**INDIAN INSTITUTE OF TECHNOLOGY TIRUPATI**
**Department of Mathematics and Statistics**
**MA633L - Numerical Analysis**
**100 Problems on Numerical Integration**

Note: Usual Notations are used. Questions may have some typos or grammatical mistakes as proofreading is not done for this.

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1. Using Trapezoidal rule, Simpson's 1/3rd rule, Simpson's 3/8 rule, integrate the following function. Find the error, if analytical solution is known.

   (a) $\int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)dx$

   (b) $\int_0^{0.8} (1 - x - 4x^3 + 2x^5)dx$

   (c) $\int_0^{4} (1 - e^{-x})dx$

   (d) $\int_0^{\pi/2} (8 + 4\cos x)dx$

   (e) $\int_{-2}^{4} (1 - x - 4x^3 + 2x^5)dx$

2. Using Composite Trapezoidal rule, Composite Simpson's 1/3rd rule, Composite Simpson's 3/8 rule, integrate the following function with $n = 2, 3, 4, 5$. Find the error, if analytical solution is known.

   (a) $\int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)dx$

   (b) $\int_0^{0.8} (1 - x - 4x^3 + 2x^5)dx$

   (c) $\int_0^{4} (1 - e^{-x})dx$

   (d) $\int_0^{\pi/2} (8 + 4\cos x)dx$

3. Using Midpoint rule and Two-point Newton-Cotes open formula, integrate the following functions. Find the error, if analytical solution is known.

   (a) $\int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)dx$

   (b) $\int_0^{0.8} (1 - x - 4x^3 + 2x^5)dx$

   (c) $\int_0^{4} (1 - e^{-x})dx$

   (d) $\int_0^{\pi/2} (8 + 4\cos x)dx$

4. Using two-point and three-point Gaussian quadrature rules, integrate following functions. Find the error, if analytical solution is known.

   (a) $\int_1^2 \left(x + \frac{1}{x}\right)^2 dx$

   (b) $\int_0^8 (-0.055x^4 + 0.86x^3 - 4.2x^2 + 6.3x + 2)dx$

   (c) $\int_0^3 xe^{2x}dx$

   (d) $\frac{1}{\sqrt{2\pi}} \int_0^a e^{-x^2} dx$

   (e) $\int_0^{0.8} (0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)dx$

5. The force on a sailboat mist can be represented by the following function

$$F = \int_0^H 200 \left(\frac{z}{5+z}\right) e^{-2z/H} dz$$

   where $z$ is the elevation above the deck and $H$ is the height of the mast. Compute the $F$ for $H = 30$ using two-point and three-point Gaussian quadrature rules. The line of action can also be determined by integration:

$$d = \frac{\int_0^H 200 \left(\frac{z^2}{5+z}\right) e^{-2z/H} dz}{\int_0^H 200 \left(\frac{z}{5+z}\right) e^{-2z/H} dz}$$

   Use the composite trapezoidal rule to compute $F$ and $d$. Use the composite Simpson's 1/3 rule to compute $F$ and $d$.

6. The root mean square current can be computed as

$$I_{RMS} = \sqrt{\frac{1}{T} \int_0^T i^2(t) dt}$$

   For $T = 1$, suppose that $i(t)$ is defined as

$$i(t) = \begin{cases} 8e^{-t/T} \sin\left(2\pi \frac{t}{T}\right) & \text{if } 0 \le t \le T/2 \\ 0 & \text{if } T/2 \le t \le t \end{cases}$$

   Evaluate the $I_{RMS}$ using two-point and three-point Gaussian quadrature rules.

7. The heat required, $\Delta H$(cal), to induce temperature change, $\Delta T(°C)$ of a material can be computed as

$$\Delta H = mC_p(T)\Delta T$$

   where $m$ is the mass $(g)$, $C_p(T)$ is the heat capacity [cal/$g°C$]. The heat capacity increases with temperature $T(°C)$ according to

$$C_p(T) = 0.132 + 1.56 \times 10^{-4}T + 2.64 \times 10^{-7}T^2$$

   . Compute the heat for $m = 1$kg, for 0 to 300$°C$ using two-point and three-point Gaussian quadrature rules.

8. The amount of mass transported through a pipe over a period of time can be computed as

$$M = \int_{t_1}^{t_2} Q(t)c(t)dt$$

where $M$ is the mass $(mg), t_1$ is the initial time $(min), t_2$ is the final time $(min), Q(t)$ is the flow rate $(m^3/min)$, and c(t) is the concentration $(mg/m^3)$. The following functional representations define the temporal variations in flow and concentration:

$$Q(t) = 9 + 5\cos^2(0.4t)$$

$$c(t) = 5e^{-0.5t} + 2e^{0.15t}$$

Determine the mass transported between $t_1 = 2$ and $t_2 = 8$ min using two-point and three-point Gaussian quadrature rules.

