x_0 = 0$

    (o) $f(x) = x^3 - x - 2$ using $x_0 = 0$

    (p) $f(x) = e^x - \dfrac{1}{1 + x^2}$ using $x_0 = -7$

    (q) $f(x) = 4x^3 - 2x^2 + 3$ using $x_0 = -1$

    (r) $f(x) = x^3 - 2$ using $x_0 = 1$

    (s) $2\sin(\pi x) + x = 0$ for $x \in [1, 2]$ using $x_0 = 1$

54. Mechanical engineers, as well as most other engineers, use thermodynamics extensively in their work. The following polynomial can be used to relate the zero-pressure specific heat of dry air c_p in $kJ/(kgK)$ to temperature in K

$$c_p = 0.99403 + 1.671 \times 10^{-4}T + 9.7215 \times 10^{-8}T^2 - 9.5838 \times 10^{-11}T^3 + 1.9520 \times 10^{-14}T^4$$

Compute the temperature that corresponds to a specific heat of $1.1kJ/(kgK)$

55. In a chemical engineering process, water vapour (H_2O) is heated to sufficiently high temperatures that a significant portion of the water dissociates, or splits apart, to form oxygen (O_2) and hydrogen (H_2):

$$H_2O \leftrightarrows H_2 + O_2$$

If it is assumed that this is the only reaction involved, the mole fraction x of H_2O that dissociates can be represented by

$$K = \frac{x}{1-x}\sqrt{\frac{2p_t}{2+x}}$$

where K is the reaction's equilibrium constant and p_t is the total pressure of the mixture. If $p_t = 3$ atm and $K = 0.05$ determine the value of x that satisfies the above equation. Using the Newton-Raphson method with a good initial guess.

56. The Redlich-Kwong equation of state is given by

$$p = \frac{RT}{v-b} - \frac{a}{v(v+b)\sqrt{T}}$$

where R is the universal gas constant= 0.518 kJ/(kg K), T is the absolute temperature (K), p is the absolute pressure (kPa) and v is the volume of a kg of gas (m^3/kg). The parameters a and b are calculated by

$$a = 0.427\frac{R^2T_c^{2.5}}{p_c}, b = 0.866R\frac{T_c}{p_c}$$

where $p_c = 4600kPa, T_c = 191K$. As a chemical engineer, you are asked to determine the amount of methane fuel that can be held in a $3-m^3$ tank at a temperature of $-40°C$ with a pressure of 65000 KPa. Using the Newton-Raphson method calculate v and then find the mass of methane contained in the tank.

57. The volume of liquid V in a hollow horizontal cylinder of radius r and length L is related to the depth of the liquid h by

$$V = L\left[r^2\cos^{-1}\left(\frac{r-h}{r}\right) - (r-h)\sqrt{2rh-h^2}\right]$$

Determine h using the Newton-Raphson method given that $r = 2m, L = 5m^3, V = 8m^3$.

58. A catenary cable is one which is hung between two points, not in the same vertical line. It is subject to no loads other than its own weight. Thus, its weight acts as a uniform load per unit length along the cable w (N/m). Based on horizontal and vertical force balances, the following differential equation model of the cable can be derived:

$$\frac{d^2y}{dx^2} = \frac{w}{T_A}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

In the course of Engineering Mathematics, you solve this equation for the height of the cable y as a function of distance x:

$$y = \frac{T_a}{w}\cosh\left(\frac{w}{T_a}x\right) + y_0 - \frac{T_a}{w}$$

Use the Newton-Raphson method to calculate a value for the parameter T_A given values for the parameters $w = 10$ and $y_0 = 5$ such that the cable has a height of $y = 15$ at $x = 50$

59. An oscillating current in an electric circuit is described by $I = 9e^{-t}\sin(2\pi t)$ where t is in seconds. Determine all values of t such that $I = 3.5$ using the Newton-Raphson method.

60. A circuit with a resistor (R), inductor L and a capacitor (C) in parallel has an impedance (Z, in Ω) which could be derived by the Kirchhoff rules as follows

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

where ω is the angular frequency/ Find the ω that results in impedance of 100Ω using the Newton-Raphson method. Assume that $R = 225\Omega, C = 0.6 \times 10^{-6}F, L = 0.5H$. Real mechanical systems may involve the deflection of nonlinear springs. A block of mass m is released at a distance h above a nonlinear spring. The resistance force F of the spring is given by

$$F = -(k_1 d + k_2 d^{3/2})$$

Conservation of energy can be used to show that

$$\frac{2k_2 d^{5/2}}{5} + \frac{1}{2}k_1 d^2 - mgd - mgh = 0$$

Solve for d using the Newton-Raphson method for the following parameter values $k_1 = 40,000g/s^2, k_2 = 40g/(s^2m^5), m = 95g, g = 9.81m/s^2, h = 0.43m$.

