MA633L-Numerical Analysis

Lecture 10 : Numerical Interpolation-Newton's Divided Difference Interpolation

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Newton's Divided Difference Interpolation

Newton's Divided Difference Interpolation

In general, the above analysis can be generalized to nth order polynomial to $n+1$ data points. The n-th order polynomial is given by (usually, it is called as Newton form of interpolation polynomial)

$$P_n(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \cdots + b_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

Newton's Divided Difference Interpolation

Before, we discuss b_i's, let us introduce another symbol called $\prod$ to denote product of numbers. Let $c_1, c_2, \cdots c_n$ be n real numbers, then their sum is denoted by symbol $\sum$. Similarly, the product of these n real numbers are denoted by the symbol $\prod$. That is,

$$\prod_{i=1}^{n} c_i = c_1 c_2 \cdots c_n$$

$$\prod_{i=0}^{n} (x - x_i) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

Newton's Divided Difference Interpolation

Using this new notation, the above polynomial P_n can be written as

$$P_n(x) = b_0 + \sum_{k=1}^{n} b_k \prod_{i=0}^{k-1} (x - x_i)$$

The following equations will be obtained for a_k's after a few manipulation.

$$\begin{aligned}
b_0 &= f[x_0] = f(x_0) && (1)\\
b_1 &= f[x_0, x_1] && (2)\\
b_2 &= f[x_0, x_1, x_2] && (3)\\
&\vdots && (4)\\
b_k &= f[x_0, x_1, x_2, \cdots x_k] && (5)\\
b_n &= f[x_0, x_1, x_2, \cdots x_n] && (6)
\end{aligned}$$

Newton's Divided Difference Interpolation

The following table will be used to get the finite divided difference formula

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
x_0	$f[x_0]$			
		$f[x_0, x_1] = \frac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1, x_2] = \frac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0, x_1, x_2, x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
		$f[x_2, x_3] = \frac{f[x_3]-f[x_2]}{x_3-x_2}$		$f[x_1, x_2, x_3, x_4] = \frac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$
x_3	$f[x_3]$		$f[x_2, x_3, x_4] = \frac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3, x_4] = \frac{f[x_4]-f[x_3]}{x_4-x_3}$		$f[x_2, x_3, x_4, x_5] = \frac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$
x_4	$f[x_4]$		$f[x_3, x_4, x_5] = \frac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4, x_5] = \frac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

Newton's Divided Difference Interpolation

The bracketed functions denotes the finite divided differences. The first finite difference is represented generally as

$$f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i}.$$

The second finite divided difference represents the difference of two first divided differences, and is expressed generally as

$$f[x_i, x_j, x_k] = \frac{f[x_j, x_k] - f[x_i, x_j]}{x_k - x_i}.$$

Newton's Divided Difference Interpolation

The third finite difference represents the difference of two second divided differences and is generally expressed

$$f[x_i, x_j, x_k, x_l] = \frac{f[x_j, x_k, x_l] - f[x_i, x_j, x_k]}{x_l - x_i}.$$

Similarly, the nth finite divided difference is generally expressed as

$$f[x_0, x_1, x_2, \cdots x_n] = \frac{f[x_1, x_2, \cdots x_n] - f[x_0, x_1 \cdots , x_{n-1}]}{x_n - x_0}.$$

These differences can be used to obtain b_k's. The quantity $f[x_0, x_1, x_2, \cdots x_k]$ is called the **divided difference of order** k

Newton's Divided Difference Interpolation

Theorem 1 (Invariance Theorem)
The divided difference $f[x_0, x_1, x_2, \cdots, x_k]$ is invariant under all permutations of arguments $x_0, x_1, x_2, \cdots, x_k$.

The another way to represent the divided difference is that,

$$f[x_0, x_1, x_2, \cdots, x_k] = \frac{f(x_k) - \sum_{i=0}^{k-1} f[x_0, x_1, x_2, \cdots, x_i] \prod_{j=0}^{i-1} (x_k - x_j)}{\prod_{j=0}^{k-1} (x_k - x_j)}$$

Newton's Divided Difference Interpolation

The general Newton form of the interpolation polynomial is given by

$$P_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \cdots, x_k] \prod_{i=0}^{k-1} (x - x_i) \tag{7}$$

