MA633L-Numerical Analysis

Lecture 12 : Error in Polynomial Interpolation

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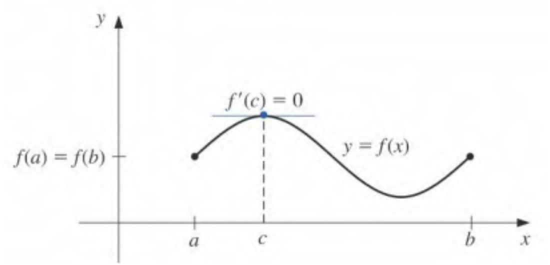
Errors in Polynomial Interpolation

Rolle's Theorem



Source: Thomas Calculus Book

Theorem 1 (Rolle's Theorem)
Suppose $f \in C[a,b]$ and f is differentiable on (a,b). If $f(a) = f(b)$, then a number $c \in (a,b)$ exists with $f'(c) = 0$.

Generalized Rolle's Theorem

Theorem 2 (Generalized Rolle's Theorem)
Suppose $f \in C[a,b]$ and f is n times differentiable on (a,b). If $f(x) = 0$, at the $n+1$ distinct numbers $a \leq x_0 < x_1 < \cdots < x_n \leq b$, then a number $c \in (x_0, x_n)$ exists with $f^{(n)}(c) = 0$.

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Theorem 3 (First Interpolation Error Theorem)
Let $f \in C^{n+1}[a,b]$, and let P_n be a polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, x_2, \cdots, x_n$ in the interval $[a,b]$. Then for each $\overline{x} \in [a,b]$, there corresponds to $\xi \in (a,b)$ such that

$$f(\overline{x}) - P_n(\overline{x}) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (\overline{x} - x_i) \quad (1)$$

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Proof:
If $\overline{x} = x_i$, the proof is obvious, as the equation reduces to zero. Assume that $\overline{x} \neq x_i$. Define a new function $\phi(t)$ in the variable t as follows:

$$\phi(t) = f(t) - P_n(t) - \left[\frac{\prod_{i=0}^{n} (t - x_i)}{\prod_{i=0}^{n} (\overline{x} - x_i)} \right] [f(\overline{x}) - P_n(\overline{x})]$$

Since x_i's are distinct and $\overline{x} \neq x_i$, the function ϕ is well defined. Also, observe that

$$\phi(x_i) = f(x_i) - P_n(x_i) - 0 = 0, \quad 0 \leq i \leq n$$

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Further,

$$\phi(\overline{x}) = f(\overline{x}) - P_n(\overline{x}) - \left[\frac{\prod_{i=0}^{n}(t - x_i)}{\prod_{i=0}^{n}(\overline{x} - x_i)}\right][f(\overline{x}) - P_n(\overline{x})] = 0$$

$\phi = 0$ at the $n+2$ points $x_0, x_1, \cdots, x_n$ and $\overline{x}$. Since $f \in C^{n+1}[a,b], P_n \in C^{\infty}[a,b], \phi \in C^{n+1}[a,b]$. Further $\phi(t) = 0$ at the $n+2$ distinct points, therefore as per the Generalized Rolle's theorem there exists a point $\xi \in (a,b)$ such that $\phi^{(n+1)}(\xi) = 0$.

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$$\phi^{(n+1)}(\xi) = 0 \implies f^{(n+1)}(\xi) - P_n^{(n+1)}(\xi) - \frac{d^{n+1}}{dt^{n+1}} \left[\frac{\prod_{i=0}^{n}(t - x_i)}{\prod_{i=0}^{n}(\overline{x} - x_i)} \right]_{t=\xi} [f(\overline{x}) - P_n(\overline{x})] = 0$$

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Since P_n is a polynomial of degree $\leq n$, $P_n^{(n+1)}(t) = 0$. Also (Prove!),

$$\frac{d^{n+1}}{dt^{n+1}} \left[\frac{\prod_{i=0}^{n}(t - x_i)}{\prod_{i=0}^{n}(\overline{x} - x_i)} \right] = \frac{(n+1)!}{\prod_{i=0}^{n}(\overline{x} - x_i)}$$

$$\phi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{(n+1)!}{\prod_{i=0}^{n}(\overline{x} - x_i)}[f(\overline{x}) - P_n(\overline{x})] = 0$$

The proof follows after rearrangements.

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Theorem 4 (Upper Bound Lemma)
Suppose that $x_0 = a, x_1 = a + h, x_2 = a + 2h, \cdots, x_i = a + ih, \cdots, x_n = a + nh = b$, where $h = (b - a)/n$. Then for any $\overline{x} \in [a, b]$

$$\prod_{i=0}^{n} |\overline{x} - x_i| \leq \frac{1}{4} h^{n+1} n!$$

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Proof:

Assume that $\overline{x} \in [x_j, x_{j+1}]$. Then you can prove that

$$|\overline{x} - x_j||\overline{x} - x_{j+1}| \leq \frac{h^2}{4}$$

Therefore,

$$\prod_{i=0}^{n} |\overline{x} - x_i| \leq \frac{h^2}{4} \prod_{i=0}^{j-1} (\overline{x} - x_i) \prod_{i=j+2}^{n} (x_i - \overline{x})$$

$$\overline{x} \leq x_{j+1} \implies \prod_{i=0}^{j-1} (\overline{x} - x_i) \leq \prod_{i=0}^{j-1} (\overline{x}_{j+1} - x_i)$$

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$$x_j \leq \overline{x} \implies \prod_{i=j+1}^{n} (x_i - \overline{x}) \leq \prod_{i=j+2}^{n} (x_i - x_j)$$

$$\prod_{i=0}^{n} |\overline{x} - x_i| \leq \frac{h^2}{4} \prod_{i=0}^{j-1} (x_{j+1} - x_i) \prod_{i=j+2}^{n} (x_i - x_j)$$

$$x_{j+1} - x_i = (j - i + 1)h, x_i - x_j = (i - j)h$$

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$$\implies \prod_{i=0}^{n} |\overline{x} - x_i| \leq \frac{h^2}{4} \left(h^j \prod_{i=0}^{j-1} (j - i + 1) \right) \left(h^{n-(j+2)+1} \prod_{i=j+2}^{n} (i - j) \right)$$

$$\implies \prod_{i=0}^{n} |\overline{x} - x_i| \leq \frac{h^{n+1}}{4} (j+1)!(n-j)! \leq \frac{h^{n+1}}{4} n!$$

Hence the proof.