61. Aerospace engineers sometimes compute the trajectories of projectiles such as rockets. A related problem deals with the trajectory of a thrown ball. The trajectory of a ball thrown by a right fielder is defined by the (x, y) coordinates and can be modeled as

$$y = (\tan\theta_0 x) - \frac{g}{2v_0^2\cos^2\theta_0}x^2 + y_0$$

Find the appropriate initial angle θ_0 if $v_0 = 30m/s$ and the distance of the catcher is $90m$. Note that the throw leaves the right fielder's hand at an elevation of $1.8m$ and the catcher receives it at $1m$.

62. You are designing a spherical tank to hold water for a small village in a developing country. The volume of liquid it can hold can be computed as

$$V = \pi h^2 \frac{3R - h}{3}$$

where V is the volume (m^3), h is the depth of water in a tank (m) and R is the tank radius (m). If $R = 3m$, what depth must the tank be filled so that it holds $30m^3$? Use three iterations of the Newton-Raphson method.

63. In control systems analysis, transfer functions are developed that mathematically relate the dynamics of a system's input to its output. A transfer function for a robotic positioning system is given by

$$G(s) = \frac{C(s)}{N(s)} = \frac{s^3 + 9s^2 + 26s + 24}{s^4 + 15s^3 + 77s^2 + 153s + 90}$$

where $G(s)$ is the system gain, $C(s)$ is the system output and $N(s)$ is the system input. s denotes the Laplace transform complex frequency. Find the roots of the numerator and denominator and factor them

$$G(s) = \frac{(s + a_1)(s + a_2)(s + a_3)}{(s + b_1)(s + b_2)(s + b_3)(s + b_4)}$$

64. The Manning equation can be written for a rectangular open channel as

$$Q = \frac{\sqrt{S}(BH)^{5/3}}{n(B + 2H)^{2/3}}$$

where Q is the flow in m^3/s, S is the slope (m/m), H is the deptn (m) and n is the manning roughness coefficient. Develop a fixed point iteration scheme to solve this equation for H when $Q = 5, S = 0.0002, B = 20$ and $n = 0.003$.

65. Give that $f(x) = -2x^6 - 1.5x^4 + 10x + 2$. Use a root location technique to determine the maximum of this function. Use bracketing methods on the interval $[0, 1]$ and the Newton-Raphson method with $x_0 = 1$. Use the Secant method with initial guess $x_0 = 0, x_1 = 1$.

66. For each of the following functions $g(x)$ determine a function f and an interval $[a, b]$ such $f : [a, b] \to [a, b]$ and $f(x) = x$.

    (a) $g(x) = x^3 - x - 1$

    (b) $g(x) = x - \tan x$

    (c) $g(x) = e^{-x} - \cos x$

67. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \sqrt{1 + x^2}$. Is it a self-map? What is the maximum of $|f'(x)|$? Does this function have a fixed point? Does the fixed point iteration converge? Justify each answer.

68. The equation $e^x - 4x^2 = 0$ has a root in the interval $[4, 5]$. Show that we cannot find this root using a fixed point iteration scheme with

$$x = \frac{1}{2}e^{x/2}$$

Can you find another alternative way to modify this so that you can locate the root?

69. The equation $e^x - 4x^2 = 0$ has a root in the interval $[0, 1]$. Show that we can find this root using the fixed point iteration with

$$x = \frac{1}{2}e^{x/2}$$

70. A civil engineer as a city planning commission member along with transportation engineers would like to find the population growth trends of a city and adjacent suburbs to design the road construction and expand the city limits. The population of the suburb area is declining with time according to

$$P_s(t) = P_{s,max}e^{-k_s t} + P_{s,min}$$

whereas the city population is growing

$$P_c(t) = \frac{P_{c,max}}{1 + [P_{c,max}/P_0 - 1]e^{-k_c t}}$$

where $P_{s,max}, P_{c,max}, P_{s,min}, P_0, k_c, k_u$ are parameters estimated from the previous population data. If $P_{s,max} = 8,00,000, k_s = 0.05/yr, k_c = 0.09/yr, P_{s,min} = 11,00,000, P_{c,max} = 32,00,000, P_0 = 1,00,000$ find the time when the suburb area population is 20% larger than the city population. Use the Newton-Raphson method with an appropriate initial guess.

71. A total charge Q is uniformly distributed around a ring-shaped conductor with radius a. A charge q is located at a distance x from the center of the ring. The force exerted on the charge by the ring is given by

$$F = \frac{1}{4\pi e_0}\frac{qQx}{(x^2 + a^2)^{3/2}}$$

where $e_0 = 8.9 \times 10^{-12} C^2/(Nm^2)$. Find the distance x when the force is $1.25N$ if $q = Q = 2 \times 10^{-5} C$ for a ring with a radius $a = 0.85m$. Use the Newton-Raphson method.