Newton's Divided Difference Interpolation

Example 2

By means of Newton's divided difference and interpolation formula, obtain the third-order Newton's interpolating polynomial to estimate $\ln 2.5$. Find the error ε_t

x	1.0	4	6	5
$f(x)$	0.0	1.386294	1.791759	1.609438

Newton's Divided Difference Interpolation

Solution: The divided difference table is given by

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
1.0	0.0			
		0.4620981		
4.0	1.386294		−0.05187311	
		0.2027326		0.007865529
6.0	1.791759		−0.0204100	
		0.1823216		
5.0	1.609438			

Newton's Divided Difference Interpolation

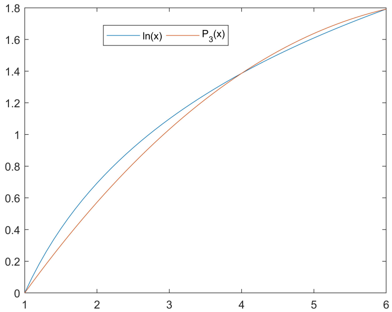

Figure 1: Newton Cubic Interpolation

Newton's Divided Difference Interpolation

The interpolating cubic polynomials is

$$\begin{aligned}
P_3(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&\quad + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
&= 0 + 0.4620981(x - 1.0) - 0.05187311(x - 1.0)(x - 4) \\
&\quad + 0.007865529(x - 1.0)(x - 4.0)(x - 6)
\end{aligned}$$

Now,

$$P_3(2.5) = 0.871802$$

and the error estimate is

$$\varepsilon_t = \frac{|0.871802 - 0.916290|}{|0.916290|} = 0.0485$$

Newton's Divided Difference Interpolation

Example 3

By means of Newton's divided difference formula and Newton's interpolating polynomial, compute $P_1(9.2), P_2(9.2)$ and $P_3(9.2)$. Estimate the error with $\ln(9.2)$.

x	8.0	9.0	9.5	11.0
$f(x)$	2.079442	2.197225	2.251292	2.397895

Newton's Divided Difference Interpolation

Solution:

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference
8.0	2.079442			
		0.117783		
9.0	2.197225		-0.006433	
		0.108134		0.000411
9.5	2.251292		-0.005200	
		0.097735		
11.0	2.397895			

Newton's Divided Difference Interpolation

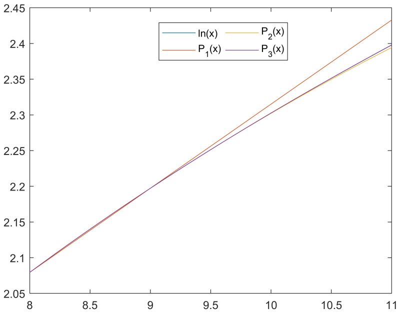

Figure 2: Newton Cubic Interpolation

Newton's Divided Difference Interpolation

The interpolating polynomials are

$$
\begin{aligned}
P_1(x) &= f[x_0] + f[x_0, x_1](x - x_0) \\
&= 2.079442 + 0.117783(x - 8.0) \\
P_2(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&= 2.079442 + 0.117783(x - 8.0) - 0.006433(x - 8.0)(x - 9.0) \\
P_3(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&\quad + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
&= 2.079442 + 0.117783(x - 8.0) - 0.006433(x - 8.0)(x - 9.0) \\
&\quad + 0.000411(x - 8.0)(x - 9.0)(x - 9.5)
\end{aligned}
$$

Newton's Divided Difference Interpolation

Now,

$$P_1(9.2) = 2.220782, \epsilon_t = 0.0007111$$

$$P_2(9.2) = 2.219238, \epsilon_t = 1.555 \times 10^{-5}$$

$$P_3(9.2) = 2.219208, \epsilon_t = 2.0349 \times 10^{-6}$$

Newton's Divided Difference Interpolation

Example 4

By means of Newton's divided difference and interpolation formula, obtain the fourth divided difference table and a polynomial of degree 4 for the following table.

x	1.0	1.3	1.6	1.9	2.2
$f(x)$	0.7651977	0.6200860	0.4554022	0.2818186	0.1103623

Calculate $P_4(1.75)$.