9. If $\int_{-\pi}^{\pi} f(x)\sin x dx = A_0[f(\pi) - f(-\pi)] + A_1\left[f\left(\frac{\pi}{2}\right) - f\left(\frac{-\pi}{2}\right)\right]$, then find $A_0$ and $A_1$.

10. If $\int_{-1}^{1} f(x)\sin\left(\frac{\pi}{2}x\right) dx = A_0 f(-1) + A_1 f(0) + A_2 f(1)$, then find $A_0, A_1$ and $A_2$.

11. If $-\int_{0}^{1} f(x)\ln(x)dx = A_0 f(0) + A_1 f(0.5) + A_2 f(1)$, then find $A_0, A_1$ and $A_2$.

12. If $\int_{0}^{1} f(x)dx = A_0 f(1/3) + A_1 f(2/3)$, then find $A_0$ and $A_1$.

13. If $\int_{0}^{\infty} e^{-x} f(x)dx = A_0 f(0) + A_1 f(2)$, then find $A_0$ and $A_1$.

14. If $-\int_{0}^{1} f(x)\ln(x)dx = A_0 f(0) + A_1 f(1/3) + A_2 f(2/3) + A_3 f(1)$, then find $A_0, A_1, A_2$ and $A_3$.

15. Using the technique taught for Gaussian quadrature, compute the weights $A_0, A_1$ and nodes $x_0, x_1$ for the following integral. $\int_{0}^{\infty} e^{-x} f(x)dx = A_0 f(x_0) + A_1 f(x_1)$.

16. Using the technique taught for Gaussian quadrature, compute the weights $A_0, A_1$ and nodes $x_0, x_1$ for the following integral. $-\int_{0}^{1} \ln x f(x)dx = A_0 f(x_0) + A_1 f(x_1)$.

17. Using the technique taught for Gaussian quadrature, compute the weights $A_0, A_1$ and nodes $x_0, x_1$ for the following integral. $\int_{0}^{1} x f(x)dx = A_0 f(x_0) + A_1 f(x_1)$.

18. Using the technique taught for Gaussian quadrature, compute the weights $A_0, A_1$ and nodes $x_0, x_1$ for the following integral. $\int_{-\pi/2}^{\pi/2} f(x)\sin x dx = A_0 f(x_0) + A_1 f(x_1)$.

19. Using the technique taught for Gaussian quadrature (Very Hard), compute the weights $A_0, A_1, A_2, A_3$ and nodes $x_0, x_1, x_2, x_3$ for the following integral. $\int_{-1}^{1} f(x)dx = A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)$.

20. Using the technique taught for Gaussian quadrature (Very Hard), compute the weights $A_0, A_1, A_2, A_3$ and nodes $x_0, x_1, x_2, x_3$ for the following integral. $\int_{-\pi}^{\pi} f(x)\sin x dx = A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2) + A_3 f(x_3)$.

21. Using the technique taught for Gaussian quadrature compute the weights $A_0, A_1, A_2$ and nodes $x_0, x_1, x_2$ for the following integral. $\int_{-\infty}^{\infty} e^{-x^2} f(x)dx = A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)$.

22. Using the technique taught for Gaussian quadrature compute the weights $A_0, A_1$ and nodes $x_0, x_1$ for the following integral. $\int_{-1}^{1} x f(x)dx = A_0 f(x_0) + A_1 f(x_1)$.

23. Using the technique taught for Gaussian quadrature compute the weights $A_0, A_1$ and nodes $x_0, x_1$ for the following integral. $\int_{-1}^{1} \sqrt{1-x^2} f(x)dx = A_0 f(x_0) + A_1 f(x_1)$.

24. Using the technique taught for Gaussian quadrature compute the weights $A_0, A_1$ and nodes $x_0, x_1$ for the following integral. $\int_{-1}^{1} f(x)dx = A_0 f(x_0) + A_1 f(x_1) + \frac{1}{6} f(-1) + \frac{1}{6} f(1)$.

25. Using the technique taught for Gaussian quadrature compute the weights $A_0, A_1, A_2$ and nodes $x_1, x_2$ for the following integral. $\int_{0}^{\infty} e^{-x} f(x)dx = A_0 f(0) + A_1 f(x_1) + A_2 f(x_2)$.