72. The activation function of a node in an artificial neural network is a function that calculates the output of the node based on its individual inputs and their weights. One of the historically important developments of neural networks is the sigmoid function. It is one of the popular activation functions in early neural networks because the gradient is strongest when the unit's output is near 0.5 and that allows efficient backpropagation training. The sigmoid function is given by

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

What should be the input signal for the sigmoid function so that $\sigma(z) = 0.5$? Use the Newton-Raphson method.

73. Determine all roots of the following polynomials using the Newton-Raphson method or Muller's method

   (a) $p(x) = x^4 - x^3 - x^2 + x - 1$
   (b) $p(x) = x^5 - 3.7x^4 - 7.4x^3 - 10.8x^2 - 10.8x - 6.8$
   (c) $p(x) = x^5 + 3.7x^4 + 7.4x^3 + 10.8x^2 + 10.8x - 6.8$
   (d) $p(x) = x^3 - x - 1$
   (e) $p(x) = x^5 - 3.7x^4 + 7.4x^3 - 10.8x^2 + 10.8x - 6.8$
   (f) $p(x) = x^3 - 3x^2 + 4$
   (g) $p(x) = x^3 + 2x - 1$
   (h) $p(x) = x^5 - 3x^3 + x^2 - 1$
   (i) $p(x) = x^3 + 3x - 1$
   (j) $p(x) = x^8 - 170x^6 + 7392x^4 - 39712x^2 + 51200$
   (k) $p(x) = x^4 - x^3 + x^2 - x + 1$
   (l) $p(x) = x^4 - 5.85x^2 + 6.3504$
   (m) $p(x) = x^7 - 28x^6 + 322x^5 - 1960x^4 + 6769x^3 - 13132x^2 + 13068x - 5040$
   (n) $p(x) = x^7 - 1$
   (o) $p(x) = x^4 - 7x^3 + 18x^2 - 20x + 8$
   (p) $p(x) = x^6 + 2x^5 + x^4 + 3x^3 + 5x^2 + x + 1$
   (q) $p(x) = 2x^4 + 24x^3 + 61x^2 - 16x + 1$
   (r) $p(x) = x^3 - 4x - 9$

74. Find the first three positive roots of the equation $x - \tan x = 0$ using Muller's method.

75. Find a positive root of $x^2 - 4x\sin(x) + (2\sin(x))^2 = 0$ accurate to two significant digits.

76. Prove that the Newton-Raphson method diverges for these functions no matter what starting real point is selected

   (a) $f(x) = x^2 + 1$
   (b) $f(x) = 7x^4 + 3x^2 + \pi$

77. What is x_2 if $x_0 = 1, x_1 = 2$ and $f(x_0) = 2, f(x_1) = 1.5$ in the Secant method?

78. Find a negative root of the equation $x^3 - 4x + 9 = 0$ using the Bisection method.

79. Find the root of the equation $cosx - xe^x$ using the Bisection method correct to four decimal places.

80. Find a real root of the equation $x^3 - 2x - 5 = 0$ using the Regula-Falsi method correct to three decimal places.

81. Find a real root of the equation $x = xe^x$ using the Regula-Falsi method correct to three decimal places.

82. Find the fourth root of 32 correct to three decimal places using the Regula-Falsi method.

83. Using the Steffensen Method for fixed point iterations, find a root of the equation

   (a) $2x - \log_{10} x = 7$
   (b) $x^3 + x^2 - 1 = 0$
   (c) $2 + \sin x - x = 0$
   (d) $x^3 - 2x - 5 = 0$
   (e) $3x^2 - e^x = 0$
   (f) $x - \cos x = 0$
   (g) $x = 0.5(2 - e^x + x^2)$
   (h) $x = 0.5(\sin x + \cos x)$
   (i) $x = \sqrt{e^x/3}$
   (j) $x = 5^{-x}$
   (k) $\cos(x - 1) = 0$

84. Find the root of the following functions (with appropriate starting values)

   (a) $f(x) = \frac{x}{2} + \frac{1}{x}$ with $x_0 = 1$
   (b) $f(x) = 10 - 2x + \sin x$
   (c) $f(x) = e^x + 1$
   (d) $x^2 - 4\sin x$
   (e) $x^3 + 6x^2 + 11x - 6$

85. Suppose x_r is a root of multiplicity m of f, where $f^{(m)}$ is continuous on an open interval containing x_r. Show that the fixed-point method has $g'(x_r) = 0$: where

$$g(x) = x - \frac{mf(x)}{f'(x)}$$

86. The iteration formula

$$x_{n+1} = x_n - \cos x_n \sin x_n + R\cos^2 x_n$$

where R is a positive constant was obtained by applying the Newton-Raphson method to some function f. What was $f(x)$?

