MA633L-Numerical Analysis

Lecture 11 : Lagrange Interpolation and Error in Polynomial Interpolation

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Lagrange Interpolation

Lagrange Interpolation

Although, the interpolating polynomial P_n associated with (x_i, f_i) is unique as per the existence and uniqueness theorem, it is possible to express the polynomial in different forms or to obtain it from different algorithms. Lagrange interpolation is one such method which will express P_n in the following form

$$P_n(x) = \sum_{i=0}^{n} \ell_i(x) f(x_i), \tag{1}$$

where

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right), \quad 0 \leq i \leq n.$$

Lagrange Interpolation

These ℓ_i's have the following property.

$$\ell_i(x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

These ℓ_i's are special polynomials called as cardinal polynomials, which has the following property. From (1), it is obvious that

$$P_n(x_j) = \sum_{i=0}^{n} \ell_i(x) f(x_i) = \ell_j(x_j) f(x_j) = f(x_j)$$

Lagrange Interpolation

From the $\ell_i(x)$ definition, we can observe that it is a product of n linear factors and it is a polynomial of degree n.

$$\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) = \left(\frac{x - x_0}{x_i - x_0} \right) \left(\frac{x - x_1}{x_i - x_1} \right) \cdots \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right) \left(\frac{x - x_{i+1}}{x_i - x_{i+1}} \right) \cdots \left(\frac{x - x_n}{x_i - x_n} \right)$$

Lagrange Interpolation: Linear

Lagrange Interpolation: Linear Interpolation

For linear interpolation, we obtain a straight line passing through (x_0, f_0) and (x_1, f_1). Therefore, the linear Lagrange interpolating polynomial P_1 is given by

$$P_1(x) = \ell_0 f_0 + \ell_1 f_1,$$

where

$$\ell_0 = \frac{x - x_1}{x_0 - x_1}, \ell_1 = \frac{x - x_0}{x_1 - x_0}$$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \tag{2}$$

Lagrange Interpolation: Quadratic

Lagrange Interpolation: Quadratic Interpolation

For Quadratic interpolation, we obtain a parabola passing through $(x_0, f_0), (x_1, f_1)$ and (x_2, f_2). Therefore, the quadratic Lagrange interpolating polynomial P_2 is given by

$$P_2(x) = \ell_0 f_0 + \ell_1 f_1 + \ell_2 f_2,$$

where

$$\ell_0 = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right)$$

$$\ell_1 = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)$$

$$\ell_2 = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)$$

Lagrange Interpolation: Quadratic Interpolation

$$P_2(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) f_0 + \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) f_1 \qquad (3)$$

$$+ \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) f_2 \qquad (4)$$

Lagrange Interpolation: Examples

Lagrange Interpolation

Example 1

By means of Lagrange's linear interpolation formula, find the value of e^1 using the information that

x	0	2
e^x	1	7.3891

Find the error ε_t.

Lagrange Interpolation

Solution: The Lagrange linear interpolation polynomial is given by

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$
$$P_1(x) = \frac{x - 2}{0 - 2} + \frac{x - 0}{2 - 0} 7.3891$$
$$P_1(x) = \frac{x}{2}(7.3891 - 1) + 1$$
$$P_1(x) = 1 + 3.19455x$$

You can observe that the polynomial obtained from Newton's linear interpolation and Lagrange linear interpolation are same. Now,

$$P_1(1) = 1 + 3.19455 = 4.19455$$

$$\varepsilon_t = \frac{|2.71828 - 4.19455|}{|2.71828|} = 0.5430$$

Lagrange Interpolation

Example 2

By means of Lagrange's quadratic interpolation formula, find the values of e^1 using the information that

x	0	2	4
e^x	1	7.3891	54.5981

Lagrange Interpolation

Find the error ε_t.