Newton's Divided Difference Interpolation

Solution:

x	$f(x)$	First Divided Difference	Second Divided Difference	Third Divided Difference	Fourth Divided Difference
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

Newton's Divided Difference Interpolation

$$
\begin{aligned}
P_4(x) = f[x_0] &+ f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&+ f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\
&+ f[x_0, x_1, x_2, x_3, x_4](x - x_0)(x - x_1)(x - x_2)(x - x_3) \\
&= 0.7651977 - 0.4837057(x - 1.0) - 0.1087339(x - 1.0)(x - 1.3) \\
&+ 0.0658784(x - 1.0)(x - 1.3)(x - 1.6) \\
&+ 0.0018251(x - 1.0)(x - 1.3)(x - 1.6)(x - 1.9)
\end{aligned}
$$

Now,

$$P_4(1.75) = 0.369042$$

Exercise

Exercise 1: Hard

1. Let $x_0 < x_1 < x_2 < \cdots < x_n$ be real, $f : \mathbb{R} \to \mathbb{R}$ be any function. If the unique polynomial

$$P_n(x) = \sum_{k=0}^{n} a_k x^k$$

passes through the data points $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$, prove that

$$a_n = \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i)}$$

Exercise

Exercise 2: Easy

2. Prove that

$$f[x_0, x_1, \cdots, x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{\prod\limits_{\substack{i=0 \\ i \neq k}}^{n} (x_k - x_i)}$$

3. Prove that $b_n = a_n$
4. Prove the Invariance Theorem

Newton's Forward Difference Formula

Newton’s Forward Difference Formula

Newton’s forward difference formula is a particular case of Newton’s divided difference formula. Since the observations in experiments can occur at random position or random time, the Newton’s divided difference formula is valid for arbitrarily spaced nodes. However, a few measurements are taken at regular interval of time. For example, weather forecasting. Let h be the difference between x_{i+1} and x_i.

$$h = x_{i+1} - x_i.$$

Newton's Forward Difference Formula

That is,

$$x_0 = x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \cdots, x_n = x_0 + nh.$$

The corresponding $f(x)$ values are given by

$$f_0, f_1, f_2, \cdots, f_n.$$

Let us define the first forward difference of f at x_i by

$$\Delta f_i = f_{i+1} - f_i,$$

and the second forward difference of f at x_i by

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i,$$

and the k-th forward difference of f at x_i by

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i,$$

Newton's Forward Difference Formula

The following equations will be obtained.

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{1}{h}(f(x_1) - f(x_0)) = \frac{1}{h}\Delta f_0$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left(\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right) = \frac{1}{2h^2}\Delta^2 f_0$$

$$f[x_0, x_1, \cdots, x_k] = \frac{1}{k!h^k}\Delta^k f_0 (Prove!)$$

Newton's Forward Difference Formula

The following table will be used to get the Newton's forward difference formula

x	$f(x)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
x_0	f_0				
		Δf_0			
x_1	f_1		$\Delta^2 f_0$		
		Δf_1		$\Delta^3 f_0$	
x_2	f_2		$\Delta^2 f_1$		$\Delta^4 f_0$
		Δf_2		$\Delta^3 f_1$	
x_3	f_3		$\Delta^2 f_2$		
		Δf_3			
x_4	f_4				

Newton's Forward Difference Formula

If $x = x_0 + rh$, then $x - x_0 = rh, x - x_1 = (r-1)h$ and the Newton's forward difference interpolation formula is given by

$$P_n(x) = \sum_{k=0}^{n} \binom{r}{k} \Delta^k f_0 \quad \text{where} \quad r = \frac{x - x_0}{h} \tag{8}$$

or

$$P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \cdots \frac{r(r-1)\cdots+(r-n+1)}{n!}\Delta^n f_0 \tag{9}$$

Newton's Forward Difference Formula

Example 5

By means of Newton's forward difference formula, compute $\cosh(0.56)$ and ε_t

x	0.5	0.6	0.7	0.8
$f(x)$	1.127626	1.185465	1.255169	1.337435

Newton's Forward Difference Formula

Solution:

x	$f(x)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0.5	1.127626			
		0.057839		
0.6	1.185465		0.011865	
		0.06704		0.000697
0.7	1.255169		0.012562	
		0.082266		
0.8	1.337435			

Newton's Forward Difference Formula

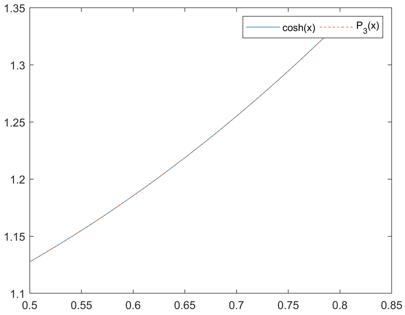

Figure 3: Newton Forward Difference

Newton's Forward Difference Formula

The interpolating polynomial is

$$P_3(x) = f(x_0) + r\Delta f_0 + + \frac{r(r-1)}{2!}\Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 f_0$$