87. Two of the four roots of $x^4 + 2x^3 - 7x^2 + 3$ are positive. Find them by using the Newton-Raphson method.

88. What happens if the Newton-Raphson method is applied to $f(x) = \tan^{-1} x$ with $x_0 = 2$? For what values of x_0, does the Newton-Raphson method converge?

89. Prove that the Newton-Raphson method to find $\frac{1}{\sqrt{R}}$ is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Rx_n}\right)$$

90. Prove that the Newton-Raphson method to find $R^{1/4}$ is

$$x_{n+1} = \frac{1}{k}\left((k-1)x_n + \frac{R}{x_n^{k-1}}\right)$$

91. Based on the order of convergence, write the following method in ascending order

    (a) Bisection Method

    (b) Regula-Falsi Method/Secant Method

    (c) Newton-Raphson Method

92. Show that $g(x) = 2^{-x}$ has a unique fixed point on $\left[\frac{1}{3}, 1\right]$. Estimate the number of iterations required to achieve 10^{-4} accuracy using the fixed point iteration method, in the computation of this unique fixed point.

93. How many iterations of the Bisection method are required to guarantee an approximation to a root of $x^5 - x^4 + x^3 - x^2 + x - 2 = 0$ in $[0,\ 2]$ that is within 10^{-4} of the true solution?

94. Consider the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = x^3 + 4x^2 - 10$. Show that it has a unique root in the interval $[1, 2]$ Which of the following $f_i(x)$ can provide a convergent fixed point iteration? Justify reason for each $f_i(x)$

    (a) $f_1(x) = x - (x^3 + 4x^2 - 10)$

    (b) $f_2(x) = 0.5\sqrt{10 - x^3}$

    (c) $f_3(x) = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

    If the sequence converges for the above $f_i(x)$, how many iterations are needed for it to converge for a given $\epsilon = 10^{-5}$?

95. Find the solution to the following problem using the Secant Method iteration.

$$f(x) = -5x^{(1/5)} - \cos(x) + 5$$

    with $x_0 = 0, x_1 = 1.5$

96. Find the solution to the following problem using the Regula-Falsi iteration.

$$f(x) = -8x^{(1/5)} - \cos(x^2) + 10$$

    on the interval $[1, 2.5]$

97. If $a_0 = 0.1, b_0 = 1.0$, how many steps of the Bisection method are needed to determine the root with an error of at most $\frac{1}{2} \times 10^{-8}$

98. Find the root or roots of the following using the Bisection and the Regula-Falsi methods: (choose the appropriate interval)

(a)
$$\frac{\ln(1+x)}{1-x^2} = 0$$

(b)
$$x^3 - 3x + 1$$

(c)
$$x^3 - 2\sin x$$

99. A circular metal shaft is being used to transmit power. It is known that at a certain critical angular velocity ω, any jarring of the shaft during rotation will cause the shaft to deform or buckle. This is a dangerous situation because the shaft might shatter under increased centrifugal force. To find this critical velocity ω, we must first compute a number x that satisfies the equation
$$\tan x + \tanh x = 0$$
This number is then used in the formula to obtain ω. Solve for x where $x > 0$.

100. When using the Regula-Falsi method with $x_0 = 1, x_1 = 2$, it computes the real root of the equation $e^x + 2^x + 2\cos(x) - 6 = 0$. Compute x_6

101. The iteration formula
$$x_{n+1} = x_n - (\cos x_n)(\sin x_n) + R\cos^2 x_n$$
where $R > 0$ constant was obtained by applying Newton-Raphson method to some function $f(x)$. What was $f(x)$.

102. Each of the following functions has $\sqrt[3]{R}, R > 0$ as a root. Find the necessary restrictions for x_0 in Newton-Raphson method

(a) $x^3 - R$
(b) $\frac{1}{x^3} - \frac{1}{R}$
(c) $x^2 - \frac{R}{x}$
(d) $x - \frac{R}{x^2}$
(e) $1 - \frac{R}{x^3}$
(f) $\frac{1}{x} - \frac{x^2}{R}$
(g) $\frac{1}{x^2} - \frac{x}{R}$
(h) $1 - \frac{x^3}{R}$

103. Newton's method to find $\sqrt{AB}$ is given by

$$\sqrt{AB} \approx \frac{A+B}{4} + \frac{AB}{A+B}$$

Show that when $x_0 = A$ or B, then the above formula is obtained for x_2 whereas when $x_0 = \frac{A+B}{2}$, the above approximation is obtained for x_1.

104. Consider the following modification of Newton-Rapshon method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}$$

Derive the rate and order of convergence for this method.

105. Consider the following modification of Newton-Rapshon method

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

where m denotes the multiplicity of the root. Prove that this sequence converges quadratically.

106. Consider the

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If this sequence converges then the limit point is a solution. Explain why or why not.