Solution: The Lagrange linear interpolation polynomial is given by

$$\ell_0 = \left(\frac{x-2}{0-2}\right)\left(\frac{x-4}{0-4}\right) = \frac{x^2-6x+8}{8}$$

$$\ell_1 = \left(\frac{x-0}{2-0}\right)\left(\frac{x-4}{2-4}\right) = \frac{x^2-4x}{-4}$$

$$\ell_2 = \left(\frac{x-0}{4-0}\right)\left(\frac{x-2}{4-2}\right) = \frac{x^2-2x}{8}$$

Lagrange Interpolation

$$
\begin{aligned}
P_2(x) &= \ell_0 f_0 + \ell_1 f_1 + \ell_2 f_2 \\
&= \frac{x^2 - 6x + 8}{8} + 7.3891 \frac{x^2 - 4x}{-4} + 54.5981 \frac{x^2 - 2x}{8} \\
&= 1 - 7.0104x + 5.1025x^2
\end{aligned}
$$

You can observe that the polynomial obtained from Newton's quadratic interpolation and Lagrange quadratic interpolation are same. Now,

$$P_2(1) = -0.9079375$$

$$\varepsilon_t = \frac{|2.71828 + 0.9079|}{|2.71828|} = 1.3340$$

Lagrange Interpolation

Example 3

By means of Lagrange's quadratic interpolation formula, find the values of $1/3$ using the information that

x	2	11/4	4
$1/x$	1/2	4/11	1/4

Find the error ϵ_t.

Lagrange Interpolation

Solution: The Lagrange linear interpolation polynomial is given by

$$\ell_0 = \left(\frac{x-2.75}{2-2.75}\right)\left(\frac{x-4}{2-4}\right) = \frac{x^2-6.75x+11}{1.5}$$

$$\ell_1 = \left(\frac{x-2}{2.75-2}\right)\left(\frac{x-4}{2.75-4}\right) = \frac{x^2-6x+8}{0.9375}$$

$$\ell_2 = \left(\frac{x-2}{4-2}\right)\left(\frac{x-2.75}{4-2.75}\right) = \frac{x^2-4.75x+5.5}{2.5}$$

Lagrange Interpolation

$$
\begin{aligned}
P_2(x) &= \ell_0 f_0 + \ell_1 f_1 + \ell_2 f_2 \\
&== \frac{x^2 - 6.75x + 11}{1.5}\frac{1}{2} + \frac{x^2 - 6x + 8}{0.9375}\frac{4}{11} + \frac{x^2 - 4.75x + 5.5}{2.5}\frac{1}{4} \\
&= 0.04545x^2 - 0.3977x + 1.1136
\end{aligned}
$$

Now,

$$P_2(3) = 0.32955$$

$$\epsilon_t = \frac{|0.32955 - 0.33333|}{|0.33333|} = 0.01134$$

Lagrange Interpolation

Example 4

The following data for the density of nitrogen gas versus temperature come from a table that was measured with high precision. Use first-through fifth-order Lagrange polynomials to estimate the density at a temperature of 330 K. What is your best estimate (use ϵ_a) among them?

$T(K)$	200	250	300	350	400	450
Density (Kg/m^3)	1.708	1.367	1.139	0.967	0.854	0.759

Lagrange Interpolation

Solution:
Since 330 lies between 300 and 350, we take $x_0 = 300, x_1 = 350$ for $P_1(x)$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

$$P_1(x) = \frac{x - 350}{300 - 350} 1.139 + \frac{x - 300}{350 - 300} 0.967$$

$$P_1(330) = \frac{330 - 350}{300 - 350} 1.139 + \frac{330 - 300}{350 - 300} 0.967$$

$$P_1(330) = 0.4556 + 0.5802 = 1.0358$$

Lagrange Interpolation

Since true value or prior approximate value of the density at a temperature 330 K is not known, we can't find ϵ_t. For $P_2(x)$, let us take $x_0 = 250, x_1 = 300, x_2 = 350.$

$$P_2(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right) f_0 + \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right) f_1 + \left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right) f_2$$

$$P_2(330) = 1.02908$$

Lagrange Interpolation

From $P_1(330) = 1.0358$ and $P_2(330) = 1.02908$, then

$$\epsilon_t = \frac{|1.0358 - 1.02908|}{1.0358} = 6.4877 \times 10^{-3}$$

For $P_3(x)$, let us take $x_0 = 250, x_1 = 300, x_2 = 350, x_3 = 400$.