Here,

$$h = 0.1, x = 0.56, x_0 = 0.5 \implies r = \frac{0.56 - 0.50}{0.1} = 0.6$$

$$\begin{aligned} P_3(0.56) &= 1.127626 + 0.6 \times 0.057839 + \frac{0.6(-0.4)}{2} \times 0.011865 \\ &+ \frac{0.6 \times -0.4 \times -1.4}{6} \times 0.000697 \\ &= 1.160944 \end{aligned}$$

$$\cosh(0.56) = 1.160941, \epsilon_t = 2.58 \times 10^{-6}$$

Newton's Forward Difference Formula

Example 6

The following information is gathered from a book of interplanetary coordinates obtained by astronomers by various means, the x coordinate of Mars in a heliocentric coordinate system at a specified date. We expect to have smooth function, but you can observe the higher differences due to error in data.

t	1250.5	1260.5	1270.5	1280.5	1290.5	1300.5
$x = f(t)$	1.39140	1.37696	1.34783	1.30456	1.24767	1.17862

t	1310.5	1320.5	1330.5	1340.5
$x = f(t)$	1.09776	1.00636	0.90553	0.79642

Newton's Forward Difference Formula

As per Newton's forward difference interpolation formula is given by

$$P_n(x) = \sum_{k=0}^{n} \binom{r}{k} \Delta^k f_0 \tag{10}$$

Suppose the error in the ith function value is given by ϵ_i. Then the above table has the numbers $\Delta^k(f_i + \epsilon_i)$ and they differ from $\Delta^k f_i$ by the amount of error $\Delta^k \epsilon_i$ which is bounded by

$$|\Delta^k \epsilon_i| \leq \epsilon 2^s$$

Newton's Forward Difference Formula

t	$f(t)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$	$\Delta^5 f_i$	$\Delta^6 f_i$
1250.5	1.39140						
		−1444					
1260.5	1.37696		−1469				
		−2913		55			
1270.5	1.34783		−1414		−3		
		−4327		52		97	
1280.5	1.30456		−1362		94		−302
		−5689		146		−205	
1290.5	1.24767		−1216		−111		408
		−6905		35		203	
1300.5	1.17862		−1181		92		−311
		−8086		127		−108	
1310.5	1.09776		−1054		−16		128
		−9140		111		20	
1320.5	1.00636		−943		4		
		−10083		115			
1330.5	0.90553		−828				
		−10911					
1340.5	0.79642						

Note, we multiplied the values by 10^5 or 10^6 to view the error in decimal locations. In the above table, if the values are accurately rounded values, then $\epsilon \leq 0.000005$, therefore, errors $\Delta^4 f_i$ should be not bigger than 8 units in the last place. But it is oscillatory.

Newton's Forward Difference Formula

If we subtract the average value of 10 on $\Delta^4 f_i$, then we get $-13, 84, -121, 82, -26, -6$, that means an error of 20 units in the last place is committed on the -121 entry, that is the error is in 1.24767. Instead of that, make change by -20 units in the last of that entry, that is 1.24767 becomes 1.24787, then the following table is produced. Still the $\Delta^5 f_i$ oscillates, but they are smaller in size as the maximum error is of 16 units which should give satisfactory results.

Newton’s Forward Difference Formula

t	$f(t)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$	$\Delta^5 f_i$	$\Delta^6 f_i$
1250.5	1.39140						
		−1444					
1260.5	1.37696		−1469				
		−2913		55			
1270.5	1.34783		−1414		17		
		−4327		72		−3	
1280.5	1.30456		−1342		14		−2
		−5669		86		−5	
1290.5	1.24787		−1256		9		8
		−6925		95		3	
1300.5	1.17862		−1161		12		−11
		−8086		107		−8	
1310.5	1.09776		−1054		4		8
		−9140		111		0	
1320.5	1.00636		−943		4		
		−10083		115			
1330.5	0.90553		−828				
		−10911					
1340.5	0.79642						

Newton’s Backward Difference Formula

Newton’s Backward Difference Formula

Newton’s forward difference formula is also a particular case of Newton’s divided difference formula. Instead of forward sloping difference, here we use backward sloping differences. Notations of h and f_i and x_i are same here. However, we introduce another notation for backward difference.