26. Using the technique taught for Gaussian quadrature compute the weights $A_0, A_1, A_2$ and nodes $x_1, x_2$ for the following integral. $-\int_{0}^{\infty} \ln x f(x)dx = A_0 f(0) + A_1 f(x_1) + A_2 f(x_2)$.

27. Using the technique taught for Gaussian quadrature compute the weights $A_0, A_1, A_2$ and node $x_1$ for the following integral. $\int_{0}^{\infty} e^{-x} f(x)dx = A_0 f(0) + A_1 f(x_1) + A_2 f(4)$.

28. A quadrature formula on the interval $[-1, 1]$ uses the quadrature formula as follows

$$\int_{-1}^{1} f(x)dx = A_0 f(-\alpha) + A_1 f(\alpha)$$

where $0 < \alpha \leq 1$. The formula is required to be exact whenever $f$ is a polynomial of degree 1. Show that $A_0 = A_1 = 1$, independent of the value of $\alpha$. Show also that there is one particular value of $\alpha$ for which the formula is exact for all polynomials of degree 2. Find this $\alpha$ and show that, for this value, the formula is exact for all polynomials of degree 3.

29. Write down the error in the approximation of $\int_0^1 x^4 dx$ and $\int_0^1 x^5 dx$ by the trapezoid and Simpson's 1/3rd rule. Find the value of the constant $C$ for which the trapezoid rule gives the correct calculation of $\int_0^1 (x^5 - Cx^4)dx$ and show that the trapezoid rule gives a more accurate result than the Simpson's 1/3rd rule when $15/14 < C < 85/74$.

30. Verify that the following formula is exact for polynomials of degree $\leq 4$

$$\int_0^1 f(x)dx \approx \frac{1}{90}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right]$$

(a) From the above formula, compute $\int_a^b f(x)dx$

(b) Calculate $\ln 2$ approximately, by applying this formula on the integral $\displaystyle\int_0^1 \frac{dt}{1+t}$

31. Find the formula $\displaystyle\int_0^1 xf(x)dx = A_0 f(0) + A_1 f(1)$ that is exact for all functions of the form $f(x) = ae^x + b\cos(\pi x/2)$

32. Suppose that the current through a resistor is described by the function

$$i(t) = (60 - t)^2 + (60 - t)\sin(\sqrt{t})$$

and the resistance is a function of the current:

$$R = 10i + 2i^{2/3}$$

Compute the average voltage over $t = 0$ to 60 using the composite Simpson's 1/3 rule.

33. If a capacitor initially holds no charge, the voltage across it as a function of time can be computed as

$$V(t) = \frac{1}{C}\int_0^t i(t)dt$$

For $C = 10^{-5}$, the data is given as follows

| $t, s$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 | 1.2 |
|---|---|---|---|---|---|---|---|
| $i, 10^{-3}A$ | 0.2 | 0.3683 | 0.3819 | 0.2282 | 0.0486 | 0.0082 | 0.1441 |

34. The work done on an object is equal to the force times the distance moved in the direction of the force. The velocity of an object in the direction of a force is given by

$$v(t) = \begin{cases} 4t & \text{if } 0 \le t \le 5 \\ 20 + (5-t)^2 & \text{if } 5 \le t \le 15 \end{cases}$$

where $v$ is in $m/s$. Determine the work if a constant force of $200N$ is applied for all $t$.

35. A rod subject to an axial load will be deformed. The area under the curve from zero stress out to the point of rupture is called the modulus of toughness of the material. It provides a measure of the energy per unit volume required to cause the material to rupture. As such, it is representative of the material's ability to withstand an impact load. Use composite numerical integration to compute the modulus of toughness for the stress-strain for the given data

| $e$ | 0.02 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 |
|---|---|---|---|---|---|---|
| $s$ | 40 | 37.5 | 43.0 | 52.0 | 60.0 | 55.50 |

36. If the velocity distribution of a fluid flowing through a pipe is known the flow rate Q (i.e., the volume of water passing through the pipe per unit time) can be computed by

$$Q = \int v dA$$

where $v$ is the velocity, and $A$ is the pipe's cross-sectional area. (To grasp the meaning of this relationship physically, recall the close connection between summation and integration.) For a circular pipe, $A = \pi r^2$ and $dA = 2\pi r dr$. Therefore,

$$Q = \int_0^r v(2\pi r) dr$$

where $r$ is the radial distance measured outward from the center of the pipe. If the velocity distribution is given by

$$v = 2\left(1 - \frac{r}{r_0}\right)^{1/6}$$

where $r_0$ is the total radius $3cm$, compute $Q$ using the composite trapezoidal rule