$$P_3(330) = 1.02889$$

From $P_2(330) = 1.02908$ and $P_3(330) = 1.02889$, then

$$\epsilon_t = \frac{|1.02908 - 1.02889|}{1.02908} = 1.846 \times 10^{-4}$$

For $P_4(x)$, let us take $x_0 = 200, x_1 = 250, x_2 = 300, x_3 = 350, x_4 = 400$.

Lagrange Interpolation

$$P_4(330) = 1.03023, \epsilon_t = \frac{|1.02889 - 1.03023|}{1.03023} = 1.3 \times 10^{-3}$$

For $P_5(x)$, let us take

$x_0 = 200, x_1 = 250, x_2 = 300, x_3 = 350, x_4 = 400, x_5 = 450.$

$$P_5(330) = 1.02902, \epsilon_t = \frac{|1.02889 - 1.02902|}{1.02889} = 1.263 \times 10^{-4}$$

The best estimate is 1.02902.

Lagrange vs Newton's Divided Difference

Criteria	Lagrange Interpolation	Newton's Divided Difference
Formula	Direct formula using Lagrange Basis Polynomials	Incremental Construction using divided differences
Efficiency	Recomputes Entire Polynomial for new points	Efficient with incremental data; adds points without recalculation
Numerical Stability	Less stable for high-degree polynomialsprone to oscillation	More stable with incremental , updates but sensitive to point spacing
Point Order Sensitivity	Independent of point order	Dependent on the order of points
Ease of Use	Easier for smaller, static datasets	More complex but practical for large or dynamic datasets
Best Application Scenarios	Static interpolation, small problems	Dynamic data interpolation, adaptive numerical methods

Vandermonde Matrix

Vandermonde Matrix

There are a few other algorithms available for interpolation with its own advantages and disadvantages. Since the interpolation conditions $P_n(x_i) = f_i$ for $0 \le i \le n$, it produces a system of $n+1$ linear equations for determining $a_0, a_1, \cdots a_n$. In matrix form this can be written as

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

Vandermonde Matrix

The coefficient matrix is called Vandermonde matrix. This system is non-singular because the system has a unique solution for any choice of $f_0, f_1, \cdots, f_n$. Hence, the Vandermonde matrix is nonzero for distinct $x_0, x_1, \cdots, x_n$.
Note that, this matrix is often ill-conditioned and the coefficients a_i may be inaccurately determined. Therefore, this approach is not recommended mostly. The main property of the square Vandermonde matrix is that its determinant is given by

$$\prod_{0 \leq i \leq j \leq n} (x_j - x_i) \tag{5}$$

Vandermonde Matrix

This same matrix is used while solving differential equations.

Remarks

1. Usually for numerical, the Newton form of the interpolation polynomial is recommended. However, when we discuss the Numerical Integration, we will use the same cardinal polynomial for our integration
2. For Newton form polynomial, if more data points are added to the form, the coefficients already computed will not have to be changed
3. Lagrange interpolating polynomial are written down at once since the coefficients in the Lagrange formula are the given f_i.

Errors in Polynomial Interpolation

Errors in Polynomial Interpolation

Let us see some errors between a function and a polynomial interpolation.