Newton's Backward Difference Formula

Definition 7

Given a sequence $\{f_n\}_{n=0}^{\infty}$, define the first backward difference of f at x_i by

$$\nabla f_i = f_i - f_{i-1},$$

and the second backward difference of f at x_i by

$$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1},$$

and the k-th backward difference of f at x_i by

$$\nabla^k f_i = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1},$$

Newton's Backward Difference Formula

However, in order to obtain the Newton's backward difference formula, the interpolating nodes are reordered from that last to first. The following equations will be obtained.

$$f[x_n] = f(x_n)$$

$$f[x_n, x_{n-1}] = \frac{1}{h}(f(x_n) - f(x_{n-1})) = \frac{1}{h}\nabla f_n$$

$$f[x_n, x_{n-1}, x_{n-2}] = \frac{1}{2h}\left(\frac{\nabla f(x_n) - \nabla f(x_{n-1})}{h}\right) = \frac{1}{2h^2}\nabla^2 f_n$$

$$f[x_n, x_{n-1}, \cdots, x_{n-k}] = \frac{1}{k!h^k}\nabla^k f_n (Prove!)$$

Newton’s Backward Difference Formula

The following table will be used to get the Newton’s backward difference formula

x	$f(x)$	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$	$\nabla^4 f_i$
x_0	f_0				
		∇f_1			
x_1	f_1		$\nabla^2 f_1$		
		∇f_2		$\nabla^3 f_3$	
x_2	f_2		$\nabla^2 f_2$		$\nabla^4 f_4$
		∇f_3		$\nabla^3 f_4$	
x_3	f_3		$\nabla^2 f_4$		
		∇f_4			
x_4	f_4				

Newton's Backward Difference Formula

If $x = x_n + rh$, then $x_n - x = -rh, x_{n-1} - x = -(r+1)h$ and the Newton's backward difference interpolation formula is given by

$$P_n(x) = \sum_{k=0}^{n} \binom{r+k-1}{k} \nabla^k f_n \quad r = \frac{x - x_n}{h} \tag{11}$$

or

$$P_n(x) = f_n + r\nabla f_n + \frac{r(r+1)}{2!}\nabla^2 f_n + \cdots \frac{r(r+1)\cdots(r+n-1)}{n!}\nabla^n f_n \tag{12}$$

Newton's Backward Difference Formula

Example 8

By means of Newton's backward difference formula, compute $\cosh(0.74)$ and ϵ_t

x	0.5	0.6	0.7	0.8
$f(x)$	1.127626	1.185465	1.255169	1.337435

Newton's Backward Difference Formula

Solution:

x	$f(x)$	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$
0.5	1.127626			
		0.057839		
0.6	1.185465		0.011865	
		0.06704		0.000697
0.7	1.255169		0.012562	
		0.082266		
0.8	1.337435			

Newton's Backward Difference Formula

The interpolating polynomial is

$$P_3(x) = f_3 + r\nabla f_3 + \frac{r(r+1)}{2!}\nabla^2 f_3 + \frac{r(r+1)(r+2)}{3!}\nabla^2 f_3$$

Here,

$$h = 0.1, x = 0.74, x_3 = 0.8 \implies r = \frac{0.74 - 0.8}{0.1} = -0.6$$

$$\begin{aligned} P_3(0.74) &= 1.337435 - 0.6 \times 0.082266 + \frac{-0.6(0.4)}{2} \times 0.012562 \\ &+ \frac{-0.6 \times 0.4 \times 1.4}{6} \times 0.000697 \\ &= 1.286528 \end{aligned}$$

$$\cosh(0.74) = 1.286524, \epsilon_t = 3.11 \times 10^{-6}$$

Newton’s Backward Difference Formula

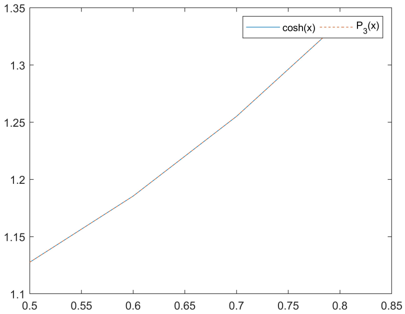

Figure 4: Newton’s Backward Difference

Thanks

Doubts and Suggestions

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MA633L-Numerical Analysis

Lecture 10 : Numerical Interpolation-Newton's Divided Difference Interpolation

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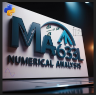

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