37. Using the following data, calculate the work done by stretching a spring that has a spring constant of $k = 300N/m$ to $x = 0.35m$. To do this, first fit the data with a polynomial and then integrate the polynomial numerically to compute the work:

| $F, 10^3N$ | 0 | 0.01 | 0.028 | 0.046 | 0.063 | 0.082 | 0.11 | 0.13 |
|---|---|---|---|---|---|---|---|---|
| $x, m$ | 0 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 |

38. Evaluate the vertical distance traveled by a rocket if the vertical velocity is given by

$$v(t) = \begin{cases} 11t & \text{if } 0 \le t \le 10 \\ 1100 - 5t & \text{if } 10 \le t \le 20 \\ 50t + 2(t-20)^2 & \text{if } 20 \le t \le 30 \end{cases}$$

39. The upward velocity of a rocket can be computed by the following formula:

$$v = u \ln\left(\frac{m_0}{m_0 - qt}\right) - gt$$

where $v$ is the upward velocity, $u$ is the velocity at which fuel is expelled relative to the rocket, $m_0$ is the initial mass of the rocket at time $t = 0$, $q$ is the fuel consumption rate, and $g$ is the downward acceleration of gravity. If $u = 1850m/s, m_0 = 160,000kg$, and $q = 2500kg/s$, determine how high the rocket will fly in $30s$.

40. The normal distribution is defined as

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Using composite trapezoid rule with $n = 4$ integrate this function from $x = -1$ to 1.

41. Use two-point and three-point Gaussian quadrature rule to find

$$\int_0^2 \frac{e^x \sin x}{1+x^2} dx$$

42. Recall that the velocity of the free-falling bungee jumper can be computed analytically as

$$v(t) = \sqrt{\frac{gm}{cd}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$

where $v(t)$ is the velocity $(m/s), t$ is the time $(s), g = 9.81m/s2, m$ is the mass $(kg), c_d$ is the drag coefficient $(kg/m)$. Use composite Simpson 1/3 rule to compute how far the jumper travels during the first 8 seconds of free fall given $m = 80kg$ and $c_d = 0.2kg/m$.

43. As specified in the following table, the earth's density varies as a function of the distance from its center $(r = 0)$:

| $r, km$ | 0 | 1100 | 1500 | 2450 | 3400 | 3630 | 4500 | 5380 | 6060 | 6280 | 6380 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\rho, g/cm^3$ | 13 | 12.4 | 12 | 11.2 | 9.7 | 5.7 | 5.2 | 4.7 | 3.6 | 3.4 | 3 |

First fit the data with a polynomial and then integrate the polynomial numerically using composite trapezoidal and composite Simpson's 1/3 rule. Use numerical integration to estimate the earth's mass (in metric tonnes) and average density (in $g/cm^3$).

44. As specified in the following table,a manufactured spherical particle has a density that varies as a function of the distance from its center $(r = 0)$:

| $r, km$ | 0 | 0.12 | 0.24 | 0.36 | 0.49 | 0.62 | 0.79 | 0.86 | 0.93 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\rho, g/cm^3$ | 6 | 5.81 | 5.14 | 4.29 | 3.39 | 2.7 | 2.19 | 2.1 | 2.04 | 2 |

Use numerical integration to estimate the particle's mass (in g) and average density (in $g/cm^3$).

45. Determine the distance traveled from the following velocity data

| $t$ | 1 | 2 | 3.25 | 4.5 | 6 | 7 | 8 | 8.5 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | 5 | 6 | 5.5 | 7 | 8.5 | 8 | 6 | 7 | 7 | 5 |

Use the trapezoidal rule. In addition, determine the average velocity

46. Water exerts pressure on the upstream face of a dam. The pressure can be characterized by

$$p(z) = \rho g(D - z)$$

where $p(z)$ is the pressure in pascals (or $N/m^2$) exerted at an elevation $z$ meters above the reservoir bottom; $\rho$ is the density of water, which for this problem is assumed to be a constant $103kg/m^3$; $g$ is the acceleration due to gravity $(9.81m/s^2)$; and $D$ is the elevation (in m) of the water surface above the reservoir bottom. According to this equation pressure increases linearly with depth. Omitting atmospheric pressure (because it works against both sides of the dam face and essentially cancels out), the total force ft can be determined by multiplying pressure times the area of the dam face. Because both pressure and area vary with elevation, the total force is obtained by evaluating

$$f_t = \int_0^D \rho g w(z)(D-z)dz$$

where $w(z)$ is the width of the dam face ($m$) at elevation $z$. The line of action can also be obtained by evaluating

$$d = \frac{\int_0^D \rho z g w(z)(D-z)dz}{\int_0^D \rho g w(z)(D-z)dz}$$

Using Simpson's rule to compute ft and d.