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Theorem 5 (Second Interpolation Error Theorem)
Let $f \in C^{n+1}[a,b]$ and $|f^{(n+1)}(x)| \leq M$. Let P_n be a polynomial of degree at most n that interpolates the function f at $n+1$ equally spaced points $x_0 = a, x_1 = a+h, x_2 = a+2h, \cdots, x_n = a+nh = b$ in the interval $[a,b]$, where $h = (b-a)/n$. Then for each $\overline{x} \in [a,b]$, there corresponds to $\xi \in (a,b)$ such that

$$|f(\overline{x}) - P_n(\overline{x})| \leq \frac{1}{4(n+1)} M h^{n+1} \tag{2}$$

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Proof:
Applying first interpolation error theorem

$$f(\overline{x}) - P_n(\overline{x}) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (\overline{x} - x_i)$$

and upper bound lemma

$$\prod_{i=0}^{n} |\overline{x} - x_i| \leq \frac{h^{n+1}}{4} n!$$

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we get that

$$
\begin{aligned}
|f(\overline{x}) - P_n(\overline{x})| &= \left| \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (\overline{x} - x_i) \right| \\
&= \frac{1}{(n+1)!} |f^{(n+1)}(\xi)| \left| \prod_{i=0}^{n} (\overline{x} - x_i) \right| \\
&\leq \frac{1}{(n+1)!} M \frac{h^{n+1}}{4} n! \\
&= \frac{1}{4(n+1)} M h^{n+1}
\end{aligned}
$$

Hence the proof.

Errors in Polynomial Interpolation

This theorem gives loose upper bound on the interpolation error for different values of n.

Theorem 6 (Third Interpolation Error Theorem)
If P_n is a polynomial of degree at most n that interpolates the function f at $n+1$ distinct points $x_0, x_1, x_2, \cdots, x_n$, then for any $\overline{x}$ that is not in the node,

$$f(\overline{x}) - P_n(\overline{x}) = f[x_0, x_1, x_2, \cdots, x_n, \overline{x}] \prod_{i=0}^{n} (\overline{x} - x_i) \tag{3}$$

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Proof:
Recall that,

$$P_n(x) = \sum_{k=0}^{n} f[x_0, x_1, \cdots, x_k] \prod_{i=0}^{k-1} (x - x_i) \tag{4}$$

Let t be any point other than a node, where $f(t)$ is defined. Let Q_{n+1} be the polynomial of degree that interpolates f at $x_0, x_1, \cdots, x_n, t$. Then, we have

$$Q_{n+1}(x) = P_n(x) + f[x_0, x_1, x_2, \cdots, x_n, t] \prod_{i=0}^{n} (x - x_i)$$

Since $Q_{n+1}(t) = f(t)$, we have

$$f(t) = P_n(t) + f[x_0, x_1, x_2, \cdots, x_n, t] \prod_{i=0}^{n} (t - x_i)$$

Hence the proof.

Errors in Polynomial Interpolation

Theorem 7 (Relation between divided difference and derivatives)
Let $f \in C^n[a,b]$ and if there $n+1$ distinct points $x_0, x_1, x_2, \cdots, x_n, \overline{x}$ in the interval $[a,b]$. Then for some $\xi \in (a,b)$,

$$f[x_0, x_1, \cdots, x_n, \overline{x}] = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \tag{5}$$

Proof:
The proof follows by combining first and third interpolation error theorems.

Inverse Interpolation

Inverse Interpolation

Consider the following table

x	1	2	3	4	5	6	7
$f(x)$	1	0.5	1.33333	0.25	0.2	0.1667	0.1429

- Earlier we have given x and the requested to find $f(x)$ using interpolation. Now, suppose $f(x)$ is given, is it possible to find x.
- For example, what is x that corresponds to $f(x) = 0.3$ from the above table. You can observe that $f(x) = 1/x$ is the function, therefore, $x = 3.33333$ provides $f(x) = 0.3$.
- This is called inverse interpolation.

Inverse Interpolation

Observations:

- For more complicated case, you may switch the $f(x)$ and x values and apply Lagrange or Newton Interpolation methods.
- Unfortunately, when we reverse the variables, no guarantee that the values along the new abscissa will be evenly spaced.

Remedy:

- Alternatively, fit an nth order interpolation polynomial $f_n(x)$ to the original data
- Since x is evenly spaced, this polynomials will not be ill-conditioned.
- Hence, finding the value of x that makes this polynomial equal to given $f(x)$, the interpolation reduces to root finding problem.

Inverse Interpolation

The above problem gives $f_2(x)$ results

$$f_2(x) = 1.08333 - 0.375x + 0.041667x^2$$

Hence finding $f(x) = 0.3$ is nothing but finding the roots of the problem

$$0.78333 - 0.375x + 0.041667x^2 = 0$$

That is, $x = 5.701458$ or 3.295842

Thanks

Doubts and Suggestions

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