Theorem 5 (First Interpolation Error Theorem)
Let $f \in C^{n+1}[a,b]$, and let P_n be a polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, x_2, \cdots, x_n$ in the interval $[a,b]$. Then for each $\overline{x} \in [a,b]$, there corresponds to $\xi \in (a,b)$ such that

$$f(\overline{x}) - P_n(\overline{x}) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (\overline{x} - x_i) \quad (6)$$

Errors in Polynomial Interpolation

This error estimation is an important theoretical result because Lagrange interpolation polynomials are used extensively for deriving numerical differentiation and integration methods. Note that the error form for the Lagrange or Newton's interpolation polynomial is similar to that of the Taylor polynomial. The nth Taylor polynomial about x_0 concentrates all the known information at x_0 and has an error term of the form

$$\frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1}.$$

However, the Lagrange or Newton interpolation polynomial of degree n uses information at the distinct numbers, $x_0, x_1, \cdots, x_n$ instead of $(x-x_0)^{(n+1)}$, it uses

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$$

Errors in Polynomial Interpolation

Example 6

Construct the Lagrange interpolating polynomial of degree 1 for $f(x) = e^x$ on the interval $[0, 1]$ with the given points $x_0 = 0, x_1 = 1$. Find a bound for the absolute error.

Solution: The interpolating polynomial is $P_1(x) = (1 - x) + e^x$. Therefore,

$$f(x) - P_1(x) = \frac{1}{(2)!} f''(\xi) x(x-1)$$

$$\max_{\xi \in [0,1]} |f''(\xi)| = \max_{\xi \in [0,1]} |e^{\xi}| \leq e, \max_{x \in [0,1]} |x(x-1)| \leq \frac{1}{4}$$

$$\implies |f(x) - P_1(x)| = |\frac{1}{(2)!} f''(\xi) x(x-1)| \leq \frac{e}{8} \approx 0.34$$

Errors in Polynomial Interpolation

Example 7

Construct the Lagrange interpolating polynomial of degree 2 for $f(x) = \cos x$ on the interval $[0, 0.9]$ with the given points $x_0 = 0, x_1 = 0.6$ and $x_2 = 0.9$. Compute $P_2(0.45)$. Find the actual error and then find the error using first interpolation error theorem

Solution: The interpolating polynomial is

$$P_2(x) = \frac{(x-0.6)(x-0.9)}{(0-0.6)(0-0.9)} + \frac{(x-0.0)(x-0.9)}{(0.6-0.0)(0.6-0.9)}\cos(0.6)$$

.

$$+\frac{(x-0.0)(x-0.6)}{(0.9-0.0)(0.9-0.6)}$$

$$P_2(0.45) = 0.8981007, \cos(0.45) = 0.9004471$$

$$f(0.45) - P_2(0.45) = 0.0023464$$

Errors in Polynomial Interpolation

$$f(x) - P_2(x) = \frac{1}{(3)!} f'''(\xi) x(x-0.6)(x-0.9)$$

$$\max_{\xi \in [0,0.9]} |f'''(\xi)| = \max_{\xi \in [0,0.9]} |\sin(\xi)| \leq \sin(0.9)$$

$$\max_{x \in [0,0.9]} |x(x-0.6)(x-0.9)| \leq \frac{1}{4}$$

Let $g(x) = x(x-0.6)(x-0.9) \implies g'(x) = 3(x^2 - x + 0.18)$,

$$g'(x) = 0 \implies x_1 = 0.235424, x_2 = 0.764575 \implies g(x_1) = 0.05704, g(x_2) = 0.01704$$

$$\implies |f(x) - P_1(x)| = |\frac{1}{(3)!} f'''(\xi) x(x-0.6)(x-0.9)| \leq \frac{1}{6} |\sin(0.9)| \times 0.05704 \approx 0.007446$$

Errors in Polynomial Interpolation

Example 8

By means of Lagrange's quadratic interpolation formula, find the values of $1/3$ using the information that

x	2	11/4	4
$1/x$	1/2	4/11	1/4

Obtain the error using the first interpolation error theorem.

$$f(x) = \frac{1}{x} \implies f'''(x) = -\frac{6}{x^4}$$

$$\left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \right| = \left| \frac{f^{(3)}(\xi)}{(3)!} (x-2)\left(x - \frac{11}{4}\right)(x-4) \right| \leq \frac{1}{16}\frac{9}{16} \approx 0.035156$$

Errors in Polynomial Interpolation

Theorem 9 (Upper Bound Lemma)