47. A wind force distributed against the side of a skyscraper is measured as

| Height $l, m$ | 0 | 30 | 60 | 90 | 120 | 150 | 180 | 210 | 240 |
|---|---|---|---|---|---|---|---|---|---|
| Force $F(l), N/m$ | 0 | 340 | 1200 | 1550 | 2700 | 3100 | 3200 | 3500 | 3750 |

Compute the net force and the line of action due to this distributed wind.

48. An 11-m beam is subjected to a load, and the shear force follows the equation

$$V(x) = 5 + 0.25x^2$$

where $V$ is the shear force, and $x$ is length in distance along the beam. We know that $V = dM/dx$, and $M$ is the bending moment. Integration yields the relationship

$$M = M_0 + \int_0^x V dx$$

If $M_0$ is zero and $x = 11$, calculate M using composite trapezoidal rule, and composite Simpson's rules. [use 1-m increments].

49. The total mass of a variable density rod is given by

$$m = \int_0^L \rho(x) A_c(x) dx$$

where $m$ is the mass, $\rho(x)$ is the density, $A_c(x)$ is the cross-sectional area, $x$ is the distance along the rod and $L$ is the the total length of the rod. The following data have been measured for a 20-m length rod. Determine the mass in grams to the best possible accuracy.

| $x, m$ | 0 | 4 | 6 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|---|---|
| $\rho, g/cm^3$ | 4.00 | 3.95 | 3.89 | 3.80 | 3.60 | 3.41 | 3.30 |
| $A_c, cm^2$ | 100 | 103 | 106 | 110 | 120 | 133 | 150 |

50. A transportation engineering study requires that you determine the number of cars that pass through an intersection traveling during morning rush hour. You stand at the side of the road and count the number of cars that pass every 4 minutes at several times as tabulated below. Use the best numerical method to determine (a) the total number of cars that pass between 7:30 and 9:15, and (b) the rate of cars going through the intersection per minute. (Hint: Be careful with units.)

| Time (hr) | 7:30 | 7:45 | 8:00 | 8:15 | 8:45 | 9:15 |
|---|---|---|---|---|---|---|
| Rate (cars per 4 min) | 18 | 23 | 14 | 24 | 20 | 9 |

51. Integration provides a means to compute how much mass enters or leaves a reactor over a specified time period, as in

$$M = \int_{t_1}^{t_2} Qcdt$$

where $t_1$ and $t_2$ are the initial and final times, respectively. Use numerical integration to evaluate this equation

| $t, min$ | 0 | 10 | 20 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|
| $Q, m^3/min$ | 4.00 | 4.8 | 5.2 | 5.0 | 4.6 | 4.3 | 4.3 | 5.0 |
| $c, mg/m^3$ | 10 | 35 | 55 | 52 | 40 | 37 | 32 | 34 |

52. The cross-sectional area of a channel can be computed as

$$A_c = \int_0^B H(y)dy$$

where $B$ is the total channel width $(m)$, $H$ is the depth $(m)$, and $y$ is the distance from the bank $(m)$. In a similar fashion, the average flow $Q(m^3/s)$ can be computed as

$$Q = \int_0^B U(y)H(y)dy$$

where $U$ is the water velocity (m/s). Use these relationships and a numerical method to determine $A_c$ and $Q$ for the following data

| $y, m$ | 0 | 2 | 4 | 5 | 6 | 9 |
|---|---|---|---|---|---|---|
| $H, m$ | 0.5 | 1.3 | 1.25 | 1.8 | 1 | 0.25 |
| $U, m/s$ | 0.03 | 0.06 | 0.05 | 0.13 | 0.11 | 0.02 |

53. The average concentration of a substance $\overline{c}(g/m^3)$ in a lake where the area $A_s(m^2)$ varies with depth $z(m)$ can be computed by integration:

$$\overline{c} = \frac{\int_0^Z c(z)A_s(z)dz}{\int_0^Z c(z)A_s(z)dz}$$

where $Z$ is the total depth $m$. Determine the average concentration based on the following data

| $z, m$ | 0 | 4 | 8 | 12 | 16 |
|---|---|---|---|---|---|
| $A, 10^6 m^2$ | 9.8175 | 5.1051 | 1.9635 | 0.3927 | 0.0 |
| $c, g/m^3$ | 10.2 | 8.5 | 7.4 | 5.2 | 4.1 |

54. The calculation of work is an important equation in science and engineering. The general formula is given by

$$\text{Work} = \text{Force} \times \text{Distance}$$

Although, this formula seems to be simple, when apply this to realistic problem, we obtain a complicated function. For example, when the force varies during the course of calculation which results in

$$W = \int_{x_0}^{x_n} F(x)dx$$

When the angle between the force and direction of the movement also varies as a function of position, we obtain that

$$W = \int_{x_0}^{x_n} F(x)\cos(\theta(x))dx$$

The following table shows the angle as a function of position $x$.

| $x$ | 0 | 1 | 2.8 | 3.9 | 3.8 | 3.2 | 1.3 |
|---|---|---|---|---|---|---|---|
| $\theta(x), deg$ | 0 | 30 | 60 | 90 | 120 | 150 | 180 |

Evaluate $W$ using the Simpson's 1/3 rule and composite Simpson's 1/3 rules for the following function

$$F(x) = 1.6x - 0.045x^2$$

55. Compute the work done if

$$F(x) = 1.6x - 0.045x^2$$

and

$$\theta(x) = -0.00055x^3 + 0.0123x^2 + 0.13x$$

where $x_0 = 0$ and $x_n = 30$. Here angle is in radians.

56. There are two Newton-Cotes formulae for $n = 2$ and $[a, b] = [0, 1]$, namely

$$\int_0^1 f(x)dx \approx af(0) + bf(1/2) + cf(1)$$

$$\int_0^1 f(x)dx \approx \alpha f(1/4) + \beta f(1/2) + \gamma f(3/4)$$

which is better?

57. Is there a formula of the form

$$\int_0^1 f(x)dx \approx \alpha[f(x_0) + f(x_1)]$$

that correctly integrates all quadratic polynomials?

58. Determine appropriate values of $A_0, A_1$ and $x_0, x_1$ so that the quadrature formula

$$\int_{-1}^{1} x^2 f(x)dx = A_0 f(x_0) + A_1 f(x_1)$$

will be correct when $f$ is a polynomial of degree 3

59. Determine appropriate values of $A_0, A_1, A_2$ and $x_0, x_1, x_2$ so that the quadrature formula

$$\int_{-1}^{1} x^2 f(x)dx = A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)$$

will be correct when $f$ is a polynomial of degree 5

60. Determine appropriate values of $c$ and $x_0, x_1, x_2$ so that the quadrature formula

$$\int_{-1}^{1} f(x)dx = c[f(x_0) + f(x_1) + f(x_2)]$$

will be correct when $f$ is a polynomial of degree 2

61. If the quadrature formula

$$\int_{-1}^{1} f(x)dx = f(\alpha) + f(-\alpha)$$

is to be exact for all quadratic polynomials, what value of $\alpha$ should be used?

62. If the quadrature formula

$$\int_{-1}^{1} f(x)dx = f(\alpha) + f(-\alpha)$$

is to be exact for all cubic polynomials, what value of $\alpha$ should be used?

63. If the quadrature formula

$$\int_{-1}^{1} f(x)dx = f(\alpha) + f(-\alpha)$$

is to be exact for all polynomials of degree 4, what value of $\alpha$ should be used?

64. If the quadrature formula

$$\int_{0}^{2} f(x)dx = f(\alpha) + f(2-\alpha)$$

is to be exact for all cubic polynomials, what value of $\alpha$ should be used?

65. Determine the coefficient $A_0, A_1$ and $A_2$ that make the formula

$$\int_{0}^{2} f(x)dx = A_0 f(0) + A_1 f(1) + A_2 f(2)$$

exact for all polynomials of degree 3.

66. Determine the coefficient $A_0, A_1$ and $A_2$ that make the formula

$$\int_{-1}^{1} f(x)dx = A_0 f(-1/2) + A_1 f(0) + A_2 f(1/2)$$

exact for all polynomials of degree 2.

67. Determine appropriate values of $A_0, A_1$ and $x_0, x_1$ so that the quadrature formula

$$\int_{-1}^{1} x^4 f(x)dx = A_0 f(x_0) + A_1 f(x_1)$$

.

68. Determine appropriate values of $A_0, A_1$ and $x_0, x_1$ so that the quadrature formula

$$\int_{0}^{1} x^4 f(x)dx = A_0 f(x_0) + A_1 f(x_1)$$

.