Suppose that $x_0 = a, x_1 = a+h, x_2 = a+2h, \cdots, x_i = a+ih \cdots, x_n = a+nh = b$, where $h = (b-a)/n$. Then for any $\overline{x} \in [a, b]$

$$\prod_{i=0}^{n} |\overline{x} - x_i| \leq \frac{1}{4} h^{n+1} n!$$

Errors in Polynomial Interpolation

Theorem 10 (Second Interpolation Error Theorem)
Let $f \in C^{n+1}[a, b]$ and $|f^{(n+1)}(x)| \leq M$. Let P_n be a polynomial of degree at most n that interpolates the function f at $n+1$ equally spaced points $x_0 = a, x_1 = a + h, x_2 = a + 2h, \cdots, x_n = a + nh = b$ in the interval $[a, b]$, where $h = (b - a)/n$. Then for each $\overline{x} \in [a, b]$, there corresponds to $\xi \in (a, b)$ such that

$$|f(\overline{x}) - P_n(\overline{x})| \leq \frac{1}{4(n+1)} M h^{n+1} \tag{7}$$

Errors in Polynomial Interpolation

Example 11

Without constructing the Lagrange interpolating polynomial for $f(x) = e^x$ on the interval $[0, 1]$ with 10 equally spaced points, find a bound for the absolute error using the second interpolation error theorem. How many points are required, if we want the error as 10^{-18}?

Solution: Given that 10 equally spaced points, therefore,
$x_i = ih, i = 0, 1, \cdots, n = 9$ where $h = 1/9$. Therefore, the error is

$$|f(\overline{x}) - P_9(\overline{x})| \le \frac{1}{4(10)} M \left(\frac{1}{9}\right)^{10}$$

$$M = \max_{\xi \in [0,1]} |f^{(10)}(\xi)| \le e \implies |f(\overline{x}) - P_9\overline{x})| \le 1.9490 \times 10^{-11}$$

14 points required for 10^{-18} accuracy (how?)

Errors in Polynomial Interpolation

This theorem gives loose upper bound on the interpolation error for different values of n.

Theorem 12 (Third Interpolation Error Theorem)
If P_n is a polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, x_2, \cdots, x_n$, then for any $\overline{x}$ that is not in the node,

$$f(\overline{x}) - P_n(\overline{x}) = f[x_0, x_1, x_2, \cdots, x_n, \overline{x}] \prod_{i=0}^{n} (\overline{x} - x_i) \tag{8}$$

If f is a polynomial of degree n, then all of the divided differences are zero for $i \geq n+1$.

Errors in Polynomial Interpolation

Example 13

Construct the Lagrange interpolating polynomial of degree 2 for $f(x) = \sin(\ln x)$ on the interval $[2, 2.6]$ with the points $x_0 = 2.0, x_1 = 2.4, x_2 = 2.6$. Find a bound for the absolute error.

Errors in Polynomial Interpolation

Solution:

The Lagrange polynomial of degree 2 is given by

$$P_2(x) = \left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)f_0 + \left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)f_1 + \left(\frac{x-x_0}{x_2-x_0}\right)$$

$$P_2(x) = \left(\frac{x-2.4}{2.0-2.4}\right)\left(\frac{x-2.6}{2.0-2.6}\right)\sin(\ln 2)$$

$$+\left(\frac{x-2}{2.4-2}\right)\left(\frac{x-2.6}{2.4-2.6}\right)\sin(\ln 2.4) + \left(\frac{x-2}{2.6-2}\right)\left(\frac{x-2.4}{2.6-2.4}\right)\sin(\ln 2.6)$$

$$f(x) - P_2(x) = \frac{1}{3!}f^{(3)}(x)(x-2)(x-2.4)(x-2.6) \leq \frac{1}{3!}f^{(3)}(2)(x-2)(x-2.4)(x-2.6)$$

$$\leq \frac{0.335765}{6}0.0169 = 9.457 \times 10^{-4}$$

Thanks

Doubts and Suggestions

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