69. Use Trapezoidal rule, Simpson's 1/3 rule, Simpson 3/8 rule, midpoint rule and two-point Newton-Cotes rule to find approximation to

$$\int_0^1 x \sin(x) dx$$

70. Use Trapezoidal rule, Simpson's 1/3 rule, Simpson 3/8 rule, midpoint rule and two-point Newton-Cotes rule to find approximation to

$$\int_0^1 f(x) dx$$

where

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1/2 \\ 1 - x & \text{if } 1/2 \le t \le 1 \end{cases}$$

71. Use Trapezoidal rule, Simpson's 1/3 rule, Simpson 3/8 rule, midpoint rule and two-point Newton-Cotes rule to find approximation to

$$\int_0^1 (1 - x^2)^{3/2} dx$$

72. Use Trapezoidal rule, Simpson's 1/3 rule, Simpson 3/8 rule, midpoint rule and two-point Newton-Cotes rule to find approximation to

$$\int_0^1 \frac{1}{x^2} \sin\left(\frac{1}{x^2}\right) dx$$

73. Use Trapezoidal rule, Simpson's 1/3 rule, Simpson 3/8 rule, midpoint rule and two-point Newton-Cotes rule to find approximation to

$$\int_0^1 \frac{1}{x^2} \sin^2(x) dx$$

74. Use Trapezoidal rule, Simpson's 1/3 rule, Simpson 3/8 rule, midpoint rule and two-point Newton-Cotes rule to find approximation to

$$\int_0^{0.1} x^{1/3} dx$$

75. Use Trapezoidal rule, Simpson's 1/3 rule, Simpson 3/8 rule, midpoint rule and two-point Newton-Cotes rule to find approximation to

$$\int_0^{\pi/4} \tan x dx$$

76. Using three-point Gaussian quadrature find an approximation to the integral

$$\int_1^3 \frac{\sin(x)}{x} dx$$

77. Using three-point Gaussian quadrature find an approximation to the integral

$$\int_0^1 \frac{\sin(\pi x)}{[x(1-x)]^{3/2}} dx$$

78. **Lobatto rule**. It is a rule to similar to Gaussian quadrature to integrate

$$\int_{-1}^1 f(x)dx$$

except that it includes the endpoints $-1$ and 1. That is

$$\int_{-1}^1 f(x)dx = A_0 f(-1) + A_1 f(x_1) + \cdots + A_{n-1} f(x_{n-1}) + A_n f(1)$$

Derive the following Lobatto rule

$$\int_{-1}^1 f(x)dx = A_0 f(-1) + A_1 f(x_1) + A_2 f(1)$$

that is exact for all cubic polynomials.

79. Determine the number of points required in the composite trapezoid rule that gives the value of

$$\int_0^1 e^{-x^2} dx$$

correct to six digits after the decimal point.

80. The determination of the condensation of a pure vapor on the outside of a cooled horizontal tube requires that the mean heat-transfer coefficient $Q$ be computed. This coefficient requires, along with other parameters, the evaluation of the integral

$$\int_0^{\pi} (\sin x)^{1/3} dx$$

Find the value of this integral using Simpson's rule with $n = 5, 10, 15, 20$.

81. Use the trapezoidal rule to numerically integrate $f(x) = \frac{1}{1+x}$ from 0 to 2 and compute the relative error.

82. Use the trapezoidal rule to numerically integrate $f(x) = 0.2 + 25x + 3x^2$ from 0 to 2 and compute the relative error.

83. Use the composite trapezoidal rule with $n = 2$ to numerically integrate $f(x) = 0.2 + 25x + 3x^2$ from 0 to 2 and compute the relative error.

84. Use the composite trapezoidal rule with $n = 10$ to numerically integrate $f(x) = e^{x^2}$ from 1 to 3. How large should we choose $n$ so that the trapezoidal rule to the same integral is certainly within 0.5 of the right value?

85. Use the composite trapezoidal rule with $n = 8$ to numerically integrate $f(x) = \sqrt{x}$ from 0 to 1. How large should we choose $n$ so that the trapezoidal rule to the same integral is certainly within $10^{-7}$ of the right value?

86. Use the composite trapezoidal rule with $n = 2$ to numerically integrate $f(x) = \frac{1}{1+x}$ from 0 to 1. How large should we choose $n$ so that the trapezoidal rule to the same integral is certainly within $10^{-7}$ of the right value?

87. Use the composite trapezoidal rule with $n = 8$ to numerically integrate $f(x) = \sin(\pi x)$ from 0 to 1. How large should we choose $n$ so that the trapezoidal rule to the same integral is certainly within $10^{-7}$ of the right value?

88. Use the composite trapezoidal rule with $n = 8$ to numerically integrate $f(x) = \sin^2(\pi x)$ from 0 to 1. How large should we choose $n$ so that the trapezoidal rule to the same integral is certainly within $10^{-7}$ of the right value?

89. Use the Simpson's 1/3 rule to numerically integrate $f(x) = 0.2 + 25x + 3x^2 + 8x^3$ from 0 to 2 and compute the relative error.

90. Use the composite Simpson's 1/3 rule with $n = 4$ to numerically integrate $f(x) = 0.2 + 25x + 3x^2 + 8x^3$ from 0 to 2 and compute the relative error.

91. Use the composite Simpson's 1/3 rule with $n = 10$ to numerically integrate $f(x) = e^{x^2}$ from 1 to 3. How large should we choose $n$ so that the composite Simpson's 1/3 rule to the same integral is certainly within 0.5 of the right value?

92. Use the composite Simpson's 1/3 rule with $n = 3$ to numerically integrate

$$f(x) = \frac{x^3 - x}{1 + x^4}$$

from 0 to 6.

93. Evaluate the double integral

$$\int_{-2}^{2} \int_{0}^{4} (x^2 - 3y^2 + xy^3) dx dy$$

analytically, and using two-point and three-point Gaussian quadrature rules.

94. Evaluate the double integral

$$\int_{-2}^{2}\int_{0}^{4}(x^2 - 3y^2 + xy^3)dxdy$$

analytically, and using trapezoidal rule, composite trapezoidal rule with $n = 2, 4$ and Simpson's 1/3 rule, Simpson 3/8 rule, Boole's rule. Compute the error for each case.

95. Evaluate the double integral analytically, and using trapezoidal rule, composite trapezoidal rule with $n = 2, 4$ and Simpson's 1/3 rule, Simpson 3/8 rule, Boole's rule. Compute the error for each case.

   (a) $\int_{0}^{\pi}\int_{0}^{\pi}\frac{\sin y}{y}dydx$

   (b) $\int_{0}^{2}\int_{0}^{4}\frac{xe^{2y}}{4-y}dydx$

   (c) $\int_{0}^{8}\int_{\sqrt{3}}^{2}\frac{1}{y^4+1}dydx$

96. Evaluate the double integral analytically, and using trapezoidal rule, composite trapezoidal rule with $n = 2, 4$ and Simpson's 1/3 rule, Simpson 3/8 rule, Boole's rule. Compute the error for each case.

   (a) $\int_{1}^{4}\int_{0}^{4}\left(\frac{x}{2}+\sqrt{y}\right)dxdy$

   (b) $\int_{-1}^{2}\int_{1}^{2}x\ln y dydx$

   (c) $\int_{0}^{1}\int_{0}^{1}\frac{y}{1+xy}dxdy$

   (d) $\int_{-1}^{1}\int_{0}^{\pi/2}x\sin(\sqrt{y})dydx$

   (e) $\int_{0}^{2}\int_{0}^{1}\frac{x}{1+xy}dxdy$

97. Find $\iint_R \sqrt{xy-y^2}dxdy$, where $R$ is the triangle with vertices $(0,0), (10,1), (1,1)$ using trapezoidal rule, composite trapezoidal rule with $n = 2, 4$ and Simpson's 1/3 rule, Simpson 3/8 rule, Boole's rule.

98. Evaluate the triple integral

$$\int_{-4}^{4}\int_{0}^{6}\int_{-1}^{3}(x^3 - 2yz)dxdydz$$

analytically, and using trapezoidal rule, composite trapezoidal rule with $n = 2, 4$ and Simpson's 1/3 rule, Simpson 3/8 rule, Boole's rule. Compute the error for each case

99. Evaluate the triple integral

$$\int_0^2 \int_0^{2\pi} \int_0^{\pi} r^2 \sin(\phi) d\phi d\theta dr$$

analytically, and using trapezoidal rule, composite trapezoidal rule with $n = 2, 4$ and Simpson's 1/3 rule, Simpson 3/8 rule, Boole's rule. Compute the error for each case

100. Find the volume and center of the mass of a diamond, the intersection of the unit sphere with the cone given in the cylindrical coordinates $z = \sqrt{3}r$ analytically, and using trapezoidal rule, composite trapezoidal rule with $n = 2, 4$ and Simpson's 1/3 rule, Simpson 3/8 rule, Boole's rule. Compute the error for each case